Developments in Mathematics

Jerzy Kąkol
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Descriptive Topology in Selected Topics of Functional Analysis

Updated and Expanded

Second Edition

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Jerzy Kąkol • Wiesław Kubiś •
Manuel López-Pellicer • Damian Sobota

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To our Friend and Teacher
Prof. Dr Manuel Valdivia

Preface to the Second Edition

We believe that the first edition (2011) of our monograph *Descriptive Topology in Selected Topics of Functional Analysis* was well received, so we decided to keep the presentation of the part corresponding to the first edition almost unchanged. The word "almost" means that only a certain number of chapters from the first edition have been expanded by adding complementary sections, this refers to new Sects. 9.5, 13.3, 16.5, and 16.6. The actual second edition is however an extension of the first one by a series of completely new Chaps. 17–21 presenting results concerning, e.g., topological spaces and groups with $\mathfrak{G}$-bases, various concepts related to networks and their applications in topology and functional analysis, and those that develop topological and analytic methods related to Grothendieck Banach spaces and Boolean algebras with the Nikodym property. We also gather some recently obtained results about distinguished spaces $C_p(X)$, providing a significant "bridge" between attractive problems from set theory and topology regarding Δ-sets, λ-sets, and $\mathbb{Q}$-sets X, and corresponding distinguished spaces $C_p(X)$.

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Poznań, Poland — Jerzy Kąkol
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Preface to the First Edition

We convoke (descriptive) topology recently applied to [Functional] Analysis of infinite-dimensional topological vector spaces, including Fréchet spaces, (LF)-spaces and their duals, Banach spaces $C(X)$ over compact spaces X, and spaces $C_p(X)$, $C_c(X)$ of continuous real-valued functions on a completely regular Hausdorff space X endowed with the pointwise and the compact-open topology, respectively. The (LF)-spaces and duals, particularly, appear in many fields of Functional Analysis/applications: Distribution Theory, Differential Equations, Complex Analysis, to name a few.

Our material, much of it in book form for the first time, carries forward the rich legacy of Köthe's *Topologische lineare Räume* (1960), Jarchow's *Locally Convex Spaces* (1981), Valdivia's *Topics in Locally Convex Spaces* (1982), and Pérez Carreras and Bonet's *Barrelled Locally Convex Spaces* (1987). We assume their (standard English) terminology. A topological vector space (tvs) must be Hausdorff and have real or complex scalar field. A *locally convex space* (lcs) is a tvs that is locally convex. Engelking's *General Topology* (1989) serves as a default reference for general topology.

Jerzy Kąkol
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Manuel López-Pellicer

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Chapter 1
Overview

Abstract The first chapter contains an overview of the material contained in the book, with several historical comments and links to the bibliography.

Let us briefly describe the organization of the book.

Chapter 2, which is essential to the sequel, contains classical results about Baire-type conditions (Baire-like, b-Baire-like, CS-barrelled, s-barrelled) on tvs. We include applications to closed graph theorems and $C(X)$ spaces. We also provide the first proof in book form of a remarkable result of Saxon from [531] (extending earlier results of Arias de Reyna and Valdivia) which states that, under Martin's axiom, every lcs containing a dense hyperplane contains a dense non-Baire hyperplane. Ours, then, is the first book to solve the first problem formally posed in Pérez Carreras and Bonet's excellent monograph. Chapter 2 also contains analytic characterizations of certain completely regular Hausdorff spaces X. For example, we show that X is pseudocompact, X is Warner bounded, or $C_c(X)$ is a (df)-space if and only if for each sequence $(\mu_n)_n$ in the dual $C_c(X)'$, there exists a sequence $(t_n)_n \subset (0, 1]$ such that $(t_n\mu_n)_n$ is weakly bounded, is strongly bounded, or is equicontinuous, respectively, [348, 349]. These characterizations help us produce a (df)-space $C_c(X)$ which is not a (DF)-space [349], solving a basic and long-standing open question. The third characterization is joined by nine more that supply tenfold an implied Jarchow request. These forge a strong link we happily claim between his book and ours.

Chapter 3 deals with the K-analyticity of a topological space E and the concept of a *resolution* generated on E, i.e. a family of sets $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ such that $E = \bigcup_\alpha K_\alpha$ and $K_\alpha \subset K_\beta$ if $\alpha \leq \beta$. Compact resolutions, i.e. resolutions $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ whose members are compact sets, naturally appear in many situations in topology and functional analysis. Any K-analytic space admits a compact resolution [579], and for many topological spaces X, the existence of such a resolution is enough for X to be K-analytic; see [117] and [127]. Many of the ideas in the book are related with the concept of compact resolution and are already in or have been inspired by papers of [117, 579], and [127]. It is an easy and elementary exercise to observe that any separable metric and complete space E

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admits a compact resolution, even swallowing compact sets. In Chap. 3, we gather some results, mostly due to Valdivia [611] about lcs (called quasi-(LB)-spaces) admitting resolutions consisting of Banach discs and their relations with the closed graph theorems.

These concepts are related with another one called a Suslin scheme, which provides a powerful tool to study structural properties of metric separable spaces; see [258] and [522]. Chapter 3 presents Hurewicz and Alexandrov's theorems as well as Calbrix–Hurewicz's theorem which yields that a regular analytic space X, i.e. a continuous image of the space $\mathbb{N}^{\mathbb{N}}$, is not σ-compact if and only if X contains a closed subset homeomorphic to $\mathbb{N}^{\mathbb{N}}$. We tried to present proofs in a transparent form. The reader is also referred among others to magnificent works of [522, 573, 579, 611], and [140].

Chapter 4 deals with the class of angelic spaces, introduced by Fremlin, for which several variants of compactness coincide. A remarkable paper of Orihuela [481] introduces a large class of topological spaces X (under the name *web-compact*) for which the space $C_p(X)$ is angelic. Orihuela's theorem covers many already known partial results providing Eberlein–Šmulian-type results. Following Orihuela [481], we show that $C_p(X)$ is angelic if X is web-compact. This yields, in particular, Talagrand's result [579] stating that for a compact space X, the space $C_p(X)$ is K-analytic if and only if $C(X)$ is weakly K-analytic. In Chap. 4, we present some quantitative versions of Grothendieck's characterization of the weak compactness for spaces $C(X)$ (for compact Hausdorff spaces X) and quantitative versions of the classical Eberlein–Grothendieck and Krein–Šmulian theorems. We follow very recent works of Angosto and Cascales [11–13]; Angosto [10]; Angosto et al. [14]; Cascales et al. [121]; Hájek et al. [200]; and Granero [283]. The last two articles [200] and [283], where in the case of Banach spaces these quantitative generalizations have been studied and presented, motivated the other mentioned papers.

In Chap. 5, we continue the study of web-compact spaces. A subclass of web-compact spaces, called strongly web-compact, is introduced, and a closed graph theorem for such spaces is provided. We prove that an own product of a strongly web-compact space need not be web-compact. This applies to show that there exists a quasi-Suslin space X such that $X \times X$ is not quasi-Suslin.

Chapter 6 studies analytic spaces. We show that a regular space X is analytic if and only if X has a compact resolution and admits a weaker metric topology. This fact, essentially due to Talagrand [577], extended Choquet's theorem [138] (every metric K-analytic space is analytic); see also [125]. Several applications will be provided. We show Christensen's theorem [140] stating that a separable metric topological space X is a Polish space if and only if X admits a compact resolution swallowing compact sets. The concept of a compact resolution swallowing compact sets is already present in the main result of [127, Theorem 1]. We study trans-separable spaces and show that a tvs with a resolution of precompact sets is trans-separable [517]. This applies to prove [127] that precompact sets are metrizable in any uniform space whose uniformity admits a $\mathfrak{G}$-base. Consequences are provided.

Chapter 6 works also with the following general problem (among some others): When does analyticity or K-analyticity of the weak topology $\sigma(E, E')$ of a dual pair (E, E') can be lifted to stronger topologies on E compatible with the dual pair? The question is essential since (as we show) there exist many weakly analytic lcs, i.e. analytic in the weak topology $\sigma(E, E')$, which are not analytic. We prove that if X is an uncountable analytic space, the Mackey dual $L_\mu(X)$ of $C_p(X)$ is weakly analytic and not analytic.

The density condition, due to Heinrich [307], studied in a series of papers of Bierstedt and Bonet [80–84] motivates a part of Chap. 6 to study the analyticity of the Mackey and strong dual of (LF)-spaces.

In Chap. 7, we show that a tvs, which is a Baire space and admits a countably compact resolution, is metrizable separable and complete. This extends a classical result of De Wilde and Sunyach [159] and Valdivia's theorem of [611]. An interesting recent applicable result due to Drewnowski (highly inspired by [335]) about continuous maps between F-spaces is presented. Namely, we show that a linear map $T : E \to F$ from an F-space E having a resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ into a tvs F is continuous if each restriction $T|K_\alpha$ is continuous. This theorem was motivated by Arias de Reyna–Valdivia–Saxon's theorem about non-Baire dense hyperplanes in Banach spaces. We provide a large class of weakly analytic metrizable and separable Baire tvs not analytic (clearly such spaces are necessarily not locally convex).

Examples of spaces of this type will be used in Chap. 8 to prove that analyticity is not a three-space property. We prove however that a metrizable topological vector space E is analytic if E contains a complete locally convex analytic subspace F such that the quotient E/F is analytic. We reprove also in Chap. 8 (using Corson's example [147]) that the Lindelöf property is not a three-space property.

Chapter 9 continues partially the study started in Chap. 3 and deals with K-analytic and analytic spaces $C_p(X)$. Some results due to Talagrand [579], Tkachuk [592], Velichko [32], and Canela [115] are presented. We extend the main result of [592] characterizing K-analytic spaces $C_p(X)$ in terms of resolutions. A remarkable Christensen's theorem [140] stating that a metrizable and separable space X is σ-compact if and only if $C_p(X)$ is analytic is proved. We show that the analyticity of $C_p(X)$ for any X implies that X is σ-compact, Calbrix [113]. A characterization of σ-compactness of a cosmic space X in terms of subspaces of $\mathbb{R}^X$ is provided, Arkhangel'skii–Calbrix [36]. Finally, we show that $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$ if and only if $C_p(X)$ admits a bounded resolution [209]. We collect also several equivalent conditions for spaces $C_p(X)$ to be a Lindelöf space over locally compact groups X; see [338].

Chapter 10, which might be a good motivation for Chaps. 11 and 12, extends the main result of Cascales–Orihuela [126] and presents the unified and direct proofs [346] of Pfister [492], Cascales–Orihuela [126], and Valdivia's [610] theorems about metrizability of precompact sets in (LF)-spaces, (DF)-spaces, and dual metric spaces, respectively. The proofs from [346] do not require the typical machinery of quasi-Suslin spaces, upper semicontinuous compact-valued maps, etc.

Chapter 11 introduces (after Cascales and Orihuela [128]) a large class of locally convex spaces under the name *the class* $\mathfrak{G}$. A lcs E is said to be in the class $\mathfrak{G}$ if its topological dual E' admits a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that sequences in each A_α are equicontinuous. The class $\mathfrak{G}$ contains among others all (LM)-spaces (hence (LF)-spaces), and dual metric spaces (hence (DF)-spaces), spaces of distributions $D'(\Omega)$, spaces $A(\Omega)$ of real analytic functions on open $\Omega \subset \mathbb{R}^n$, etc. We show in Chap. 11 the main result of [128], with a simpler proof from [215], stating that every precompact set in a lcs in the class $\mathfrak{G}$ is metrizable. This general result covers many already known theorems for (DF)-spaces, (LF)-spaces, and dual metric spaces, respectively.

In Chap. 12, we continue the study of spaces in the class $\mathfrak{G}$. We prove that the *weak** dual $(E', \sigma(E', E))$ of a lcs E in the class $\mathfrak{G}$ is K-analytic if and only if $(E', \sigma(E', E))$ is Lindelöf if and only if $(E, \sigma(E, E'))$ has countable tightness if and only if each finite product $(E', \sigma(E', E))^n$ is Lindelöf; see [118]. Developing the argument producing upper semicontinuous maps, we show also that every quasibarrelled space in the class $\mathfrak{G}$ has countable tightness both for the weak and the original topologies. This extends a classical result of Kaplansky for metrizable lcs; see [240]. Although (DF)-spaces belong to the class $\mathfrak{G}$, concrete examples of (DF)-spaces without countable tightness are provided.

On the other hand, there are many Banach spaces E for which E endowed with the weak topology $\sigma(E, E')$ is not Lindelöf. We show, however, Khurana [366], that every weakly compactly generated (WCG) Fréchet space E is weakly K-analytic, i.e. $(E, \sigma(E, E'))$ is K-analytic. This extends Talagrand's corresponding result for (WCG) Banach spaces; see [575] and also [198] and [482]. In general, for (WCG) lcs, this result fails, as Hunter and Lloyd [318] have shown. It is natural to ask (Corson [147]; see also [409]) if every weakly Lindelöf Banach space is a (WCG) Banach space, i.e. E admits a weakly compact set whose linear span is dense in E.

The first example of a non (WCG)-Banach space whose weak topology is Lindelöf was provided by Rosenthal [525]. We present an example due to Pol [502], showing that there exists a Banach space $C(X)$ over a compact scattered space X such that $C(X)$ is weakly Lindelöf and $C(X)$ is not a (WCG) Banach space. This example answers also (in the negative) some questions of Corson [147], Problem 7 posed by Benyamini, Rudin, and Wage from [78]. Talagrand, inspired and motivated by several results of Corson, Lindestrauss, and Amir, continued this line of research in his remarkable papers; see, e.g. [575, 576, 579], and [578]. Chapter 12 contains also the proof of Amir–Lindenstrauss's theorem that every non-separable reflexive Banach space contains a complemented separable subspace [408]. Several consequences are provided. This subject, related to (WCG) Banach spaces and Amir-Lindenstrauss's theorem, will be continued in Chaps. 23, 24, and 25, where Banach spaces with a rich family of projections onto separable subspaces are studied.

In Chap. 13, the class of Banach spaces having the property (C) is studied. This property, isolated by Corson [147], provides a large subclass of Banach spaces E whose weak topology is Lindelöf. We collect some results of Corson [147], Pol [503, 503, 504, 508], and Frankiewicz et al. [243].

Chapters 14 and 15 deal with topological (vector) spaces satisfying some sequentially conditions. We study Fréchet–Urysohn spaces, i.e. spaces E such that for each $A \subset E$ and each $x \in \overline{A}$, there exists a sequence in A converging to x. The main result states that every sequentially complete Fréchet–Urysohn lcs is a Baire space.

Since every infinite-dimensional Montel (DF)-space E is non-metrizable and sequential (i.e. every sequentially closed set in E is closed), the following question arises: Is every Fréchet–Urysohn space in the class $\mathfrak{G}$ metrizable?

In Chap. 15, we prove that a lcs in the class $\mathfrak{G}$ is metrizable if and only if E is b-Baire-like if and only if E is Fréchet–Urysohn; see [118] and [119]. Consequently, no proper (LB)-space E is Fréchet–Urysohn (since E contains the space φ, i.e. the $\aleph_0$-dimensional vector space with the finest locally convex topology). We prove that if a (DF) or (LM)-space E is sequential, then E is either metrizable or Montel (DF); see [346]. In [638], Webb introduced the property C_3 (i.e. sequential closure of any set is sequentially closed), which characterizes metrizability for (LM)-spaces but not for (DF)-spaces. We distinguish a variant of the property C_3 called property C_3^- (i.e. sequential closure of any vector subspace is sequentially closed) and characterize both (DF)-spaces and (LF)-spaces with the property C_3^- as being of the form M, φ, or $M \times \varphi$, where M is metrizable [346].

In Chap. 16, we apply the concept of the tightness to study distinguished Fréchet spaces. Valdivia provided a non-distinguished Fréchet space whose *weak** bidual is quasi-Suslin but not K-analytic; see [611]. Using the concept of the tightness, we show that Köthe's echelon non-distinguished Fréchet space $\lambda_1(A)$ serves the same purpose [220], and we provide another (much simpler) proof of the deep result of Bastin and Bonet stating that for $\lambda_1(A)$, there exists a locally bounded discontinuous linear functional over the space $(\lambda_1', \beta(\lambda_1', \lambda_1))$; see [216]. The basic fact [220] is that a (DF)-space is quasibarrelled if and only if its tightness is countable. We show that a Fréchet space is distinguished if and only if its strong dual has countable tightness. This approach to study distinguished Fréchet spaces leads to a rich supply of (DF)-spaces whose *weak** duals are quasi-Suslin but not K-analytic, for example, spaces $C_c(\kappa)$ for κ a cardinal of uncountable cofinality. The small cardinals $\mathfrak{b}$ and $\mathfrak{d}$ will be used to improve the analysis of Köthe's example; see [221] and [220]. The bounding cardinal $\mathfrak{b}$ (introduced by Rothberger) is the smallest infinite-dimensionality for metrizable barrelled spaces; see [536] for detail.

In general, a quasibarrelled E belongs to the class $\mathfrak{G}$ if and only if E admits a $\mathfrak{G}$-base, i.e. a family $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of neighbourhoods of zero in E such that every neighbourhood of zero in E contains some U_α; see [220]. This concept provides spaces $C_c(X)$, different from what Talagrand presented in [579], whose *weak** dual is not K-analytic but does have compact resolutions. We show that the *weak** dual of any space in the class $\mathfrak{G}$ is quasi-Suslin [221]. An immediate consequence is that every space with a $\mathfrak{G}$-base enjoys this property. In Chap. 16, we show that $C_c(\omega_1)$ may or may not have a $\mathfrak{G}$-base. The existence of a $\mathfrak{G}$-base for $C_c(\omega_1)$ depends on the ZFC-consistent axiom system. $C_c(\omega_1)$ has a $\mathfrak{G}$-base if and only if $\aleph_1 = \mathfrak{b}$. Several interesting examples of (DF)-spaces which admit and do not admit $\mathfrak{G}$-bases will be

also provided. Recently, the concept of lcs with $\mathfrak{G}$-bases has been extended to the class of topological spaces; see, for example, [261, 266, 267], and [57]. Note that Banakh in [57] topological spaces possessing a (locally uniform) $\mathfrak{G}$-base has been studied under the name a ω^ω-base.

In Chap. 17, we present a number of recent results related to the study of spaces $C_p(X)$ on Tychonoff spaces X that are distinguished, i.e. spaces $C_p(X)$ that are large subspaces of $\mathbb{R}^X$, equivalently, spaces whose strong dual $L_\beta(X)$ of $C_p(X)$ carries the finest locally convex topology. Most of results concerning distinguished spaces $C_p(X)$ have been published in [207, 211, 213, 232, 332, 333, 406], and [224]. We prove, among the others, that the locally convex space $C_p(X)$ is distinguished if and only if X is a Δ-space in the sense of Reed. This theorem apparently provides (with several applications) a significant "bridge" between the still attractive problems from the set theory and related with Δ-sets, λ-sets, and $\mathbb{Q}$-sets X and corresponding distinguished spaces $C_p(X)$.

As an application of this characterization theorem, we obtain the following results: If X is a Čech-complete space such that $C_p(X)$ is distinguished, then X is scattered. If X is the compact space $[0, \omega_1]$, then $C_p(X)$ is not distinguished. Nevertheless, for every Eberlein compact scattered space X, the space $C_p(X)$ is distinguished.

Chapter 18 deals with several concepts related with networks and applications both in topology and functional analysis. Especially, we will discuss these concepts for topological groups with $\mathfrak{G}$-bases. Recall that topologists are seeking several possible general sequential concepts, which together under an additional topological condition, force a topological space X to be metrizable. Following this line of research, Tsaban and Zdomskyy [605] introduced a property (the strong Pytkeev property) that implies the countable tightness and, in the frame of Fréchet–Urysohn topological groups, yields the metrizability.

It turned out that for many classes of lcs E in functional analysis the weak topology of the space E is a generalized metric space, we refer to recent papers [58, 265, 268] and references therein. The concept of network is one of the well-recognized good tools, coming from the pure set-topology, which turned out to be of great importance to study successfully renorming theory in Banach spaces; see the survey paper [129], especially [129, Theorem 13] for $\sigma(E, E')$-slicely networks.

In this chapter, we show also that for any topological group G with the strong Pytkeev property, there exists an ordered family $\{U_\alpha : \alpha \in \mathbf{M}\}$ of subsets of G, where $\mathbf{M} \subset \mathbb{N}^\mathbb{N}$, which is an "almost" $\mathfrak{G}$-base (of neighbourhoods of the unit) of G.

We prove that a wide class of topological groups with a $\mathfrak{G}$-base has the strong Pytkeev property. We study the strong Pytkeev property for several well-known classes of lcs including (DF)-spaces and (LM)-spaces.

In Chap. 19, we study Banach spaces $C(K)$ that are Grothendieck. We provide two characterizations of Grothendieck spaces $C(K)$: in terms of operators onto the Banach space c_0 and in terms of sequences of Radon measures related to the classical Josefson–Nissenzweig theorem from Banach space theory. The latter approach will be further studied and extended in Chap. 20 in the context of the so-called ℓ_1-Grothendieck property. We also present in detail a proof of Grothendieck's theorem

asserting that for every extremely disconnected compact space K, the space $C(K)$ is Grothendieck—for this purpose, we present proofs of Rosenthal's "disjointification" lemma for sequences of measures and Dieudonné–Grothendieck's characterization of weakly compact subsets of dual spaces $C(K)'$. We also briefly discuss the possible minimal density of a space $C(K)$ with the Grothendieck property. For basic information concerning Grothendieck $C(K)$-spaces, we refer the reader to monographs [150] and [44].

Chapter 20 is devoted to the study of a weak variant of the Grothendieck property for Banach spaces $C(K)$, called the ℓ_1-Grothendieck property. As in the case of the original Grothendieck property, we characterize the ℓ_1-Grothendieck property in terms of operators onto the space c_0 (endowed here with the pointwise topology) and in terms of sequences of (finitely supported) Radon measures related to Josefson–Nissenzweig's theorem. We present a construction of a separable compact space K such that $C(K)$ has the ℓ_1-Grothendieck property but it does not have the Grothendieck property. Generalizing the classical result of Cembranos and Freniche, we show that spaces of the form $C(K \times L)$ have never even the ℓ_1-Grothendieck property. We also study the ℓ_1-Grothendieck property for spaces $C(K)$, where K is the limit of an inverse system based on simple extensions of totally disconnected compact spaces. The latter issue appears to shed some light on classical constructions of Efimov spaces.

In Chap. 21, we study the Nikodym property for Boolean algebras. The property is closely related to the barrelledness of the subspace of $C(St(\mathcal{A}))$, where $St(\mathcal{A})$ denotes the Stone space of $\mathcal{A}$, spanned by the set of all characteristic functions corresponding via the Stone duality to elements of $\mathcal{A}$, and hence to the classical Banach–Steinhaus theorem. We present a proof of a variant of Nikodym's uniform boundedness principle, asserting that every Boolean algebra with Haydon's Subsequential Completeness Property has the Nikodym property. We introduce the strong Nikodym property and show a proof of Valdivia's result asserting that every σ-field has the strong Nikodym property as well as discuss its various consequences.

In Chap. 22, we continue the subject developed in Chap. 12 related to (WCG) Banach spaces and the Amir–Lindestrauss theorem. We discuss Banach spaces that have a rich family of norm-one projections onto separable subspaces. Probably, the most general class of Banach spaces with "many" projections specifies the separable complementation property. Recall that a Banach space E has the separable complementation property (SCP) if for every separable subspace D of E, there exists a bounded linear projection with a separable range containing D. This property seems to be too general for proving any reasonable structural properties of Banach spaces. A strengthening of the (SCP) is the notion of a projectional skeleton defined and studied in Sect. 22.5. Banach spaces with a projectional skeleton have good stability properties as well as some nice structural ones. For instance, they have an equivalent locally uniformly convex norm, and they admit a bounded injective linear operator into some $c_0(\Gamma)$ space.

A natural property of a projectional skeleton is *commutativity*. It turns out that this is equivalent to the existence of a countably norming Markushevich basis. A space with this property is often called a *Plichko space*. We study this class of spaces

in Sect. 22.7. A more special property of a projectional skeleton gives the class of (WLD) Banach spaces. We present selected results concerning the complementation property in general Banach spaces, in order to motivate the study of projectional skeletons and projectional resolutions. This part contains some information about Plichko spaces, stability of this class, and some natural examples.

In order to simplify several arguments and constructions of projections, we present in Sect. 22.1 the method of elementary substructures coming from logic. Using this method, we prove, for example, a result on the Lindelöf property of the topology induced by a certain norming subspace of the dual, induced by the projectional skeleton. In the case of (WLD) spaces, this is the well-known result on the Lindelöf property of the weak topology. The method of elementary substructures is also used in the next chapters for proving topological properties, e.g. countable tightness.

Chapter 23 discusses selected properties of Banach spaces of type $C(X)$, where X is a linearly ordered compact space, called shortly a compact line. In particular, we present Nakhmanson's theorem stating that if X is a compact line such that $C_p(X)$ is a Lindelöf space, then X is second countable.

Compact lines are relatively easy to investigate, yet they form a rich class of spaces and provide several interesting examples. A very special case is the smallest uncountable well-ordered space $\omega_1 + 1$, which appears several times in the previous chapters. Its space of continuous functions turns out to be a canonical example for several topological and geometric properties of Banach spaces. More complicated compact lines provide examples related to Plichko spaces.

Chapter 24 presents several classes of non-metrizable compact spaces which correspond to well-known classes of Banach spaces with many projections. In particular, we discuss the class of Valdivia compact spaces and its subclasses: Corson and Eberlein compact spaces. We discuss a general class of compact spaces obtained by limits of continuous retractive sequences. We also introduce the notion of a retractional skeleton, dual to projectional skeletons in Banach spaces. The last section of Chap. 24 contains an overview of Eberlein compact spaces with some classical results and examples relevant to the subject of previous chapters.

Finally, Chap. 25 deals with complementably universal Banach spaces. Assuming the Continuum Hypothesis, there exists a complementably universal Banach space of density $\aleph_1$ for the class of Banach spaces with a projectional resolution of the identity. Similar methods produce a universal pre-image for the class of Valdivia compacta of weight $\aleph_1$.

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1.1 General Comments and Historical Facts

The earliest approach to K-analytic spaces seems to be due to Choquet [137], who called a topological space X K-analytic if it is a $K_{\sigma\delta}$-set in some compact space. Rogers proved [521] that if X is a completely regular Hausdorff space, the last condition is equivalent to the following one:

(*) *There exists a Polish space Y and an upper semicontinuous compact-valued map from Y covering X.*

The property (*) can be seen in Martineau [433]; see also Frolik [257], Rogers [521], Stegall [573], and Sion [560, 561]. In our book by a K-analytic space X, we mean a topological space satisfying condition (*). Since every Polish space is a continuous image of the space $\mathbb{N}^{\mathbb{N}}$, we note an equivalent another way to look at K-analytic spaces as the image under an upper semicontinuous compact-valued map of the space $\mathbb{N}^{\mathbb{N}}$. The images under upper semicontinuous compact set-valued maps of Polish spaces are also known in the literature as K-Suslin spaces; see also [257, 433, 611], and [521]. K-analytic and analytic spaces are also useful topological objects to study nice properties of topological measures; see [550] and [249]. For example, every semifinite topological measure, which is inner regular for closed sets for a K-analytic space, is inner regular for compact sets, and every semifinite Borel measure is inner regular for compact sets for an analytic space. Every K-analytic space X admits a compact resolution, i.e. a family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact sets covering X such that $K_\alpha \subset K_\beta$ if $\alpha \leq \beta$; see [117, 127, 579], and [180] (in [180] formally, this term was used for the first time). In the frame of angelic spaces X, the existence of a compact resolution implies that X is necessarily a K-analytic space, [117]. Talagrand [579] already observed this for spaces $C_p(X)$ over compact spaces X; see also [125].

Many interesting topological problems in infinite-dimensional topological vector spaces might be motivated by some results from the theory of $C_p(X)$ spaces. Let us mention, for example, one of them due to Velichko [32, Theorem I.2.1]:

The space $C_p(X)$ is σ-compact, i.e. covered by a sequence of compact sets, if and only if X is finite.

Tkachuk and Shakhmatov [598] extended this result to σ-countably compact spaces $C_p(X)$; see also [28] for a general approach including both cases. Clearly, if $(K_n)_n$ is an increasing sequence of compact sets covering $C_p(X)$, then the sets $K_\alpha := K_{n_1}$, where $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, form a compact resolution for $C_p(X)$. On the other hand, we can prove that $C_p(X)$ has a fundamental sequence of bounded sets only if X is finite; see Chap. 2.

Natural questions arise: Characterize completely regular Hausdorff spaces X for which $C_p(X)$ admits a compact resolution. When does $C_p(X)$ admit a resolution

consisting of topologically bounded sets? Recently, a related problem has been solved by Tkachuk [592], who proved the following:

$C_p(X)$ is K-analytic if and only if $C_p(X)$ admits a compact resolution.

We provide another approach to solve this problem. We show that if $C_p(X)$ admits a resolution consisting of tvs-bounded sets, i.e. sets that are absorbed by any neighbourhood of zero in $C_p(X)$, then $C_p(X)$ is angelic. Since angelic spaces with compact resolutions are K-analytic [117], this yields Tkachuk's result and provides more applications.

The class of weakly Lindelöf determined Banach spaces ((WLD) Banach spaces introduced in [17]) provides a larger class of weakly Lindelöf Banach spaces containing the weakly compactly generated Banach spaces ((WCG) Banach spaces). The study of (WLD) Banach spaces was motivated by results of Gul'ko [296] about weakly K-countably determined Banach spaces (called also weakly countably determined (WCD), or weakly Lindelöf Σ-spaces, or Vašák spaces, [579, 632]; see also [196] and [441]). Recall that according to [483], a Banach space E is (WLD) if and only if its closed unit ball in E' is Corson compact in $\sigma(E', E)$. Quite recently Cascales, Namioka, and Orihuela [123] have shown that if E is a Banach space satisfying Corson's property (C) and E admits a projectional generator, then E is (WLD). We reprove this result in Sect. 22.6 using the notion of a projectional skeleton. See also [124] and [198] (and references) for more details.

In 1961, Corson [147] started a systematic study of certain topological properties of the weak topology of Banach spaces. This line of research provided more general classes such as reflexive Banach spaces, weakly compactly generated Banach spaces [8, 168, 525], and the class of weakly K-analytic and weakly K-countably determined Banach spaces. For another approach to study geometric and topological properties of non-separable Banach spaces, we refer to [300].

In his fundamental paper [147], Corson asked if (WCG) Banach spaces are exactly those Banach spaces whose weak topology is Lindelöf. The first example of a non (WCG)-Banach space whose weak topology was Lindelöf was provided by Rosenthal [525]. We refer the reader to the monographs of [196, 198], and [199] for many essential information about (WCG) Banach spaces.

One can ask for which compact spaces X the space $C_p(X)$ is Lindelöf. This problem was first studied by Corson [147]; see also [149]. The class of Corson compact spaces X, i.e. homeomorphic to a compact subset of a Σ-product of real lines, provides examples of Lindelöf spaces $C_p(X)$; see [7] and [295]. On the other hand, for every weakly K-analytic Banach space E, the closed unit ball in E' in the topology $\sigma(E', E)$ is a Corson compact set [296]. There exist, however, examples due to Talagrand, Haydon, and Kunen (under CH) of Corson compact spaces X such that the Banach space $C(X)$ is not weakly Lindelöf; see [466]. Also, in [18], it was shown that for a Corson compact space X, the Banach space $C(X)$ is (WLD) if and only if every positive regular Borel measure on X has separable support. In general, the claim that for every Corson compact space X the space $C(X)$ is a (WLD) Banach space is independent of the usual axioms of the set theory [18].

There exist concrete Banach spaces $C(X)$ over compact scattered spaces X which are weakly Lindelöf but not (WCG); see, for example, [503] and [579].

Nevertheless, for a compact space X, the Banach space $C(X)$ is (WCG) if and only if X is Eberlein compact [8]; see also [196, 198]. For a compact space X the space $C(X)$ is weakly K-analytic if and only if $C_p(X)$ is K-analytic [579]. This distinguishes the class of Talagrand compact spaces, i.e. such compact X for which $C_p(X)$ is K-analytic.

This line of research between topology and functional analysis inspired several specialists (mainly from functional analysis) in developing new technics from descriptive topology to study concrete problems and classes of spaces in linear functional analysis; see [198] for many references. For example, one may ask the following:

(i) Is a Banach space E weakly Lindelöf if its *weak** dual $(E', \sigma(E', E))$ has countable tightness?
(ii) If $\sigma(E, E')$ is Lindelöf, is the unit ball in E' of countable tightness in $\sigma(E', E)$?

Question (ii) is related to Banach spaces satisfying the property (C) of Corson. This property, introduced by Corson [147], provided a large subclass of Banach spaces E whose weak topology is Lindelöf. Papers of Corson and Pol described the property (C) in terms of countable tightness-type conditions for the topology $\sigma(E', E)$.

Corson's paper [147] concerning the property (C) and results of Pol from [502, 504, 508], or [505] motivated several articles, for example, [243, 482, 496, 498], and [118], to study concrete classes of the weakly Lindelöf Banach spaces. This subject of research has been continued by many specialists; we refer the reader to articles [115, 117, 128, 440] and [441]; see also [118, 119, 125, 213, 216, 220, 221, 221], and [180].

Many important spaces in functional analysis are defined as certain (DF)-spaces, or (LB)-spaces, or (LF)-spaces, i.e. inductive limits of a sequence of Banach (Fréchet) spaces, or their strong duals; see, for example, [80, 82, 83, 322, 439, 491], and [611] as good sources of information.

A significant difference to the Banach space case is that the strong dual of a Fréchet space is not metrizable in general. The strong dual of Fréchet spaces are (DF)-spaces, introduced by Grothendieck [287]. Clearly, any (LB)-space is a (DF)-space. One can ask, among others, for which (LF)-spaces E their Mackey dual $(E', \mu(E', E))$, or the strong dual $(E', \beta(E', E))$, is K-analytic or even analytic. It was known already [128] that the precompact dual of any separable (LF)-space is analytic. In [546, 547], the class of (SWCG) Banach spaces E for which the Mackey topology $\mu(E', E)$ arises in a natural way was introduced.

There are many interesting topological problems related to the above classes of lcs. Let us mention a few of them strictly connected with the topic of the book. Floret in [239], motivated by earlier works of Grothendieck, Fremlin, De Wilde, and Pryce, asked if the compact sets in any (LF)-space are metrizable and if any (LF)-space is weakly angelic. Although the first question (as we have already mentioned in the Preface) has been answered positively for (DF)-spaces and dual metric spaces, both questions for (LF)-spaces have been solved (also positively) by Cascales and Orihuela in [126]. In [481], Orihuela answered the second question also for dual

metric spaces. Therefore, it was natural to ask about a possible large class of lcs (clearly including (LF)-spaces and dual metric spaces) for which both questions have also positive answers. Such a class of lcs, called the class $\mathfrak{G}$, was introduced in [127] by Cascales and Orihuela. A lcs E belongs to $\mathfrak{G}$ if the *weak** dual E' admits a resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ consisting of $\sigma(E', E)$-relatively countably compact sets such that each sequence in any K_α is equicontinuous.

Spaces in the class $\mathfrak{G}$ enjoy interesting topological properties, for example:

(i) The weak topology of $E \in \mathfrak{G}$ is angelic and every precompact set in E is metrizable.
(ii) For $E \in \mathfrak{G}$, the density of E and $(E', \sigma(E', E))$ coincide if $(E, \sigma(E, E'))$ is a Lindelöf Σ-space [127]; this extends a classical result of Amir–Lindestrauss for (WCG) Banach spaces.
(iii) For a compact space X, the space $C_p(X)$ is K-analytic if and only if X is homeomorphic to a weakly compact set of a locally convex space in the class $\mathfrak{G}$; see [127]. Recall that a compact space X is Eberlein compact if and only if X is homeomorphic to a weakly compact subset of a Banach space.
(iv) A lcs in $\mathfrak{G}$ is metrizable if and only if it is Fréchet–Urysohn. A barrelled lcs E in the class $\mathfrak{G}$ (e.g. any (LF)-space E) is metrizable if and only if E is Fréchet–Urysohn if and only if E is Baire-like if and only if E does not contain φ, i.e. the $\aleph_0$-dimensional vector space with the finest locally convex topology [119].
(v) Every quasibarrelled space E in $\mathfrak{G}$ has countable tightness and the same also holds true for $(E, \sigma(E, E'))$, [118]. This extends to classical Kaplansky's result; see [240].

On the other hand, there is a large and important class of lcs that do not belong to $\mathfrak{G}$. A lcs $C_p(X)$ belongs to the class $\mathfrak{G}$ if and only if X is countable, i.e. $C_p(X)$ is metrizable; see [119]. Therefore, many results for spaces $C_p(X)$ (also presented in the book) require methods and technics different from those applied to study the class $\mathfrak{G}$; we refer the reader to the excellent works about $C_p(X)$ theory in [28] and [31].

Topologists used to say that a topological space X is sequential if every sequentially closed set in in X is closed. Clearly, from the definition, we have metrizable $\Rightarrow$ Fréchet–Urysohn $\Rightarrow$ sequential $\Rightarrow$ k-space. Probably, the first proof that an (LF)-space if metrizable if and only if it is Fréchet–Urysohn was presented in [328]. Cascales–Orihuela's result [126] that (LF)-spaces are angelic proved that any (LF)-space is sequential if and only if it is a k-space.

Nyikos [472] observed that the (LB)-space φ is sequential and not Fréchet–Urysohn. Much earlier, Yosinaga [644] proved that the strong dual of any Fréchet–Schwartz space (equivalently, every Silva space) is sequential. Next, Webb [638] extended this result to strong duals of Fréchet–Montel spaces (equivalently, to Montel-(DF)-spaces). He proved that these spaces are Fréchet–Urysohn only when finite-dimensional.

Since Montel (DF)-spaces form a part of the class of (LB)-spaces, one can ask if the Nyikos/Yosinaga/Webb result extends further within (LB)-spaces, or (DF)-spaces. It turns out [346] that the answers are negative.

The strong dual of a metrizable lcs E is sequential if and only if E is a dense subspace of either a Banach space or a Fréchet Montel space.

Apparently, any proper (LB)-space is not Fréchet–Urysohn, since proper (LB)-spaces contain a copy of φ; see [534]. As we have already mentioned, Fréchet–Urysohn (DF)-spaces or (LF)-spaces are metrizable, since they belong to the class $\mathfrak{G}$. At this point, there emerges a disparity. Namely, Webb [638] introduced the property C_3 (sequential closure of each set is sequentially closed) that characterizes metrizability for (LF)-spaces [328] but not for (DF)-spaces; see [346, Assertions 5.2 and 5.3]. A variant property C_3^- (the sequential closure of every linear subspace is sequentially closed) defined in [346] characterized both the barrelled (DF)-spaces and (LF)-spaces as being of the form M, φ, or $M \times \varphi$, where M is a metrizable lcs. The main result of [346] characterizing both (DF)-spaces and (LF)-spaces that are sequential as being either metrizable or Montel (DF)-spaces provides an answer to topological group questions of Nyikos [472, Problem 1].

Chapter 2
Elementary Facts about Baire and Baire-Type Spaces

Abstract This chapter contains classical results about Baire-type conditions (Baire-like, b-Baire-like, CS-barrelled, s-barrelled) on tvs. We include applications to closed graph theorems and $C(X)$-spaces. We also provide the first proof in book form of a remarkable result of Saxon (extending earlier results of Arias de Reyna and Valdivia), which states that, under Martin's axiom, every lcs containing a dense hyperplane contains a dense non-Baire hyperplane. This part also contains analytic characterizations of certain completely regular Hausdorff spaces X. For example, we show that X is pseudocompact, is Warner bounded, or $C_c(X)$ is a (df)-space if and only if for each sequence $(\mu_n)_n$ in the dual $C_c(X)'$ there exists a sequence $(t_n)_n \subset (0, 1]$ such that $(t_n\mu_n)_n$ is weakly bounded, strongly bounded, or equicontinuous, respectively. These characterizations help us produce a (df)-space $C_c(X)$ that is not a (DF)-space, solving a basic and long-standing open question.

2.1 Baire Spaces and Polish Spaces

Let A be a subset of a non-void Hausdorff topological space X. We shall say that A is *nowhere dense* (or rare) if its closure $\overline{A}$ has a void interior. Clearly, every subset of a nowhere dense set is nowhere dense.

A is called of the *first category* if A is a countable union of nowhere dense subsets of X. Clearly, every subset of a first category set is again of the first category. A is said to be of the *second category* in X if it is not of the first category. If A is of the second category and $A \subset B$, then B is of the second category.

The classical Baire category theorem states the following.

Theorem 2.1.1 *If E is either a complete metric or a locally compact Hausdorff space, then the intersection of countably many dense, open subsets of X is dense in E.*

Proof We show only that the intersection of countably many dense open sets in every metric complete space (E, d) is non-void. If this were false, then $E = \bigcup_n E_n$, where each E_n is a closed subset with an empty interior. Hence, there exists x_1 and $0 < \epsilon_1 < 1$ such that $B(x_1, \epsilon_1) \subset E \setminus E_1$, where $B(x_1, \epsilon_1)$ is the open ball at

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x_1 with radius ϵ_1. Next there exist $x_2 \in B(x_1, 2^{-1}\epsilon_1)$ and $0 < \epsilon_2 < 2^{-1}\epsilon_1$ such that $B(x_2, \epsilon_2) \subset E \setminus E_2$. Continuing this way, one obtains a shrinking sequence of open balls $B(x_n, \epsilon_n)$ with a radius less than 2^{-n} disjoint with E_n. Clearly, $(x_n)_n$ is a Cauchy sequence in (E, d), so it converges to $x \in E \setminus E_n$, $n \in \mathbb{N}$, and we reached a contradiction. Similarly, one gets that each open subset of (E, d) is of the second category. □

This deep theorem is a principal one in analysis and topology providing many applications for the closed graph theorems and the uniform boundedness theorem.

A topological space X is called a *Baire space* if every non-void open subset of X is of the second category (equivalently, if the conclusion of the Baire theorem holds). Clearly, every Baire space is of the second category. Although there exist topological spaces of the second category which are not Baire spaces, we note the following fact for tvs. All tvs considered in the sequel are assumed to be real or complex, if nothing more will be mentioned.

Proposition 2.1.2 *If a tvs E is of the second category, E is a Baire space.*

Proof Let A be a non-void open subset of E. If $x \in A$, then there exists a balanced neighbourhood of zero U in E such that $x + U \subset A$. Since $E = \bigcup_n nU$ and E is of the second category, there exists $m \in \mathbb{N}$ such that mU is of the second category. Then U is of the second category too. This implies that $x + U$ is of the second category and A, containing $x + U$, is also of the second category. □

We shall need also the following classical fact; see [491, 10.1.26]. For a completely regular Hausdorff space X by $C_c(X)$ and $C_p(X)$, we denote the space of real-valued continuous functions on X endowed with the compact-open and pointwise topologies, respectively.

Proposition 2.1.3 *Let X be a paracompact and locally compact topological space. Then $C_c(X)$ is a Baire space.*

Proof Since X is a paracompact locally compact space, X can be represented as the topological direct sum of a disjoint family $\{X_t : t \in T\}$ of locally compact σ-compact spaces X_t, and we have a topological isomorphism of $C_c(X)$ and the product $\prod_{t\in T} C_c(X_t)$. It is known that each space $C_c(X_t)$ is a Fréchet space, i.e. a metrizable and complete lcs, and since products of Fréchet spaces are Baire spaces (see Theorem 14.3.6) for an alternative proof, we conclude that $\prod_{t\in T} C_c(X_t)$ is a Baire space. □

The following general fact is a simple consequence of the above definitions.

Proposition 2.1.4 *If E is a tvs and F is a vector subspace of E, then F is either dense or nowhere dense in E. If F is dense in E and F is Baire, then E is Baire.*

Proof Assume that F is not dense in E. Let G be its closure in E, a proper closed subspace of E. If G is not nowhere dense in E, then there exists a balanced neighbourhood of zero U in E and a point $x \in G$ such that $x + U \subset G$. Then $E = \bigcup_n nU \subset G$, providing a contradiction. The other part is clear. □

Recall that every *Čech-complete* space E, i.e. E can be represented as a countable intersection of open subsets of a compact space, is a Baire space. Arkhangel'skii proved [26] that if E is a topological group and F is a Čech-complete subspace of E, then either F is nowhere dense in E or E is Čech-complete as well. This, combined with Proposition 2.1.4, shows that if a tvs E contains a dense Čech-complete vector subspace, then E is Čech-complete.

A subset A of a topological space X is said to have the *Baire property* in X if there exists an open subset U of X such that $U \setminus A$ and $A \setminus U$ are of the first category.

Let $D(A)$ be the set of all $x \in X$ such that each neighbourhood $U(x)$ of x intersects A in a set of the second category. Set $O(A) := \text{int } D(A)$.

Proposition 2.1.5 *A subset A of a topological space X has the Baire property if and only if $O(A) \setminus A$ is of the first category.*

Proof Assume A has the Baire property, and let U be an open set in X such that $U \setminus A$, and $A \setminus U$ are sets of first category. Note that $D(A) \subset \overline{U}$. Indeed, if $x \in D(A) \setminus \overline{U}$, then (by definition) the set

$$(X \setminus \overline{U}) \cap A \ (= A \setminus \overline{U})$$

is of the second category. By the assumption $A \setminus \overline{U} \subset A \setminus U$ is of the first category, a contradiction. Since $\overline{U} \setminus U$ is nowhere dense, one concludes that $\overline{U} \setminus A$ is of the first category. Finally, since

$$O(A) \setminus A \subset D(A) \setminus A \subset \overline{U} \setminus A,$$

then $O(A) \setminus A$ is of first category as claimed.

Now assume that $O(A) \setminus A$ is a set of the first category. It is enough to prove that $A \setminus O(A)$ is of the first category. Let $C(A)$ be the union of the family $\mathfrak{L} := \{A_i : i \in I\}$ of all the open subsets of X that intersect A in a set of first category. Note that $O(A) = X \setminus \overline{C(A)}$. We show that

$$A \cap \overline{C(A)} \ (= A \setminus O(A))$$

is of the first category. Let $\{A_i : i \in J\}$ be a maximal pairwise disjoint subfamily of $\mathfrak{L}$. Then (as it is easily seen) $A \cap (\bigcup_{i \in J} A_i)$ is of the first category. Then the set $A \cap \overline{\bigcup_{i \in J} A_i}$ is also of the first category. By the maximality condition, we deduce that $\bigcup_{i \in I} A_i \subset \overline{\bigcup_{i \in J} A_i}$, which completes the proof. □

Since $A \cap \overline{C(A)}$ is of the first category, we note the following simple fact; see, for example, [611, p.4].

Proposition 2.1.6 *Let E be a topological space and let B be a subset of E which is the union of a sequence $(U_n)_n$ of subsets of E. Then $D(B) \setminus \bigcup\{O(U_n) : n \in \mathbb{N}\}$ is nowhere dense. Therefore, $O(B) \setminus \bigcup\{O(U_n) : n \in \mathbb{N}\}$ is also nowhere dense.*

Proof Assume that the interior A of the closed set $D(B) \setminus \bigcup\{O(U_n) : n \in \mathbb{N}\}$ is non-void. Then $A \cap B$ is of the second category. Hence, there exists $m \in \mathbb{N}$ such that $A \cap U_m$ is of the second category. Therefore, as $U_m \cap \overline{C(U_m)}$ is of the first category, we have $A \not\subset \overline{C(U_m)}$, and hence $A \cap O(U_m)$ is non-void, a contradiction. □

Every Borel set in a topological space E has the Baire property. This easily follows from the following well-known fact; see [550].

Proposition 2.1.7 *Let E be a topological space. The family of all subsets of E with the Baire property forms a σ-algebra.*

Now we are ready to formulate the following useful fact.

Proposition 2.1.8 *Let U be a subset of a topological vector space E. If U is of the second category and has the Baire property, $U - U$ is a neighbourhood of zero.*

Proof Since U is of the second category, we have $O(U) \neq \emptyset$. If $x \notin U - U$, then clearly $(x + U) \cap U = \emptyset$. The Baire property of U implies that $(x + O(U)) \cap O(U)$ is a set of the first category. On the other hand, since non-void open subsets of $O(U)$ are of the second category, it follows that $(x + O(U)) \cap O(U) = \emptyset$. Then $x \notin O(U) - O(U)$. This proves that $U - U$ contains the neighbourhood of zero $O(U) - O(U)$. □

It is clear that the above fact has a corresponding variant for topological groups (called Philips' lemma); see, for example, [522].

In general, even the self-product $X \times X$ of a Baire space X need not be Baire; see [142, 485], and [236]. Nevertheless, the product $\prod_{i \in I} X_i$ of metric complete spaces is Baire; see [491]. Also the product $\prod_{i \in I} X_i$ of any family $\{X_i : i \in I\}$ of separable Baire spaces is a Baire space; see [611].

In [20], Arias de Reyna proved the following:

Theorem 2.1.9 (Arias de Reyna) *The Hilbert space $\ell^2(\omega_1)$ contains a family $\{X_t : t < \omega_1\}$ of different Baire subspaces such that for all $t, u < \omega_1$, $t \neq u$, the product $X_t \times X_u$ is not Baire.*

In [613], Valdivia generalized this result by proving the same conclusion in each space $c_0(I)$ and $\ell^p(I)$, for uncountable set I, and $0 < p < \infty$. Lemma 1 in [20] has been improved by Drewnowski in [179].

Every Polish space is a Baire space. A topological space E is called a *Polish space* if E is separable and if there exists a metric d on E generating the same topology such that (E, d) is complete.

Proposition 2.1.10 *(i) The intersection of any countable family of Polish subspaces of a topological space E is a Polish space.*

(ii) Every open (closed) subspace V of a Polish space E is a Polish space. Hence, a subspace of a Polish space which is a G_δ-set is a Polish space.

Proof (i) Let $(E_n)_n$ be a sequence of Polish subspaces of E, and let $G := \bigcap_n E_n$. Then the product $\prod_n E_n$ (endowed with the product topology) and the diagonal

$\Delta \subset \prod_n E_n$ (as a closed subset) are Polish spaces. Since Δ is homeomorphic to the intersection G, the conclusion follows.

(ii) Let d be a complete metric on E. Let V be open and $V^c := E \setminus V$. Define the function $d(x, V^c)$ by $d(x, V^c) := \inf\{d(x, y) : y \in V^c\}$. Set $\xi(x) := d(x, V^c)^{-1}$ and

$$D(x, y) := |\xi(x) - \xi(y)| + d(x, y)$$

for all $x, y \in V$. It is easy to see that $D(x, y)$ defines a complete metric on V giving the original topology of E restricted to V. Hence, V is a Polish space. □

The following characterizes Polish subspaces of a Polish space; see [550].

Proposition 2.1.11 *A subspace F of a Polish space E is Polish if and only if F is a G_δ-set in E.*

Proof If F is a G_δ-set in E, then F is a Polish space by the previous proposition. To prove the converse, let d (resp. d_1) be a compatible (resp. complete) metric on E (resp. F). For each $x \in F$ and each $n \in \mathbb{N}$, there exists $0 < t_n(x) < n^{-1}$ such that $z \in F$ and $d(x, z) < t_n(x)$ imply $d_1(x, z) < n^{-1}$. Define $U_n(x) := \{z \in E : d(x, z) < t_n(x)\}$, $U_n := \bigcup_{x \in F} U_n(x)$, and $W := \bigcap_n U_n$. Then $W \subset F$. Indeed, if $y \in W$, then there exists a sequence $(x_n)_n$ in F with $d(x_n, y) < t_n(x_n) < n^{-1}$. Therefore, $x_n \to y$. Fix $m \in \mathbb{N}$. Since $d(x_m, y) < t_m(x_m)$, there exists $k_m \in \mathbb{N}$ such that $k_m^{-1} + d(x_m, y) < t_m(x_m)$. Then, if $n > k_m$, one gets

$$d(x_n, x_m) \leq d(x_n, y) + d(y, x_m) < n^{-1} + d(y, x_m) < k_m^{-1} + d(x_m, y) < t_m(x_m).$$

By construction, the inequality $d(x_n, x_m) < t_m(x_m)$ yields $d_1(x_n, x_m) < m^{-1}$. Clearly, $(x_n)_n$ is Cauchy in (F, d_1), and from the completeness, it follows that $y \in F$. □

Corollary 2.1.12 *A topological space E is a Polish space if and only if it is homeomorphic to a G_δ-set contained in the compact space* $[0, 1]^{\mathbb{N}}$.

Proof Since the space $[0, 1]^{\mathbb{N}}$ is a metric compact space, it is a Polish space and the previous Proposition 2.1.10 applies. To get the converse, assume that E is a Polish space. Hence, it is separable and metrizable, and consequently E is homeomorphic to a subspace of $[0, 1]^{\mathbb{N}}$. Proposition 2.1.11 applies to complete the proof. □

2.2 A Characterization of Baire Topological Vector Spaces

The following characterization of Baire tvs is due to Saxon [529]; see also [491, Theorem 1.2.2].

Theorem 2.2.1 (Saxon) *The following are equivalent for a tvs E:*

(i) E is a Baire space.
(ii) Every absorbing balanced and closed subset of E is a neighbourhood of some point.

The scalar field $\mathbb{K}$ is either the reals or the complexes. After a little preparation motivated by annular regions in the complex plane, we present a single proof that simultaneously solves both the real and complex cases.

Claim 2.2.2 Let U be a non-empty set in a tvs E. If $U - U \neq E$, then $\bigcap_{n=1}^{\infty} n^{-1}U$ has an empty interior.

Proof Let $x \in E \setminus (U - U)$. If V is a non-empty open set contained in $\bigcap_n n^{-1}U$, then the open neighbourhood of zero $V - V$ contains $m^{-1}x$ for some sufficiently large $m \in \mathbb{N}$ and is contained in $n^{-1}(U - U)$ for every $n \in \mathbb{N}$. In particular, $m^{-1}x \in m^{-1}(U - U)$, which implies $x \in U - U$, a contradiction. □

For $0 < r_1 < r_2$, we define the annulus Λ_{r_1,r_2} in $\mathbb{K}$ by writing

$$\Lambda_{r_1,r_2} = \{t \in \mathbb{K} : r_1 \leq |t| \leq r_2\}.$$

For each $\varepsilon > 0$, let β_ε denote the open ball $\{t \in \mathbb{K} : |t| < \varepsilon\}$. For $0 < \varepsilon < 1$, compactness provides a finite subset Γ_ε of $\Lambda_{\varepsilon,1}$ such that

$$\Gamma_\varepsilon + \beta_{\varepsilon^2} \supset \Lambda_{\varepsilon,1}.$$

Claim 2.2.3 If $0 < \delta < \varepsilon < 1$ and $z + \beta_\varepsilon \subset \Lambda_{\varepsilon,1}$, then $\Gamma_\delta \cdot (z + \beta_\varepsilon) \supset \Lambda_{\delta,\varepsilon}$

Proof

$$\begin{aligned}\Gamma_\delta \cdot (z + \beta_\varepsilon) &= \bigcup_{t\in\Gamma_\delta} t \cdot (z + \beta_\varepsilon) = \bigcup_{t\in\Gamma_\delta} \left(t \cdot z + \beta_{\varepsilon|t|}\right) \supset \bigcup_{t\in\Gamma_\delta} (t \cdot z + \beta_{\varepsilon\delta}) \\ &= \left(\Gamma_\delta + \beta_{\delta\varepsilon/|z|}\right) \cdot z \supset \left(\Gamma_\delta + \beta_{\delta^2}\right) \cdot z \supset \Lambda_{\delta,1} \cdot z = \Lambda_{\delta|z|,|z|} \supset \Lambda_{\delta,\varepsilon}.\end{aligned}$$

□

Claim 2.2.4 Let $(B_n)_n$ be a sequence of subsets of a tvs E. Fix $0 < r_1 < r_2$ and $y \in E$. If $\Lambda_{r_1,r_2} \cdot y \subset \bigcap_n B_n$, then there exists $\delta > 0$ such that $\{ty : 0 < |t| \leq \delta\} \subset \bigcup_n n^{-1}B_n$.

Proof For each n, we have $\Lambda_{r_1/n,r_2/n} \cdot y \subset n^{-1}B_n$. There is a natural number m such that $r_1/n \leq r_2/(n+1)$ for every $n \geq m$. The claim follows for $\delta = r_2 m^{-1}$. □

Now we are ready to prove Theorem 2.2.1

Proof Only the implication (ii) ⇒ (i) needs a proof. Indeed, if there exists an absorbing, balanced, and nowhere dense set B, then $E = \bigcup_n nB$ is of the first category, and not Baire. We assume that E is not Baire and construct such a set B.

Since E is of the first category, its topology is non-trivial, and E contains a closed balanced neighbourhood U of zero with $U - U \neq E$. Since U is also of the first category in E, there exists a sequence $(A_n)_n$ of closed nowhere dense sets in E whose union is U. With the notation of Claim 2.2.3, we observe that each set

$$B_n := \bigcup_{j,k \leq n} \Gamma_{1/k} \cdot A_j$$

is closed and nowhere dense, being a finite union of such sets. Furthermore, each B_n is contained in the balanced set U.

We wish to see that $A := \bigcup_n n^{-1} B_n$ is nowhere dense. Suppose some non-empty open set W is contained in $\overline{A}$. Because each finite union $\bigcup_{n<k} n^{-1} B_n$ is nowhere dense, the closure of each $\bigcup_{n \geq k} n^{-1} B_n$ must contain W. But the closure of the k^{th} union is clearly contained in $k^{-1}U$ for each $k \in \mathbb{N}$, so that $W \subset \bigcap_k k^{-1}U$, which contradicts Claim 2.2.2.

Next, we prove that A absorbs any given point y in E. Since U absorbs y, there exists $r > 0$ such that the set $\beta_r \cdot y$ is contained in U and is thus covered by $(A_n)_n$. We may harmlessly assume that $r \leq 1$. The set $\beta_r \cdot y$ either has the trivial topology or is a topological copy of the subset β_r of $\mathbb{K}$. Either way, $\beta_r \cdot y$ is of the second category in itself, and there exist $p \in \mathbb{N}$, $\varepsilon > 0$, and $z \in \beta_r$ such that

$$z + \beta_\varepsilon \subset \beta_r \quad \text{and} \quad (z + \beta_\varepsilon) \cdot y \subset A_p.$$

In fact, re-choosing z and ε if needed, we may additionally insist that $z \neq 0$, and then we may yet again refine the choice of ε in the interval $(0, 1)$ so that

$$z + \beta_\varepsilon \subset \Lambda_{\varepsilon,1} \quad \text{and} \quad (z + \beta_\varepsilon) \cdot y \subset A_p.$$

Let q be a natural number larger than both p and ε^{-1}. For $n \geq q$, we apply Claim 2.2.3 to obtain

$$B_n \supset \Gamma_{1/q} \cdot A_p \supset \Gamma_{1/q} \cdot (z + \beta_\varepsilon) \cdot y \supset \Lambda_{1/q,\varepsilon} \cdot y.$$

Now Claim 2.2.4 shows that, for some $\delta > 0$,

$$\{ty : 0 < |t| \leq \delta\} \cdot y \subset \bigcup_{n \geq q} n^{-1} B_n \subset A.$$

Therefore, A absorbs y, given that zero is (obviously) in A. The balanced core of A, i.e. the largest balanced set A_0 contained in A, is absorbing and nowhere dense because A has these properties. Therefore, $B := \overline{A_0}$ is a closed, balanced, absorbing nowhere dense set in E, as promised. □

Theorem 2.2.1 provides the following:

Corollary 2.2.5 *Every Hausdorff quotient of a Baire tvs is a Baire space.*

We also have the following:

Corollary 2.2.6 *If a Baire tvs E is covered by a sequence $(E_n)_n$ of vector subspaces of E, then E_m is dense and Baire for some $m \in \mathbb{N}$.*

Proof By hypothesis, there exists $m \in \mathbb{N}$ such that E_m is of the second category in E; therefore, it cannot be nowhere dense in E. By Proposition 2.1.4 E_m is dense in E, thus of the second category in itself, thus Baire by Proposition 2.1.2. □

When $\dim(E)$ is infinite, E_m may satisfy $\dim(E/E_m) = \dim(E)$, an extreme.

Proposition 2.2.7 *Every infinite-dimensional Baire tvs E contains a dense Baire subspace F whose dimension equals the codimension in E.*

Proof By $(x_t)_{t\in T}$, denote a Hamel basis of E. Fix a partition $(T_n)_n$ of T such that card T= card T_n for all $n \in \mathbb{N}$. Set $E_n := \operatorname{span}\{x_t : t \in \bigcup_{i=1}^{n} T_i\}$. Then $(E_n)_n$ covers E and $\dim E = \dim E_n = \dim(E/E_n)$ for $n \in \mathbb{N}$. By Corollary 2.2.6, there exists a dense Baire subspace $F := E_m$ of E, as desired. □

At the other algebraic extreme, hyperplanes of E are also Baire when closed, and those that contain F are dense and Baire.

2.3 Arias de Reyna–Valdivia–Saxon's Theorem

For many years, the following question remained: *When does an infinite-dimensional Baire tvs E admit a non-Baire (necessarily dense) hyperplane?* In 1966, Wilansky/Klee conjectured: *Never, for E a Banach space*. This conjecture was denied by Arias de Reyna in 1980, [491, Theorem 1.2.12], who proved, under Martin's axiom, the answer: *Always, when E is a separable Banach space*. In 1983, Valdivia [612] proved, under Martin's axiom, the more general answer: *Always, when E is a separable tvs.* In 1987, Pérez Carreras and Bonet [491, Question 13.1.1] repeated the question for E a (not necessarily separable) Banach space.

Finally, in 1991, Saxon [531] provided a complete answer in the general locally convex setting. He proved

Theorem 2.3.1 (Arias de Reyna–Valdivia–Saxon) *Assume* $\mathfrak{c}$*-A. Every tvs E with infinite-dimensional dual contains a non-Baire hyperplane. Consequently, (1) every infinite-dimensional lcs admits a non-Baire hyperplane and (2) A lcs E admits a dense non-Baire hyperplane iff $E' \neq E^*$.*

For a tvs E by the dual of E, we mean its topological dual E', a linear subspace of its algebraic dual E^*. A subset A of a tvs E is called *bornivorous* if A absorbs every bounded set in E. Recall that a lcs E is *barrelled* (*quasibarrelled*), if every closed absolutely convex and absorbing (and bornivorous) subset of E is a neighbourhood

of zero of E, or equivalently, if every bounded set in the weak dual $(E', \sigma(E', E))$ (strong dual $(E', \beta(E', E))$) is equicontinuous.

The axiom $\mathfrak{c}$-A ($\mathfrak{c}$-additivity, where $\mathfrak{c} := 2^{\aleph_0}$) proclaims that: *The union of less than* $\mathfrak{c}$ *subsets of* $\mathbb{R}$, *each of measure zero, itself has measure zero*.

Note that CH $\Rightarrow$ Martin's axiom $\Rightarrow$ $\mathfrak{c}$-A, and the converse implications fail in general; see [247, Corollary 32(G)(c)].

To prove Theorem 2.3.1, we will need the following technical fact from [491, Theorem 1.2.11].

Lemma 2.3.2 *Let* e_1 *and* e_2 *be the canonical unit vectors in the real Euclidean space* $\mathbb{R}^2$ *endowed with its usual inner product* $(.,.)$ *and the corresponding norm* $\|.\|$. *By* $m(x) := \arccos((x, e_1)\|x\|^{-1})$, *for* $x \neq 0$, *we denote the angle between* x *and* e_1. *If* $0 < b < 1$, q *is a positive integer,* $\|u\|, \|v\| > q^{-1}$ *and* $\|u - v\| < bq^{-1}$, *then* $|m(u) - m(v)| < 2b$.

Now we prove Theorem 2.3.1.

Proof It is enough to prove the initial claim. Clearly, (1) then follows and (2) as well. Indeed, if E is a lcs with $E' \neq E^*$, then E contains a non-Baire hyperplane H by (1). If (a) E is Baire, then all its closed hyperplanes are Baire, and H must be dense. If (b) E is non-Baire, then so are all its hyperplanes, including dense hyperplanes, which exist by the hypothesis $E' \neq E^*$. Conversely, no dense hyperplanes exist if $E' = E^*$.

We prove the real scalar case only. One may then easily dispatch the complex case by a standard procedure. We also assume, without loss of generality, that E is Baire.

The hypothesis implies a biorthogonal sequence $(x_n, h_n)_n \subset E \times E'$, so the map $T : E \longrightarrow \mathbb{R}^{\mathbb{N}}$ defined by $T(x) := (h_n(x))_n$ is continuous and linear. Therefore, E admits a quotient $F := E/Q$ of dimension at most $\mathfrak{c}$ isomorphic to a subspace of $\mathbb{R}^{\mathbb{N}}$ containing the canonical unit vectors and endowed with a vector topology finer than the one inherited from the usual product topology. Hence, each unit vector e_n and each coordinate functional f_n belongs to F and F', respectively. For each $n \in \mathbb{N}$ set $M_n := \operatorname{span}\{e_i : i \leq n\}$. If $(w_{n,k})_k$ is an enumeration of the countable set $\{\sum_{i=1}^{n} a_i e_i : a_i \in \mathbb{Q},\ a_n \neq 0\}$, then $(w_{n,k})_k$ is dense in M_n for all $n \in \mathbb{N}$. Set

$$U_i := \bigcap_{n=1}^{i} \{x \in F : |f_n(x)| < 2^{-i}\}$$

for each $i \in \mathbb{N}$. Then

$$\bigcap_i U_i = \{0\},\ U_{i+1} + U_{i+1} \subset U_i,\ i \in \mathbb{N}. \tag{2.1}$$

For each fixed $n \in \mathbb{N}$, we choose a sequence $(\epsilon_{n,k})_k$ of numbers such that

$$0 < 2\epsilon_{n,k+1} \leq \epsilon_{n,k} < 2^{-n} \wedge |f_n(w_{n,k})|.$$

Note that

$$w_{i,k} \notin V_{i,k} := \bigcap_{n=1}^{i} \{x \in F : |f_n(x)| < \epsilon_{i,k}\} \subset U_i$$

for all $i, k \in \mathbb{N}$.

Moreover,

$$V_{n,k+1} + V_{n,k+1} \subset V_{n,k} \subset U_n, \ \ n, k \in \mathbb{N} \tag{2.2}$$

and

$$L_n := \bigcup_k (w_{n,k} + V_{n,k}) \subset F \setminus \{0\}, \ \ n \in \mathbb{N}. \tag{2.3}$$

Each L_n is dense and open in F $(= M_n \oplus (\bigcap_{i \le n} f_i^{\perp}))$. To complete the proof, we need only find a non-Baire hyperplane G in F; indeed, the hyperplane H in E satisfying $H \supset Q$ and $G = H/Q$ would also be non-Baire by Corollary 2.2.5. Note that $\aleph_0 \le \dim(F) =: \alpha \le \mathfrak{c}$. From the previous section, we know that the Baire space F contains a dense Baire hyperplane P. Let

$$B := \left\{ y_\beta : \beta \text{ is an ordinal} < \alpha \right\}$$

be a Hamel basis for P. If A is the absolutely convex envelope of B, then $\overline{A} \cap P$ is a barrel in the dense barrelled subspace P and, consequently, $\overline{A}$ is a neighbourhood of zero in F. Hence, we can find a point $z \in F \setminus P$ such that $2z \in \overline{A}$. We observe from (2.3) that the set $F \setminus L$ is of the first category in F, where $L := \bigcap_n L_n$. The proof will be complete if we find a dense hyperplane G contained in $F \setminus L$. The space G will take the form

$$G := \operatorname{span}\{y_\gamma + a_\gamma z : \gamma < \alpha\}$$

for scalars a_γ suitably chosen with each $|a_\gamma| \le 1$.

Formula (2.3) ensures that L misses

$$\{0\} = \operatorname{span}\emptyset = \operatorname{span}\{y_\gamma + 1 \cdot z : \gamma \ is \text{ an ordinal } < 0\}.$$

Zorn's lemma provides a maximal subspace M of F of the form

$$M := \operatorname{span}\{y_\gamma + a_\gamma z : \gamma < \beta\}$$

subject to the conditions that L misses M, the ordinal β does not exceed α, and each $|a_\gamma| \le 1$.

Claim. We have $\beta = \alpha$.

Indeed, assume that $\alpha \neq \beta$. The set $\{\gamma : \gamma < \beta\}$ and the family $\mathfrak{F}$ of all its finite subsets have less than $\mathfrak{c}$ elements. Now we will apply Lemma 2.3.2 to our situation: Identify y_β and z with the unit vectors e_1 and e_2, respectively, and $\mathbb{R}^2$ with the linear span X of y_β and z.

Let $h : M + X \to X$ be the projection onto X along M. Let $\mathfrak{D}$ be the family of all subspaces of the form $X + \operatorname{span}\{y_\gamma + a_\gamma z : \gamma \in J\}$ for $J \in \mathfrak{F}$. Note that $\mathfrak{D}$ covers $M + X$ and $|\mathfrak{D}| < \mathfrak{c}$. Next observe that

$$L \cap (M + X) = \bigcup_{n\in\mathbb{N}, D\in\mathfrak{D}} L_{D,n},$$

where $L_{D,n} := L \cap D \cap \{x : \|h(x)\| > n^{-1}\}$. This follows from the fact that $h(x) \neq 0$ if $x \in L \cap (M + X)$, because then $x \notin M$ (since M misses L).

Fix arbitrary $n \in \mathbb{N}$ and $D \in \mathfrak{D}$ as well as $0 < t < 1$. Since D is finite-dimensional, we deduce from (2.1) that $(D \cap U_i)_i$ is a base of neighbourhoods of zero in D. Since the restriction map $h|D$ is continuous, there exists $r > 1$ such that $\|h(x)\| < tn^{-1}$ for $x \in D \cap U_{r-1}$. Note that

$$L_{D,n} \subset L_r \cap D \cap \{x : \|h(x)\| > n^{-1}\} =$$

$$\bigcup_k D \cap (w_{r,k} + V_{r,k}) \cap \{x : \|h(x)\| > n^{-1}\}.$$

Assume for the moment that x and y are in the kth set. Then $\|h(x)\| > n^{-1}$ and $\|h(y)\| > n^{-1}$. Moreover, $2^{k-1}(x - y) \in D$. Since $x - y \in V_{r,k} + V_{r,k}$, we use (2.2) and (2.3) and $k - 1$ more steps to obtain

$$2^{k-1}(x - y) \in V_{r,1} + V_{r,1} \subset U_r + U_r \subset U_{r-1}.$$

Hence, by continuity of $h|D$, we have

$$\|h(x - y)\| < 2^{1-k} tn^{-1}.$$

Now we can apply Lemma 2.3.2 to obtain that

$$|m(h(x)) - m(h(y))| < 2^{2-k} t.$$

This proves that the set $m(h(L_{D,n}))$ is covered by a sequence of intervals whose union has measure less than $4t = \sum_k 2^{2-k} t$. Since the scalar t was arbitrary, we conclude that $m(h(L_{D,n})$ has measure zero. Now the axiom $\mathfrak{c}$-A declares that

$$C := m(h(L \cap (M + X)))$$

has measure zero. Therefore, we can find an angle θ such that $0 < \theta < \pi/4$ and $\theta \notin C$ and $\pi - \theta \notin C$. Now set $a_\beta := \tan\theta$ and $v = y_\beta + a_\beta z$. Hence, $0 < a_\beta < 1$. Note that

$$m(h(x + bv)) = m(bv) = m(v) = \theta, \quad m(h(x - bv)) = m(-v) = \pi - \theta$$

if $x \in M$ and $b > 0$. This implies that $x \pm bv \notin L$. We know already that M misses L. Now $M + \text{span}\{v\}$ also misses L, which contradicts the maximality of M, proving the Claim.

Finally, we prove that $G := M$ is dense in F. Since G is a hyperplane, it is enough to show that G is not closed. Now $tz \notin M$ for each $t \neq 0$. Since $2z \in \overline{A}$, there exists a net $(z_u)_u = \left(\sum_{\gamma\in J_u} b_{u,\gamma}\, y_\gamma\right)_u$ in A which converges to $2z$, where J_u is finite and $\sum_{\gamma\in J_u} |b_{u,\gamma}| \leq 1$. The corresponding net $\left(\sum_{\gamma\in J_u} b_{u,\gamma} a_\gamma z\right)_u$ has an adherent point pz in the compact interval $\{bz : |b| \leq 1\}$. Let U and V be neighbourhoods of $2z$ and pz, respectively. V contains a cofinal subnet of the second net, and U contains points of the corresponding cofinal subnet of the first net. Hence, $U + V$ contains points of a subnet of $\left(\sum_{\gamma\in J_u} b_{u,\gamma}\left(y_\gamma + a_\gamma z\right)\right)_u \subset M$. It follows that $2z + pz \in \overline{M}\backslash M$, and $G = M$ is not closed. The proof is complete. □

Remark 2.3.3 If E is a tvs with a separable quotient F such that $F' \neq F^*$, one can find points $w_{n,k}$ in F and balanced open neighbourhoods U_i and $V_{i,k}$ of zero satisfying (2.1) and (2.2) such that each L_n defined as in (2.3) is dense (and open) in F. It then follows from the proof of Theorem 2.3.1 that, under assumption of c-A, there exists in E a dense non-Baire hyperplane.

2.4 Locally Convex Spaces with Some Baire-Type Conditions

A new line of research concerning Baire-type conditions started with the Amemiya–Kōmura theorem (see [491, Theorem 8.2.12] and [607]), stating that if $(A_n)_n$ is an increasing sequence of absolutely convex closed subsets covering a metrizable and barrelled lcs E, then there exists $m \in \mathbb{N}$ such that A_m is a neighbourhood of zero in E.

Saxon [528], motivated by the Amemiya–Kōmura result, defined a lcs E to be *Baire-like* if for any increasing sequence $(A_n)_n$ of closed absolutely convex subsets of E covering E there is an integer $n \in \mathbb{N}$ such that A_n is a neighbourhood of zero. If the sequence $(A_n)_n$ is required to be *bornivorous*, i.e. for every bounded set B in E, there exists A_m that absorbs B, then Ruess defines E to be *b-Baire-like*. Clearly, for a lcs, Baire implies Baire-like, Baire-like implies b-Baire-like and barrelled, and (b)-Baire-like implies (quasi)barrelled.

The main purpose of the research started by Saxon was to study stable lcs properties inherited by products and small-codimensional subspaces of Baire spaces. The

Baire-like property is such an example: Although products [613] and countable-codimensional subspaces (even hyperplanes) [20] of Baire locally convex spaces need not be fully Baire, they are always Baire-like, since countable-codimensional subspaces of Baire-like spaces are Baire-like, and topological products of Baire-like spaces are Baire-like; see [528].

These weak Baire-ness/strong barrelledness properties have well occupied other authors and books; see [230, 391, 491, 534, 539, 611], and [538]. Notwithstanding, our selection/treatment is unique.

Metrizable barrelled spaces are Baire-like (Amemiya–Kōmura) and include all metrizable (LF)-spaces [491, Proposition 4.2.6]. Conversely, as we shall soon see, Baire-like (LF)-spaces must be metrizable.

(i) Let E be a vector space, and let $(E_n, \tau_n)_n$ be an increasing sequence of vector subspaces of E covering E, each E_n endowed with a locally convex topology τ_n, such that $\tau_{n+1}|E_n \leq \tau_n$ for each $n \in \mathbb{N}$. Then on E, there exists the finest locally convex topology τ such that $\tau|E_n \leq \tau_n$ for each $n \in \mathbb{N}$. If τ is Hausdorff, we say that (E, τ) is the *inductive limit space* of the sequence $(E_n, \tau_n)_n$, and the latter is a *defining sequence* for (E, τ).

(ii) If each (E_n, τ_n) is metrizable, then (E, τ) is an (LM)-*space*.

(iii) If each (E_n, τ_n) is a Fréchet space (a Banach space), then (E, τ) is an (LF)-*space* (an (LB)-*space*).

(iv) If $\tau_{n+1}|E_n = \tau_n$ for each $n \in \mathbb{N}$, then $\tau|E_n = \tau_n$ for each $n \in \mathbb{N}$, and (E, τ) is the *strict inductive limit* of $(E_n, \tau_n)_n$, and (E, τ) is complete, if each (E_n, τ_n) is complete; see [491, Proposition 8.4.16].

(v) An (LF)-space E is called *proper* if it has a defining sequence of proper subspaces of E.

The Baire-likeness of some concrete normed vector-valued function spaces was studied by several specialists; see [181] and [182] for details. We note only that the normed spaces of Pettis or Bochner integrable functions are not Baire spaces but Baire-like; see also [609] for more examples of normed Baire-like spaces that are not Baire. We have also the following:

Proposition 2.4.1 *Every metrizable lcs E is b-Baire-like.*

Proof Let $(U_n)_n$ be a decreasing base of absolutely convex neighbourhoods of zero in E. Assume that E is not b-Baire-like. Then there exists a bornivorous sequence $(A_n)_n$ of absolutely convex closed sets such that $U_n \not\subset nA_n$ for each $n \in \mathbb{N}$. Choose

$$x_n \in U_n \setminus nA_n.$$

Since the null sequence $(x_n)_n$ is bounded, it is contained in mA_k for some $k, m \in \mathbb{N}$. Hence, for all $n \geq \max\{m, k\}$, we have

$$\{x_n : n \in \mathbb{N}\} \subset mA_k \subset nA_n,$$

a contradiction. □

Proposition 2.4.2 *Every barrelled b-Baire-like space E is Baire-like.*

Proof Let $(A_n)_n$ be an increasing sequence of absolutely convex closed subsets of E covering E. The proof will be finished if we show that $(A_n)_n$ is bornivorous. Indeed, then we apply that E is b-Baire-like, and some A_m will be a neighbourhood of zero.

Assume, by way of contradiction, that there exists a bounded set $B \subset E$ such that $B \not\subset nA_n$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, select

$$x_n \in n^{-1}B \setminus A_n.$$

Since each set A_n is closed, for each $n \in \mathbb{N}$, there exists a closed and absolutely convex neighbourhood of zero U_n such that

$$U_{n+1} + U_{n+1} \subset U_n, \quad x_n \notin \overline{A_n + U_n}.$$

Set

$$U := \bigcap_n \overline{A_n + U_n}.$$

Then U is a barrel in E and is a neighbourhood of zero to which almost all elements of the null sequence $(x_n)_n$ must belong, a contradiction. □

Proposition 2.4.2 shows that every metrizable barrelled space is Baire-like, the Amemiya–Kōmura result. A more general fact, due to Saxon, is known [528, Theorem 2.1]: A barrelled lcs that does not contain a (isomorphic) copy of φ (i.e. an $\aleph_0$-dimensional vector space with the finest locally convex topology) is Baire-like. Saxon actually proved the following [528, Corollary 2.2], which is even a bit more general; see also [327].

Theorem 2.4.3 *Let E be ℓ^∞-barrelled, i.e. every $\sigma(E', E)$-bounded sequence in E' is equicontinuous. Assume that E is covered by an increasing sequence $(A_n)_n$ of absolutely convex closed subsets of E such that no set A_n is absorbing in E. Then E contains a copy of φ.*

Proof By considering subsequences, we may assume that the span of each A_n is a proper subspace of the span of A_{n+1}. Arbitrarily select $(x_n)_n$ in E such that $x_n \in A_{n+1} \setminus \operatorname{span}(A_n)$. We show that the span $\mathbb{S}$ of the necessarily linearly independent sequence $(x_n)_n$ is a copy of φ. It suffices to show that, if p is an arbitrary seminorm on $\mathbb{S}$, there exists a continuous seminorm q on E such that $p \le q|\mathbb{S}$. Without loss of generality, we may assume that

$$p\left(\sum_{j=1}^{n} a_j x_j\right) = \sum_{j=1}^{n} |a_j| p(x_j), \quad p(x_n) \ge 1, \quad n \in \mathbb{N}.$$

We proceed inductively to find a sequence $(f_n)_n$ in E' such that

$$f_n \in A_n^\circ, \tag{2.4}$$

and if $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n$ are scalars, then

$$\max_{1\le r\le n} |f_r(\sum_{j=1}^{n} a_j x_j)| \ge (1+2^{-n})p\left(\sum_{j=1}^{n} a_j x_j\right). \tag{2.5}$$

The Hahn–Banach separation theorem provides $f_1 \in A_1^\circ$ such that

$$|f_1(x_1)| \ge (1+2^{-1})p(x_1).$$

Let $k \in \mathbb{N}$ and assume there exist $f_1, f_2, \ldots, f_k$ in E' such that the above conditions are satisfied for $n \le k$. Define

$$D := \left\{ x = \sum_{j=1}^{k} a_j x_j : \max_{1\le r\le k} |f_r(x + x_{k+1})| < (1+2^{-k-1})p(x + x_{k+1}) \right\}.$$

If D is empty, we complete the induction step by letting $f_{k+1} = 0$. Assume that D is non-empty. For $x \in D$, we have

$$\begin{aligned}(1+2^{-k})p(x) - \max_{1\le r\le k} |f_r(x_{k+1})| &\le \max_{1\le r\le k} |f_r(x + x_{k+1})| \\ &< (1+2^{-k-1})[p(x) + p(x_{k+1})].\end{aligned}$$

This yields

$$2^{-k-1}p(x) \le \max_{1\le r\le k} |f_r(x_{k+1})| + (1+2^{-k-1})p(x_{k+1}).$$

Hence, $\gamma := \sup_{x\in D}\ p(x) < \infty$. For $A := A_{k+1} + \text{span}\{x_1, \ldots, x_k\}$, we have

$$A_{k+1} \subset A,\ \ x_{k+1} \notin \text{span}A = \text{span}A_{k+1}.$$

Moreover, A is absolutely convex and closed. The Hahn–Banach theorem provides $f_{k+1} \in A^\circ \subset A_{k+1}^\circ$ such that

$$f_{k+1}(x_{k+1}) = (1+2^{-k-1})(\gamma + p(x_{k+1})).$$

Thus, $x \in D$ implies that

$$f_{k+1}(x + x_{k+1}) = (1+2^{-k-1})(\gamma + p(x_{k+1})) \ge (1+2^{-k-1})p(x + x_{k+1}).$$

To prove (2.5) for n=k+1, consider an arbitrary element $z = y + ax_{k+1} \in \mathbb{S}$ with $y = \sum_{j=1}^{k} a_j x_j$ and

$$|f_r(z)| < (1 + 2^{-k-1})p(z)$$

for $1 \leq r \leq k$. By the induction assumption, $a \neq 0$ and $a^{-1}y \in D$. Therefore, by the above, we have

$$|f_{k+1}(a^{-1}z)| \geq (1 + 2^{-k-1})p(a^{-1}z).$$

Thus, (2.4) and (2.5) hold for $n \leq k + 1$; the induction is complete. Now fix $x \in E$. Since $f_n \in A_n^\circ$, the fact that $x \in A_n$ for almost all $n \in \mathbb{N}$ means that $|f_n(x)| \leq 1$ for almost all $n \in \mathbb{N}$. Thus, $(f_k)_k$ is $\sigma(E', E)$-bounded and equicontinuous by hypothesis on E. The formula

$$x \mapsto q(x) := \sup_k |f_k(x)|$$

defines a continuous seminorm on E. By (2.5), we have $p \leq q|\mathbb{S}$. This shows that p is continuous on $\mathbb{S}$. We proved that each seminorm on $\mathbb{S}$ is continuous, which shows that $\mathbb{S}$ is as desired. □

This yields the following:

Corollary 2.4.4 (Saxon) *Every barrelled lcs that does not contain a copy of φ is a Baire-like space.*

Hence, if a barrelled lcs E admits a finer metrizable locally convex topology, then E is Baire-like. The converse implication fails. Indeed, let $E = (E, \vartheta)$ be an uncountable product of metrizable and complete lcs. Clearly, E is a non-metrizable Baire lcs for which every finer locally convex topology ξ is non-metrizable. For if we assume that ξ has a countable base $(U_n)_n$ of absolutely convex neighbourhoods of zero, then, because ϑ is barrelled, $(\overline{U_n}^{\vartheta})_n$ is a countable base of neighbourhoods of zero for ϑ, an impossibility.

Corollary 2.4.5 *Let E be a lcs such that each $\sigma(E', E)$-bounded sequence in E' is equicontinuous and E is covered by a strictly increasing sequence of closed subspaces. Then E contains a copy of φ.*

In fact, an ℓ^∞-barrelled space E contains a complemented copy of φ if and only if E is covered by a strictly increasing sequence of closed subspaces; see [534, Theorem 1].

Corollary 2.4.6 *Every proper (LB)-space E contains a copy of φ.*

Proof We may assume that the unit ball U_n in the n^{th} defining Banach space (E_n, τ_n) is contained in U_{n+1} for each $n \in \mathbb{N}$. No $\overline{U_n}$ is absorbing in the barrelled space E. Otherwise, E_n would be a dense barrelled subspace of E, consequently

complete by the open mapping theorem, so that $E_n = E$, a contradiction. The theorem applies with $A_n = \overline{U_n}$. □

Hence, no proper (LB)-space is metrizable. Nevertheless, metrizable and even normable proper (LF)-spaces do exist; see [491] for details and references.

Corollary 2.4.7 *For an (LF)-space E, the following conditions are equivalent:*

(i) *E is Baire-like.*
(ii) *E is metrizable.*
(iii) *E does not contain φ.*

Proof (i) $\Rightarrow$ (ii): Let $(E_n)_n$ be a defining sequence of Fréchet spaces for E, and for each $n \in \mathbb{N}$, let $\mathcal{F}_n$ be a countable base of neighbourhoods of zero in the Fréchet space E_n. The set $\mathcal{F}$ of all unions of finite subsets of $\bigcup_n \mathcal{F}_n$ is countable. Let $\mathcal{M}$ be the set of all absolutely convex closed neighbourhoods of zero in E. Given any $A \subset E$, let $\overline{\text{ac}}A$ denote the absolutely convex closed envelope of A in E. The set

$$\mathcal{N} := \mathcal{M} \cap \{\overline{\text{ac}}A : A \in \mathcal{F}\}$$

is countable. Given an arbitrary U in $\mathcal{M}$, for each $n \in \mathbb{N}$, there exists $U_n \in \mathcal{F}_n$ with $U_n \subset U$, and we set

$$A_n := \bigcup_{j=1}^{n} U_j \in \mathcal{F}.$$

Clearly, the increasing sequence $(n \cdot \overline{\text{ac}}A_n)_n$ covers E, and each $\overline{\text{ac}}A_n \subset U$. If E is Baire-like, some $\overline{\text{ac}}A_n$ is in $\mathcal{M}$, hence in $\mathcal{N}$, which proves that the countable $\mathcal{N}$ is a base of neighbourhoods of zero in E; i.e. E is metrizable.

(ii) $\Rightarrow$ (iii) is clear since φ is non-metrizable.

(iii) $\Rightarrow$ (i) follows from Corollary 2.4.4 □

We provide later a general result that in particular shows also that non-metrizable (LF)-spaces are not Baire-like.

We shall need also in the sequel the following result due to Valdivia [607]; see also [491, Proposition 8.2.27].

Proposition 2.4.8 (Valdivia) *Let F be a dense barrelled (quasibarrelled) subspace of a lcs E, and assume that $(B_n)_n$ is an increasing sequence of absolutely convex sets such that each point (each bounded set) of F is absorbed by some B_n. Then $\bigcap_{\epsilon>0} \bigcup_n (1+\epsilon)\overline{B_n} = \overline{\bigcup_n B_n}$, where the closure is in E. In particular, $E = \bigcup_n n\overline{B_n}$, and if $(B_n)_n$ is a covering of F, then $E = \bigcup_n \overline{B_n}$.*

Proof We prove only the barrelled case. It is enough to show that

$$\overline{\bigcup_n B_n} \subset \bigcup_n (1+\epsilon)\overline{B_n}$$

for each $\epsilon > 0$. If there exists $\epsilon > 0$ and $x \notin \bigcup_n (1+\epsilon)\overline{B_n}$, we may choose for each n an absolutely convex neighbourhood of zero U_n in E such that

$$x \notin (1+\epsilon)\, B_n + 2U_n.$$

Set

$$U := \bigcap_n \overline{\epsilon B_n + U_n}$$

and observe that $U \cap F$ is a barrel in the barrelled space F. Thus, $U \cap F$ is a neighbourhood of zero in F, and U is a neighbourhood of zero in E by density of F. But for each n, we have

$$x \notin (1+\epsilon)\, B_n + 2U_n \supset \overline{B_n + \epsilon B_n + U_n} \supset B_n + \overline{\epsilon B_n + U_n} \supset B_n + U,$$

so that $x \notin \bigcup_n B_n + U$. Hence, $x \notin \overline{\bigcup_n B_n}$. □

We note the following:

Corollary 2.4.9 *For any completely regular Hausdorff space X, the space $C_p(X)$ is b-Baire-like.*

Proof Let $(A_n)_n$ be a bornivorous increasing sequence of absolutely convex closed subsets of $C_p(X)$ covering $C_p(X)$. For each $n \in \mathbb{N}$, let B_n be the closure of A_n in the space $\mathbb{R}^X$. Since the space $C_p(X)$ is dense in $\mathbb{R}^X$ and $C_p(X)$ is quasibarrelled by [322, Theorem 2], we apply Proposition 2.4.8 to get $\mathbb{R}^X = \overline{C_p(X)} = \bigcup_n B_n$. Since $\mathbb{R}^X$ is a Baire space, and the sets B_n are closed and absolutely convex, there exists $m \in \mathbb{N}$ such that B_m is a neighbourhood of zero in $\mathbb{R}^X$. Consequently, A_m is a neighbourhood of zero in $C_p(X)$. □

Corollary 2.4.10 *If E is a barrelled space covered by an increasing sequence of absolutely convex complete subsets, then E is complete.*

A lcs E is called *locally complete* if every bounded closed absolutely convex set B is a Banach disc, i.e. the linear span of B endowed with the Minkowski functional norm $\|x\|_B := \inf\{\epsilon > 0 : \epsilon^{-1}x \in B\}$ is a Banach space. A lcs E is *docile* if every infinite-dimensional subspace of E contains an infinite-dimensional bounded set; see [346].

It is known by Saxon–Levin–Valdivia's theorem that every countable-codimensional subspace of a barrelled space is barrelled; see [491, Theorem 4.3.6]. Following Theorem 2.4.11 is due to Saxon; see [533]. In fact, Saxon in [530] proved even that F is totally barrelled. An extension of this result to barrelled lcs E such that $(E', \mu(E', E))$ is complete was obtained by Valdivia; see [491, Proposition 4.3.11].

Theorem 2.4.11 (Saxon) *If F is a subspace of an (LF)-space E such that* $\dim(E/F) < 2^{\aleph_0}$, *then F is barrelled.*

For the proof, we need the following:

Lemma 2.4.12 (Sierpiński) *Every denumerable set $\mathbb{S}$ admits $2^{\aleph_0}$ denumerable subsets $\mathbb{S}_\iota$ ($\iota \in I$ and $|I| = 2^{\aleph_0}$) that are almost disjoint; i.e. if ι and $\jmath$ are distinct members of the indexing set I, then $\mathbb{S}_\iota \cap \mathbb{S}_\jmath$ is finite.*

Proof Take $\mathbb{S}$ to be all rational numbers, let I be the irrationals, and for each $\iota \in I$, choose $\mathbb{S}_\iota$ to be a sequence of rationals that converges to ι. □

Lemma 2.4.13 *Let A be a closed absolutely convex set having span F in a docile locally complete space E. Then* dim (E/F) *is either finite or at least $2^{\aleph_0}$.*

Proof Assume dim (E/F) is infinite. Then there is an infinite-dimensional subspace G of E that is transverse to F. By docility there is an infinite-dimensional bounded absolutely convex set B in G, and, trivially, its span is transverse to F. Let x_1 be an arbitrary non-zero member of B. The bipolar theorem provides $f_1 \in E'$ such that

$$f_1 \in A^\circ \text{ and } f_1(x_1) = 1 \cdot 2^1.$$

Now the span of $B_1 := f_1^\perp \cap B$ is infinite-dimensional and transverse to the span of $A_1 := A + \text{span}\{x_1\}$, and A_1 is absolutely convex and closed. Let x_2 be an arbitrary non-zero member of B_1 and use the bipolar theorem as before to obtain $f_2 \in E'$ such that

$$f_2 \in A_1^\circ \text{ and } f_2(x_2) = 2 \cdot 2^2.$$

By setting $B_2 = f_2^\perp \cap B_1$ and $A_2 = A_1 + sp\{x_2\}$ and continuing inductively, we obtain $(x_n)_n \subset B$ and $(f_n)_n \subset E'$ such that

$$(f_n)_n \subset A^\circ;\ f_i\left(x_j\right) = 0 \text{ for } i \neq j;\ \text{and } f_n(x_n) = n \cdot 2^n \text{ for all } n \in \mathbb{N}.$$

Since $(x_n)_n$ is bounded and E is locally convex, the series $\sum_n 2^{-n} x_n$ is absolutely, hence subseries, convergent. Let $\{\mathbb{N}_\iota\}_{\iota \in I}$ be a collection of $\mathfrak{c}$ denumerable subsets of $\mathbb{N}$ that are almost disjoint (Lemma 2.4.12), and for every $\iota \in I$ set

$$y_\iota = \sum_{n \in \mathbb{N}_\iota} 2^{-n} x_n.$$

We claim that the set $\{y_\iota\}_{\iota \in I}$ is linearly independent and has span transverse to F. Indeed, if y is any (finite) linear combination in which the coefficient of some y_ι is a non-zero scalar a, then we have

$$f_n(y) = n \cdot a \text{ for all but finitely many } n \in \omega_\iota.$$

Therefore, $(f_n)_n$ is unbounded at y, whereas $(f_n)_n \subset A^\circ$ implies that $(f_n)_n$ is bounded at all points of F, since they are absorbed by A. We have thus shown that

$\{y_\iota\}_{\iota\in I}$ consists of $\mathfrak{c}$ linearly independent vectors whose span is transverse to F, as desired. □

This implies that every infinite-dimensional locally complete docile space E has dimension at least $2^{\aleph_0}$. Indeed, take $A = F = \{0\}$. In particular, every infinite-dimensional Fréchet space has dimension at least $2^{\aleph_0}$; see also [422].

We are ready to prove Theorem 2.4.11.

Proof Let B be a barrel in F, let $(E_n)_n$ be a defining sequence of Fréchet spaces for E, let A be the closure of B in E, and let G be the span of A. In each Fréchet space E_n, the intersection $A \cap E_n$ is closed and has span of codimension less than $2^{\aleph_0}$ in E_n; by Lemma 2.4.13, the codimension is finite. Since the countable union of finite sets is countable, the codimension of G is countable in E. Therefore, G is barrelled by the Saxon–Levin–Valdivia theorem, and A is a neighbourhood of zero in G. It follows that $B = A \cap F$ is a neighbourhood of zero in F. □

The following observation will be used below; see [491, Proposition 4.1.6].

Proposition 2.4.14 *If E is a barrelled lcs covered by an increasing sequence $(E_n)_n$ of vector subspaces, then a linear map $f : E \to F$ from E into a lcs F is continuous if and only if there exists $m \in \mathbb{N}$ such that the restrictions $f|E_n : E_n \to F$ are continuous for all $n \geq m$, where the topology of each E_n is that induced by E.*

There is an interesting application of Baire-like spaces for closed graph theorems; see [528]. It is known that the class of barrelled spaces is the largest class of lcs for which the closed graph (open mapping) theorem holds vis-a-vis Fréchet spaces. Bourbaki [98, Theorem III.2.1] observed that the class of barrelled spaces is also the largest one for which the uniform boundedness theorem holds.

Saxon [528] showed that the Grothendieck factorization theorem for linear maps from a Baire lcs into an (LF)-space with closed graph remains true for linear maps from a Baire-like space into an (LB)-space. We have the following:

Theorem 2.4.15 (Saxon) *Let F be a Baire-like space. Let E be an (LB)-space with a defining sequence $(E_n)_n$ of Banach spaces. Then every linear map $f : F \to E$ with closed graph is continuous.*

Proof Via equivalent norms, we may assume that the unit ball B_n for E_n is contained in B_{n+1} and that $(B_n)_n$ covers E. Thus, $(f^{-1}(B_n))_n$ covers F, and there is some $m \in \mathbb{N}$ such that $\overline{f^{-1}(B_n)}$ is a neighbourhood of zero in F for each $n \geq m$. Hence, the closure of $f^{-1}(V_n)$ in $f^{-1}(E_n)$ is a neighbourhood of zero whenever V_n is a neighbourhood of zero in the Banach space E_n and $n \geq m$; i.e. the restriction of f to $f^{-1}(E_n)$ is almost continuous as a mapping into the Banach space E_n. Since the Banach topology is finer than induced by E, the graph of this restriction is also closed. By Ptak's closed graph theorem [491], this mapping is continuous, and all the more so when the range space E_n is given the coarser topology induced by E. Since F is barrelled, Proposition 2.4.14 ensures that the unrestricted f is also continuous. □

We complete this section with a characterization of the Baire-likeness of spaces $C_c(X)$ for some concrete spaces X.

In what follows, X is a completely regular Hausdorff topological space. If $\mathcal{F} := \{f \in C(X) : f(X) \subset [0, 1]\}$, the subspace

$$\{(f(x) : f \in \mathcal{F}) : x \in X\}$$

is homeomorphic to X; we identify X with this subspace, and the closure of X in $[0, 1]^{\mathcal{F}}$ is the Čech–Stone compactification of X, denoted by βX. Taking into account the restrictions to βX of the coordinate projections of $[0, 1]^{\mathcal{F}}$, we have $f \in \mathcal{F}$, and therefore each uniformly bounded $f \in C(X)$ has a unique continuous extension to βX.

By the *realcompactification* υX of X, we mean the subset of βX such that $x \in \upsilon X$ if and only if each $f \in C(X)$ admits a continuous extension to $X \cup \{x\}$. From the regularity of X, it follows that each $f \in C(X)$ admits a continuous extension to υX. Therefore, the closure in $\mathbb{R}^{C(X)}$ of $\{(f(x) : f \in C(X)) : x \in X\}$ is homeomorphic to υX.

By definition, X is called *realcompact* if $X = \upsilon X$. From the continuity of the coordinate projections, it follows that X is realcompact if and only if X is homeomorphic to a closed subspace of a product of the real lines. Clearly, closed subspaces of a realcompact space are realcompact, and any product of realcompact spaces is realcompact. The intersection of a family of realcompact subspaces of a space is realcompact, because this intersection is homeomorphic to the diagonal of a product.

Recall that a subset $A \subset X$ of a topological space X is *topologically bounded* if the restricted map $f|A$ is bounded for each $f \in C(X)$; otherwise, we will say that A is *topologically unbounded*.

If X is completely regular and Hausdorff, then $A \subset X$ is topologically bounded if and only if for each locally finite family $\mathcal{F}$, the family $\{F \in \mathcal{F} : F \cap A \neq \emptyset\}$ is finite. If X is a topologically bounded set of itself, then X is called *pseudocompact*.

Recall here that a lcs E is said to be *bornological*, if every bornivorous absolutely convex subset of E is a neighbourhood of zero.

The link between properties of $C_c(X)$ and $C_p(X)$ and topological properties of X is illustrated by Nachbin [459], Shirota [557], De Wilde–Schmets [158], and Buchwalter–Schmets [106], respectively.

Proposition 2.4.16 (Nachbin–Shirota) *$C_c(X)$ is barrelled if and only if X is a μ-space, i.e. every topologically bounded subset of X has compact closure.*

Proposition 2.4.17 (Nachbin–Shirota, De Wilde–Schmets) *The space $C_c(X)$ is bornological if and only if $C_c(X)$ is the inductive limit of Banach spaces if and only if X is realcompact.*

Proposition 2.4.18 (Buchwalter–Schmets) *$C_p(X)$ is barrelled if and only if every topologically bounded subset of X is finite.*

A good sufficient condition for $C_c(X)$ to be a Baire space is hard to locate. Let us mention the following one: If X is a locally compact and paracompact space, $C_c(X)$ is Baire. The argument uses the well-known fact stating that X can be written as the topological direct sum of locally compact, σ-compact (hence hemicompact) disjoint subspaces $\{X_t : t \in T\}$ of X. Since $C_c(X)$ is isomorphic to the product $\prod_t C_c(X_t)$ of Fréchet spaces, the space $C_c(X)$ is a Baire space; see, for example, [491]. In [293], Grenhage and Ma defined and studied the *moving off property* and proved that, if X is locally compact or first countable, $C_c(X)$ is Baire if and only if X has the moving off property.

Combining results of Lehner [400, Theorem III.2.2, Theorem III.3.1] and Proposition 2.4.16, we note the following:

Proposition 2.4.19 (Lehner) *$C_c(X)$ is Baire-like if and only if for each decreasing sequence $(A_n)_n$ of closed non-compact subsets of X, there exists a continuous function $f \in C(X)$, which is unbounded on each A_n.*

This yields the following useful Proposition 2.4.20; see [400] and also [329]. Recall that a topological space X is of *pointwise countable type* if each $x \in X$ is contained in a compact set $K \subset X$ of *countable character* in X, i.e. having a countable basis of open neighbourhoods. All first countable spaces, as well as Čech-complete spaces (hence locally compact spaces), are of pointwise countable type; see [195] and [32].

Proposition 2.4.20 (Lehner) *(i) If X is a locally compact space and $C_c(X)$ is barrelled, then $C_c(X)$ is a Baire-like space.*

(ii) If X is a space of pointwise countable type and $C_c(X)$ is Baire-like, X is a locally compact space.

Proof (i) Assume $C_c(X)$ is a barrelled space. Then by Proposition 2.4.16, the space X is a μ-space. Having in mind Proposition 2.4.19, we need only to find for a decreasing sequence $(A_n)_n$ of closed not topologically bounded sets in X a continuous function $f \in C(X)$ which is unbounded on each A_n. By the assumption, on each A_n, there exists $f_n \in C(X)$ unbounded on A_n. Note that the proof will be completed if we find a number $m \in \mathbb{N}$ such that f_m is unbounded on each A_n. Assume that for each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that f_n is bounded on A_{k_n}. Since $(A_n)_n$ is decreasing, we may assume that f_n is bounded on A_{n+1} for each $n \in \mathbb{N}$.

Two cases are possible:

(a) $A := \bigcap_n A_n$ is non-compact. Since X is a μ-space, there exists a continuous function $f \in C(X)$, which is unbounded on A, and the proof is finished.

(b) A is compact. Since X is locally compact, there exists an open set V_0 such that $A \subset V_0$ whose closure W_0 is compact. But f_1 is bounded on $A_2 \cup W_0$ and unbounded on A_1, so $A_1 \not\subset A_2 \cup W_0$. Select

$$x_1 \in A_1 \setminus (A_2 \cup W_0).$$

Then there exists an open neighbourhood V_1, $x_1 \in V_1$, whose closure W_1 is compact and

$$V_1 \subset X \setminus (A_2 \cup W_0).$$

By a simple induction, we select a sequence $(x_n)_n$ in X, and a pairwise disjoint sequence $(V_n)_n$ of open neighbourhoods of x_n whose closure W_n is compact for each $n \in \mathbb{N}$. Since $(V_n)_n$ is an open cover of

$$\{x_n : n \in \mathbb{N}\} \cup A,$$

we deduce that this cover does not admit a finite subcover. We conclude that $\{x_n : n \in \mathbb{N}\} \cup A$ is not compact. Set $L := \{x_n : n \in \mathbb{N}\}$.

Claim. $L \cup A$ is closed in X. Indeed, if $x \in \overline{L} \setminus A$, then $x \in \overline{\{x_k : k > n\}}$ for all $n \in \mathbb{N}$. Since $\{x_k : k > n\} \subset A_n$ for each $n \in \mathbb{N}$, so $x \in A$. This yields that $L \cup A$ is closed. As every topologically bounded set in X is relatively compact, so there exists a continuous function $f \in C(X)$ unbounded on $L \cup A$. Therefore, f is unbounded on each A_n.

(ii) Fix $x \in X$. By the assumption, there exists a compact set K, $x \in K$, and a decreasing basis $(U_n)_n$ of open neighbourhoods of K. Set

$$H_n := \{f \in C(X) : \sup_{h \in U_n} |f(h)| \leq n\}.$$

Then, as it is easily seen, the family $\{H_n : n \in \mathbb{N}\}$ covers $C(X)$. Since $C_c(X)$ is Baire-like, there exist a compact set $D \subset X$, $\epsilon > 0$, and $n \in \mathbb{N}$ such that

$$\{f \in C(X) : \sup_{d \in D} |f(d)| < \epsilon\} \subset H_n.$$

Observe that $U_n \subset D$. Indeed, if $z \in U_n \setminus D$, we can find a continuous function $f \in C(X)$ such that $f(z) = n + 1$ and $f(d) = 0$ for all $d \in D$. Hence, $f \in H_n$ so $f(z) \leq n$, which provides a contradiction. □

We already mentioned that the product of two normed Baire spaces need not be a Baire space. Nevertheless, particular products of Baire spaces are Baire; see the proof of Proposition 2.1.3 and a part below Proposition 2.1.8. On the other hand, any product of metrizable and separable Baire spaces is a Baire space; see [485]. This fact applies to get the following interesting result concerning the products of Baire spaces $C_p(X)$; see [590, Theorem 4.8] and [423].

Theorem 2.4.21 (Tkachuk) *Let $\{C_p(X_t) : t \in A\}$ be a family of Baire spaces. Then the product $\prod_{t \in A} C_p(X_t)$ is a Baire space.*

Proof Note that $\prod_{t \in A} C_p(X_t)$ is isomorphic to $C_p(X)$, where $X = \bigoplus_{t \in A} X_t$. We will need here the following useful fact stating that $C_p(X)$ is a Baire space if and

only if $T_D(C_p(X))$ is a Baire space for each countable set $D \subset X$, where as usual $T_D(f) := f|D$ for $f \in C(X)$ means the restriction map; see [423, Theorem 3.6]. Let $D \subset X$ be a countable set, and set $D_t := D \cap X_t$ for each $t \in A$. Clearly, $W_D := \{t \in A : D_t \neq \emptyset\}$ is countable. Since by the assumption each $C_p(X_t)$ is a Baire space, $T_{D_t}(C_p(X_t))$ is a Baire space. On the other hand, as it is easily seen,

$$T_D(C_p(X)) = \prod_{t \in W_D} T_{D_t}(C_p(X_t)).$$

As each space $T_{D_t}(C_p(X_t)$ is a metrizable and separable Baire space, the product $T_D(C_p(X))$ is a Baire space. Hence, $C_p(X)$ is Baire. □

2.5 Strongly Realcompact Spaces X and Spaces $C_c(X)$

This section deals with the class of strongly realcompact spaces, introduced and studied in [350]; see also [599]. The following well-known characterization of realcompact spaces will be used in the sequel; see [275] or [195].

Proposition 2.5.1 *A completely regular Hausdorff space X is realcompact if and only if for every element $x \in X^* := \beta X \setminus X$, there exists $h_x \in C(\beta X)$, $h_x(X) \subset]0, 1]$, i.e. which is positive on X, and $h_x(x) = 0$.*

Proof Assume the condition holds. Then $X = \bigcap\{h_y^{-1}]0, 1] : y \in \beta X \setminus X\}$. As each $h_y^{-1}]0, 1]$ is a realcompact subspace of βX (since $h_y^{-1}]0, 1]$ is homeomorphic to $(\beta X \times]0, 1]) \cap G(h_y)$, where $G(h_y)$ means the graph of h_y), then X is also realcompact. Conversely, if X is realcompact and $x_0 \in \beta X \setminus X = \beta X \setminus \upsilon X$, there exists a continuous function $f : X \to \mathbb{R}$, which cannot be extended continuously to $X \cup \{x_0\}$. From

$$f(x) = \max(f(x), 0) + \min(f(x), 0) =$$

$$1 + \max(f(x), 0) - (1 - \min(f(x), 0)),$$

we know that one of the functions $g_1(x) = 1 + \max(f(x), 0)$, $g_2(x) = 1 - \min(f(x), 0)$, cannot be extended continuously to $X \cup \{x_0\}$. Hence, there exists a continuous function $g : X \to [1, \infty[$, which cannot be extended continuously to $X \cup \{x_0\}$. Let $\widehat{h}$ be a continuous extension of the bounded function $h := 1/g$ to βX. If $\widehat{h}(x_0) \neq 0$, we reach to a contradiction. Hence, $\widehat{h}(x_0) = 0$. □

We shall say that X is *strongly realcompact* [350] if for every sequence $(x_n)_n$ of elements in X^*, there exists $f \in C(\beta X)$, which is positive on X and vanishes on some subsequence of $(x_n)_n$. Clearly, every strongly realcompact space is realcompact. It is known, [634, Exer. 1B. 4], that if X is locally compact σ-compact, then X^* is a zero set in βX, so X is strongly realcompact.

A subset $A \subset X$ is said to be C-embedded ((C^*)-embedded) if every real-valued continuous (bounded and continuous) function on A can be extended to a continuous function on the whole space X.

For strongly realcompact spaces, we note the following property. The proof presented below (see [350]) uses some argument (due to Negrepontis [465]) from [274, Therem 2.7].

Proposition 2.5.2 *If X is strongly realcompact, every infinite subset D of X^* contains an infinite subset S, which is relatively compact in X^* and C^*-embedded in βX.*

Proof Let $(x_n)_n$ be an injective sequence in D (i.e. $x_n \neq x_m$ if $n \neq m$), and let $f : \beta X \to [0, 1]$ be a continuous function that is positive on X and vanishes on a subsequence of $(x_n)_n$. Set

$$S = \{x_n : n \in \mathbb{N}\} \cap f^{-1}\{0\}, \quad Y_n = \{x \in \beta X : |f(x)| \geqslant n^{-1}\}, n \in \mathbb{N},$$

$$X_1 = S \cup \bigcup_n Y_n.$$

Note that the space X_1 is regular and σ-compact; hence, it is normal; see Lemma 6.1.3. Since S is closed in X_1, it is C^*-embedded in X_1. Therefore, S is C^*-embedded in βX_1. As $X \subset X_1 \subset \beta X$, we conclude that $\beta X_1 = \beta X$. □

This implies that, if X is a strongly realcompact space, every infinite closed subset of X^* contains a copy of the space $\beta\mathbb{N}$. On the other hand, in [76, Example 1.11] Baumgartner and van Douwen provided a separable first countable locally compact realcompact space X (hence strongly realcompact by Theorem 2.5.5) for which X^* contains a discrete countable subset that is not C^*-embedded in βX. This result with [76, Theorem 1.2] can be used to distinguish an example of a locally compact realcompact space X such that X^* contains a sequence $(x_n)_n$ for which does not exist $f \in C(\beta X)$ which is positive on X and vanishes on $(x_n)_n$.

The space $\mathbb{Q}$ of the rational numbers is not strongly realcompact, but by [624], one gets that $\mathbb{Q}^*$ is a $\beta\mathbb{N}$-space, i.e. if D is a countable discrete subset of $\mathbb{Q}^*$, and $\overline{D}$ (the closure in $\mathbb{Q}^*$) is compact, then $\overline{D} = \beta D$. Hence, D is C^*-embedded in $\beta\mathbb{Q}$. It is known (see [275]) that $\mathbb{Q}^*$ contains a countable subset that is not C^*-embedded in $\beta\mathbb{Q}$.

A filter (filterbasis) $\mathcal{F}$ on a topological space X is said to be *unbounded* if there exists a continuous real-valued function f on X which is unbounded on each element of $\mathcal{F}$. We call f unbounded on $\mathcal{F}$.

In order to prove Theorem 2.5.5, we need the following two lemmas.

Lemma 2.5.3 *A filter $\mathcal{F}$ on a topological space X is unbounded if and only if there exists $x \in \bigcap_{F\in\mathcal{F}} \overline{F} \setminus \upsilon X$, where the closure is taken in βX.*

Proof Set $K := \bigcap_{F\in\mathcal{F}} \overline{F}$, and assume by contradiction that $K \subset \upsilon X$. Then for each continuous real-valued function f on X, there exists an open $U_f \subset \beta X$ such that $K \subset U_f$, and the restriction $f|U_f \cap X$ is bounded. Note that there exists $F \in \mathcal{F}$ contained in U_f. Indeed, otherwise the family of sets $\{\overline{F} \setminus U_f : F \in \mathcal{F}\}$ satisfies the finite intersection property, which leads a point in $K \setminus U_f$. This is a contradiction. We proved that there exists $F \in \mathcal{F}$ contained in U_f. This shows that $\mathcal{F}$ is not unbounded.

To prove the converse, assume that there exists $x \in K \setminus \upsilon X$. It is known (see [275]) that $\upsilon X = \bigcap_{f\in C(X)} \upsilon_f(X)$, where

$$\upsilon_f(X) := \{x \in \beta X : f^\beta(x) \neq \infty\}.$$

Then there exists $f \in C(X)$ for which the extension $f^\beta : \beta X \to \mathbb{R}_\infty$ has property $f^\infty(x) = \infty$, where $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ (the Alexandroff one-point compactification). Since $x \in \overline{F}$ for each $F \in \mathcal{F}$, the map f is unbounded on $\mathcal{F}$. □

Lemma 2.5.4 *Each unbounded filterbasis $\mathcal{F}$ on a topological space X is contained in an unbounded ultrafilter $\mathcal{U}$ on X.*

Proof If $\mathcal{M} := \{M \subset X : \exists F \in \mathcal{F};\ F \subset M\}$, then $\mathcal{M}$ is an unbounded filter on X. By Lemma 2.5.3, there exists $x \in \bigcap_{F\in\mathcal{M}} \overline{F} \setminus \upsilon X$. Let $\mathcal{A}$ be the family of all filters $\mathcal{G}$ on X containing $\mathcal{M}$ and such that $x \in \bigcap_{F\in\mathcal{G}} \overline{F}$. Order $\mathcal{A}$ by inclusion. Since there exists a maximal chain in $\mathcal{A}$, its union $\mathcal{U}$ is an ultrafilter on X containing $\mathcal{F}$ such that $x \in \bigcap_{F\in\mathcal{U}} \overline{F}$, we use Lemma 2.5.3 to conclude that $\mathcal{U}$ is unbounded on X. □

We are ready to prove the following characterization of strongly realcompact spaces; parts (i) and (ii) were proved in [350]; part (iii) is from [599].

Theorem 2.5.5 *(i) A topological space X is strongly realcompact if and only if X is realcompact and X^* is countably compact. Hence, every locally compact realcompact space is strongly realcompact.*

(ii) Every strongly realcompact space of pointwise countable type is locally compact.

(iii) A realcompact space X is strongly realcompact if and only if for each sequence $(\mathcal{F}_n)_n$ of unbounded filters (filterbases) on X, there exist a continuous real-valued function on X and a subsequence $(\mathcal{F}_{n_k})_k$ such that f is unbounded on each $\mathcal{F}_{n_k}$.

Proof (i) Suppose X is strongly realcompact. Let $P \subset X^*$ be an infinite set, and let $(x_n)_n$ be an injective sequence in P. There exists a continuous function $f : \beta X \to [0, 1]$, which is positive on X and zero on some subsequence $(x_{k_n})_n$ of $(x_n)_n$. Then

$$\{x_{k_n} : n \in \mathbb{N}\} \subset f^{-1}(0) \subset X^*.$$

Hence,

$$\{x_{k_n} : n \in \mathbb{N}\}^d \subset f^{-1}(0).$$

Note that $\{x_{k_n} : n \in \mathbb{N}\}^d$ is non-empty, where A^d is the set of all accumulation points of a set A. This proves that P has an accumulation point in X^*.

To prove the converse, assume that X is realcompact and every infinite subset of X^* has an accumulation point in X^*.

Let $(x_n)_n$ be a sequence in X^*. If $P = \{x_n : n \in \mathbb{N}\}$ is finite, then (since X is realcompact) there exists a continuous function $f : \beta X \to [0, 1]$, which is positive on X and zero on a subsequence of $(x_n)_n$. If $P = \{x_n : n \in \mathbb{N}\}$ is infinite, take $p \in P^d \setminus X$. Then there exists a continuous function $f : \beta X \to [0, 1]$, which is positive on X and vanishes on p. Note that for every $r > 0$, the set $P \cap f^{-1}([0, r))$ is infinite, since $f^{-1}([0, r))$ is a neighbourhood of the point $p \in P^d$.

We consider two possible cases.

Case 1. The set $P \cap f^{-1}(0)$ is infinite. Then f is positive on X and zero on some subsequence of the sequence $(x_n)_n$.

Case 2. The set $P \cap f^{-1}(0)$ is finite. As for every $r > 0$ the set $P \cap f^{-1}([0, r))$ is infinite, there exists an injective sequence $(t_n)_n$ in P such that the sequence $(f(t_n))_n$ is strictly decreasing and converges to zero. Set $P_0 = \{t_n : n \in \mathbb{N}\}$, $s_0 = 1$ and $s_k \in (f(t_{k+1}), f(t_k))$ for all $k \in \mathbb{N}$. Then $(s_k)_k$ is decreasing and converges to zero. Set

$$F_k = f^{-1}([s_k, s_{k-1}])$$

for $k \in \mathbb{N}$. Then F_k are compact and $t_k \in F_n$ if and only if $k = n$. Moreover,

$$X \subset f^{-1}((0, 1]) = \bigcup_k F_k,$$

and

$$P_0 \cap F_k = \{t_k\}, \quad k \in \mathbb{N}.$$

If $f(x) = c > 0$, then $x \in f^{-1}((2^{-1}c, 1])$. Since $f(t_k) \to 0$, we have that $x \notin P_0^d$. Hence, if $x \in P_0^d$, then $x \in f^{-1}(0)$. Hence, $x \notin \bigcup_k F_k$. We showed that

$$P_0^d \cap \left(\bigcup_k F_k \right) = \emptyset.$$

Since X is realcompact, for every $k \in \mathbb{N}$, there exists a continuous function $f_k : \beta X \to [0, 1]$, which is positive on X and zero on t_k. Next set

$$T_n^k = f_k^{-1}([n^{-1}, 1]), \quad n, k \in \mathbb{N}.$$

Then

$$X \subset f_k^{-1}((0,1]) = \bigcup_n T_n^k$$

and $t_k \notin T_n^k$ for all $k, n \in \mathbb{N}$. Moreover,

$$X \subset \bigcup_k F_k \cap X \subset \bigcup_k \bigcup_n F_k \cap T_n^k, \quad P_0 \cap (F_k \cap T_n^k) \subset P_0 \cap F_k = \{t_k\}.$$

As $t_k \notin F_k \cap T_n^k$, so

$$P_0 \cap (F_k \cap T_n^k) = \emptyset$$

for all $n, k \in \mathbb{N}$. Hence, $P_0 \cap W = \emptyset$, and

$$P_0^d \cap W \subset P_0^d \cap \bigcup_k F_k = \emptyset,$$

where $W = \bigcup_k \bigcup_n F_k \cap T_n^k$. Therefore, $\overline{P_0} \cap W = \emptyset$. We showed that there exist an infinite subset P_0 of P and an infinite sequence of compact sets $(K_n)_n$ such that

$$X \subset \bigcup_n K_n \subset \beta X, \quad \left(\bigcup_n K_n\right) \cap \overline{P_0} = \emptyset.$$

For every $n \in \mathbb{N}$, let $g_n : \beta X \to [0,1]$ be a continuous function such that

$$g_n | K_n = 1, \quad g_n | \overline{P_0} = 0.$$

Put $g = \sum_n 2^{-n} g_n$. The function $g : \beta X \to [0,1]$ is continuous, positive on X, and zero on some subsequence of the sequence $(x_n)_n$. This shows that for every sequence $(x_n)_n$ in X^*, there exists a continuous function on βX which is positive on X and vanishes on some subsequence of $(x_n)_n$.

(ii) Assume X is a strongly realcompact space of pointwise countable type and X is not locally compact. Then there exist $x_0 \in X$ for which does not exist a relatively compact open neighbourhood and a compact set K with $x_0 \in K$ which admits a countable (decreasing) basis $(U_n)_n$ of neighbourhoods of K. For every $n \in \mathbb{N}$, choose $x_n \in (\overline{U_n} \setminus X)$, where the closure is taken in βX. Note that

$$(\beta X \setminus K) \cap \{x_n\}^d = \emptyset.$$

Indeed, let $x \in (\beta X \setminus K)$. Let $V \subset \beta X$ be an open neighbourhood of K such that $x \in (\beta X \setminus \overline{V})$. Then there exists $n_0 \in \mathbb{N}$ such that $U_{n_0} \subset V \cap X$, so $\overline{U_{n_0}} \subset \overline{V}$. Since

$$\{x_n\}^d \subset \overline{U_{n_0}} \subset \overline{V},$$

$x \in \beta X \setminus \{x_n\}^d$. Hence, $\{x_n\}^d \subset K$. This shows that X is not strongly realcompact, a contradiction.

(iii) Assume X is strongly realcompact and each $\mathcal{F}_n$ is an unbounded filterbasis on X. For each $n \in \mathbb{N}$, there exists an accumulation point of $\mathcal{F}_n$, say $x_n \in \beta X \setminus \upsilon X$. Since X is a realcompact space, $\upsilon X = X$. As X is strongly realcompact, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and a positive continuous function $g \in C(X)$ such that $g(x) \leq 1$ and $g^\beta(x_{n_k}) = 0$ for all $k \in \mathbb{N}$. Then $f := g^{-1} \in C(X)$. Hence, f is unbounded on each $\mathcal{F}_{n_k}$ since

$$x_{n_k} \in \bigcap_{F \in \mathcal{F}_{n_k}} \overline{F} \setminus \upsilon_f(X).$$

This proves one direction of the claim (iii).

To prove the converse, assume that $(x_n)_n$ is a sequence in $\beta X \setminus X$. Then for each $n \in \mathbb{N}$, there exists a filter $\mathcal{F}_n$ on X, which converges to x_n in the space βX. But

$$x_n \in \bigcap_{F \in \mathcal{F}_n} \overline{F}, \quad x_n \notin X = \upsilon X.$$

This shows that each $\mathcal{F}_n$ is unbounded on X. By the assumption, there exists a subsequence $(\mathcal{F}_{n_k})_k$ of $(\mathcal{F})_n$ and $f \in C(X)$ which is unbounded on each $\mathcal{F}_{n_k}$. Set

$$g(x) := (1 + |f(x)|)^{-1}$$

for each $x \in X$. Clearly, the function g is positive on X and is continuous and bounded. Therefore, there exists a continuous extension g^β of g to βX, and clearly $g^\beta(x_{n_k}) = 0$. This proves that X is strongly realcompact. □

Example 2.5.6 There is a strongly realcompact space that is not locally compact. The space $\mathbb{R}^{\mathbb{N}}$ is realcompact and not strongly realcompact.

Proof Let P be a countably and non-empty subset of $\mathbb{N}^*$. Note that the subspace $X := \mathbb{N} \cup P$ of $\beta\mathbb{N}$ is a Lindelöf space. Hence, it is a realcompact space. On the other hand, since every countably and closed subset of $\beta\mathbb{N}$ is finite (see [634, p.71]), one gets that the space $X^* = \mathbb{N}^* \setminus P$ is countably compact. Now Theorem 2.5.5 applies to deduce that X is strongly realcompact. Note that X is not locally compact. The second statement follows directly from Theorem 2.5.5. □

If D is an absolutely convex subset of $C_c(X)$, a *hold* K of D is a compact subset of βX such that $f \in C(X)$ belongs to D if its continuous extension $f^\beta : \beta X \to \beta\mathbb{R}$

is identically zero on a neighbourhood of K. The intersection $k(D)$ of all holds of an absolutely convex set D in $C_c(X)$ is again a hold. $k(D)$ is called a *support* of D. If, moreover, D is bornivorous, $k(D)$ is contained in υX; see [491, Lemma 10.1.9].

We need also the following fact due to Valdivia [607]; see also [491].

Lemma 2.5.7 (Valdivia) *Let E be a barrelled space. Let $(A_n)_n$ be an increasing sequence of absolutely convex closed subsets of E covering E. Then for every bounded set $B \subset E$, there exists $m \in \mathbb{N}$ such that $B \subset mA_m$.*

Proof Assume that for each $n \in \mathbb{N}$ there exists $x_n \in B \setminus nA_n$. Then the sequence $(y_n)_n$, $y_n := n^{-1}x_n$, converges to zero in E, and $y_n \notin A_n$ for all $n \in \mathbb{N}$. Let $(U_n)_n$ be a decreasing sequence of absolutely convex neighbourhoods of zero in E such that $U_{n+1} + U_{n+1} \subset U_n$ and $y_n \notin A_n + U_n$ for all $n \in \mathbb{N}$. Then

$$y_n \notin \overline{A_n + U_{n+1}}$$

for all $n \in \mathbb{N}$. Set

$$U := \bigcap_n \overline{(A_n + U_n)}.$$

The set U is closed absolutely convex and absorbing in E and $y_n \notin U$ for all $n \in \mathbb{N}$. Hence, U is a barrel in E. Since E is barrelled, U is a neighbourhood of zero in E. This proves that U contains almost all elements of the sequence $(y_n)_n$, a contradiction. □

We are prepared to prove the following result from [350].

Theorem 2.5.8 (Kąkol–Śliwa) *(i) If X is a strongly realcompact space, $C_c(X)$ is Baire-like and bornological.*

(ii) If X is locally compact, $C_c(X)$ is Baire-like and bornological if and only if X is realcompact.

(iii) If X is a space of pointwise countable type, $C_c(X)$ is bornological and Baire-like if and only if X is strongly realcompact.

Proof (i) Let $(D_n)_n$ be a bornivorous sequence in $C_c(X)$, i.e. every bounded set in $C_c(X)$ is absorbed by some D_n. We may assume that every bounded set in $C_c(X)$ is contained in some D_n. Note that $k(D_{m_0}) \subset X$ for some $m_0 \in \mathbb{N}$. Indeed, otherwise for every $n \in \mathbb{N}$, there exists $x_n \in k(D_n) \setminus X$. Since X is strongly realcompact, there exists a continuous function $f : \beta X \to [0, 1]$, which is positive on X and zero on some subsequence of $(x_n)_n$. Since $(D_n)_n$ is increasing, we may assume that $f(x_n) = 0$, $n \in \mathbb{N}$. The sets

$$A_m = \{y \in \beta X : f(y) > m^{-1}\}$$

are open in βX and compose an increasing sequence covering X. Since $x_n \notin \overline{A_n}$ for $n \in \mathbb{N}$, where the closure is taken in βX, we have $k(D_n) \not\subset \overline{A_n}$ for

every $n \in \mathbb{N}$. This implies that $\overline{A_n}$ is not a hold of D_n for any $n \in \mathbb{N}$. Hence, there exists a sequence

$$f_n \in C_c(X) \setminus D_n$$

such that the extension $f_n^\beta = 0$ on some neighbourhood of $\overline{A_n}$. As the sequence $(f_n)_n$ converges to zero in $C_c(X)$, so there exists $p \in \mathbb{N}$ such that $f_n \in D_p$ for all $n \in \mathbb{N}$, a contradiction.

We proved that there exists $m_0 \in \mathbb{N}$ such that $k(D_{m_0}) \subset X$. Next we show that there exist $m \geqslant m_0$, $r > 0$, such that

$$\{f \in C(X) : \sup_{x \in k(D_m)} |f(x)| < r\} \subset D_m, \tag{2.6}$$

and this will show that $C_c(X)$ is b-Baire-like.

To show (2.6), it is enough to prove that there exist $r > 0$, and $n \geqslant m_0$, such that

$$\{f \in C(X) : \sup_{x \in X} |f(x)| < r\} \subset D_n;$$

see [549, Theorem II.1.4]. Assume this fails. Then there exists a sequence

$$f_n \in C(X) \setminus D_n$$

such that $|f_n(x)| < n^{-1}$ for every $x \in X$ and $n \in \mathbb{N}$. Since $(f_n)_n$ converges to zero in $C_c(X)$ and $(D_n)_n$ is bornivorous, we reach a contradiction.

On the other hand, since X is realcompact, by Proposition 2.4.17 the space $C_c(X)$ is the inductive limit of Banach spaces; therefore, $C_c(X)$ is both barrelled and bornological. By Proposition 2.4.2, any barrelled b-Baire-like space is Baire-like. Hence, $C_c(X)$ is Baire-like. An alternative (shorter) proof of (i): Part (i) can be also proved by using Theorem 2.5.5 (iii). Indeed, since a sequence $(A_n)_n$ of unbounded subsets of X provides a sequence $((A_n))_n$ of unbounded filterbases, we apply (iii) of Theorem 2.5.5.

(ii) Assume X is locally compact and $C_c(X)$ is Baire-like and bornological. Then X is realcompact. The rest follows from Theorem 2.5.5.

(iii) Assume X is of pointwise countable type, and the space $C_c(X)$ is bornological and Baire-like. Then X is realcompact by Proposition 2.4.17. We prove that X is locally compact. Let $x \in X$. Since X is of pointwise countable type, there exist a compact set K in X containing x, a decreasing basis $(U_n)_n$ of open neighbourhoods of the set K. Then the absolutely convex and closed sets

$$W_n = \{f \in C(X) : \sup_{x \in U_n} |f(x)| \leqslant n\}$$

cover $C_c(X)$. By assumption, there exist $n \in \mathbb{N}$, $\epsilon > 0$, and a compact subset S of X such that

$$\{f \in C(X) : \sup_{y \in S} |f(y)| < \epsilon\} \subset W_n.$$

Hence, $U_n \subset S$. We proved that X is locally compact. Theorem 2.5.5 applies to deduce that X is strongly realcompact. For the converse, we apply again Theorem 2.5.5 and the previous case. □

From Theorem 2.5.5, we know that a realcompact space X, for which $\beta X \setminus X$ is countably compact, is strongly realcompact. As concerns the converse to Theorem 2.5.8, we note only the following fact.

Proposition 2.5.9 *If $C_c(X)$ is a Baire space and X is realcompact, then $\beta X \setminus X$ is pseudocompact, i.e. its image under any real-valued continuous function is bounded.*

Proof Assume that $X^* := \beta X \setminus X$ is not pseudocompact. Then there exists a locally finite sequence $(U_n)_n$ of open disjoint subsets in X^*. Then, by regularity, we obtain a sequence $(V_n)_n$ of open non-empty sets in βX such that

$$\emptyset \neq V_n \cap X^* \subset \overline{V_n} \cap X^* \subset U_n,$$

where the closure is taken in βX. Since X is realcompact, $\overline{A_n} = \overline{V_n}$, where $A_n := V_n \setminus X^* = V_n \cap X$. As every topologically bounded set in X is relatively compact by Proposition 2.4.16 (since $C_c(X)$ is barrelled), then A_n is not topologically bounded in X for $n \in \mathbb{N}$. Since $C_p(X)$ is a Baire space, then by [400], there exists a continuous function $f \in C(X)$ and a subsequence $(A_{n_k})_k$ such that $f|A_{n_k}$ is unbounded for each $k \in \mathbb{N}$.

Let $\mathbb{R}_\infty$ be the Alexandrov one-point compactification of $\mathbb{R}$ and let $f^\infty : \beta X \to \mathbb{R}_\infty$ be the continuous extension of f. As each $f|A_{n_k}$ is unbounded, there exists a sequence $(x_k)_k$ such that $f^\infty(x_k) = \infty$ for each $k \in \mathbb{N}$ and

$$x_k \in \overline{A_{n_k}} \setminus X \subset U_{n_k}.$$

Since the sequence $(U_{n_k})_k$ is locally finite in X^*, we deduce that the sequence $(x_k)_k$ has an adherent point $x \in X$. But $f^\infty(x_k) = \infty$ for each $k \in \mathbb{N}$, so $f(x) = f^\infty(x) = \infty$. This provides a contradiction, since $f(X) \subset \mathbb{R}$. We proved that $\beta X \setminus X$ is pseudocompact. □

We need the following fact following from [436, Theorem 5.3.5].

Lemma 2.5.10 *If there exists an infinite family $\mathcal{K}$ of non-empty compact subsets of X such that (i) for every compact set L of X, there exists $K \in \mathcal{K}$ with $K \cap L = \emptyset$. (ii) Any infinite subfamily of $\mathcal{K}$ is not discrete. Then $C_c(X)$ is not a Baire space.*

There are several examples of barrelled spaces $C_c(X)$, which are not Baire; see, for example, [400, 590], and [423]. Next example from [350] is motivated by [76, Example 1.11].

Example 2.5.11 (Kąkol–Śliwa) There exists a locally compact and strongly realcompact space X such that bornological Baire-like $C_c(X)$ is not Baire.

Proof Let X be the set $\mathbb{R}$ of reals endowed with a topology defined as follows:

(a) For every $t \in \mathbb{Q}$, the set $\{t\}$ is open in X.
(b) For every $t \in \mathbb{R} \setminus \mathbb{Q}$, there exists a sequence $(t_n)_n \subset \mathbb{Q}$, which converges to t such that the sets

$$V_n(t) = \{t\} \cup \{t_m : m \geqslant n\},$$

$n, m \in \mathbb{N}$, form a base of neighbourhoods of t in X.
(c) For all dense sets $A, B \subset \mathbb{Q}$ in the natural topology of $\mathbb{R}$, the set $\overline{A} \cap \overline{B}$ is non-empty, where the closure is taken in X.

As the space X is a locally compact and realcompact space, then X is strongly realcompact by Theorem 2.5.5.

We prove that X satisfies conditions (i) and (ii) from Lemma 2.5.10. Let $(P_n)_n$ be a sequence of pairwise disjoint finite subsets of $\mathbb{R} \setminus \mathbb{Q}$ such that for any subsequence $(P_{n_k})_k$, the set $\bigcup_k P_{n_k}$ is dense in $\mathbb{R}$. Set

$$P_n = \{t_k^n : 1 \leqslant k \leqslant m_n\}$$

for all $n \in \mathbb{N}$. For all $n, m \in \mathbb{N}$, the set

$$K_{n,m} = \bigcup_{1 \leqslant k \leqslant m_n} V_m(t_k^n)$$

is non-empty and compact in X.

Claim 1. $\mathcal{K} = \{K_{n,m} : n, m \in \mathbb{N}\}$ satisfies (i).

Indeed, any compact subset L of X is contained in a set of the form

$$V_1(t^1) \cup \cdots \cup V_1(t^m) \cup \{p_1, \ldots, p_k\},$$

where $m, k \in \mathbb{N}$ and $t^1, t^2, \ldots, t^m \in \mathbb{R} \setminus \mathbb{Q}$, $p_1, \ldots, p_k \in \mathbb{Q}$. In fact, the set $L \cap (\mathbb{R} \setminus \mathbb{Q})$ is finite, since the open cover

$$\{V_1(t) : t \in L \cap (\mathbb{R} \setminus \mathbb{Q})\} \cup \{\{q\} : q \in L \cap \mathbb{Q}\}$$

of L has a finite subcover. On the other hand,

$$(L \setminus \Big(\bigcup \{V_1(t) : t \in L \cap (\mathbb{R} \setminus \mathbb{Q})\}\Big)$$

is finite, since $X^d \subset (\mathbb{R} \setminus \mathbb{Q})$. Hence, $\mathcal{K}$ indeed satisfies (i).

Claim 2. $\mathcal{K}$ satisfies (ii).

Indeed, otherwise some infinite subfamily $\{K^n : n \in \mathbb{N}\}$ of $\mathcal{K}$ is discrete. But then sets $A = \bigcup_n K^{2n}$ and $B = \bigcup_n K^{2n+1}$ are disjoint and closed in X. Applying the property of $(P_n)_n$, one gets that $A \cap \mathbb{Q}$ and $B \cap \mathbb{Q}$ are dense in $\mathbb{R}$. Hence, by (c), the set $\overline{(A \cap \mathbb{Q})} \cap \overline{(B \cap \mathbb{Q})}$ is non-empty, a contradiction. We proved that $C_c(X)$ is not Baire. □

In [590], Tkachuk provided an example of a countable space X such that X has exactly one non-isolated point, X has no infinite topologically bounded sets (hence $C_p(X)$ is barrelled by Proposition 2.4.18), and $C_p(X)$ is not Baire. We show another example of this type.

Set $X := \mathbb{N}^2 \cup \{x\}$ for $x \notin \mathbb{N}^2$, where all points of $\mathbb{N}^2$ are isolated in X, and set $X_n := \{(m, n) : m \in \mathbb{N}\}$ for each $n \in \mathbb{N}$. The basis $\mathcal{B}(x)$ at x is formed by the sets

$$\{U \subset X : x \in U,\ |\{n : |(X \setminus U) \cap X_n| = \infty\}| < \infty\}.$$

The space X (originally defined by Arens) is completely regular and Hausdorff. Since X is countable, X is Lindelöf.

Example 2.5.12 The space $C_p(X)$ for the Arens space X is barrelled and not Baire.

Proof If $K \subset X$ is compact, K is finite (so X is hemicompact). Indeed, note that $|K \cap X_n| < \infty$ for each $n \in \mathbb{N}$, and

$$|\{n \in \mathbb{N} : K \cap X_n \neq \emptyset\}| < \infty.$$

Clearly, $C_p(X)$ is metrizable since X is countable. By Propositions 2.4.18 and 2.4.16, the space $C_c(X) = C_p(X)$ is barrelled.

We show that $C_p(X)$ is not Baire. First, observe that for each $f \in C(X)$, there exists $n \in \mathbb{N}$ such that the restriction $f|X_n$ is bounded. Indeed, otherwise, if there exists $f \in C(X)$, which is unbounded on each X_n, then f is unbounded on each open neighbourhood of the point x. This implies that f is discontinuous at x, a contradiction. Now set

$$B_{m,n} := \{f \in C(X) : \sup_{y \in X_m} |f(y)|| \leq n\}$$

for each $n, m \in \mathbb{N}$. The sets $B_{m,n}$ are absolutely convex and closed. Clearly, $C(X) = \bigcup_n B_{m,n}$. Assume that $C_p(X)$ is a Baire space. Then there exist $m, n \in \mathbb{N}$ such that $B_{m,n}$ is a neighbourhood of zero in $C_p(X)$. Hence, there exists a compact set $K \subset X$ and $\epsilon > 0$ such that

$$\{f \in C(X) : \sup_{y \in K} |f(y)| < \epsilon\} \subset B_{m,n}. \tag{2.7}$$

We claim that $X_m \subset K$ (what will provide a contradiction). If there exists $y \in X_m \setminus K$, then there exists $f \in C(X)$ such that $f(z) = 0$ for each $z \in K$ and $f(y) > n$. This yields a contradiction with (2.7). □

The following problem is motivated by the previous results.

Problem 2.5.13 Characterize a strongly realcompact space X in terms of topological properties of $C_c(X)$ (or $C_p(X)$).

2.6 Pseudocompact Spaces, Warner Boundedness, and Spaces $C_c(X)$

In this section, we characterize pseudocompact spaces. For example, we show that X is pseudocompact, X is a Warner-bounded set, or $C_p(X)$ is a (df)-space if and only if for each sequence $(\mu_n)_n$ in the dual $C_c(X)'$ of $C_c(X)$, there exists a sequence $(t_n)_n \subset (0, 1]$ such that $(t_n\mu_n)_n$ is weakly bounded, is strongly bounded, or is equicontinuous, respectively.

This result will be used to provide an example of a (df)-space $C_c(X)$, which is not a (DF)-space; this solves an open question; see [348]. Parts of this section will be used to present concrete examples of quasi-Suslin spaces that are not K-analytic.

Buchwalter [105] called a topological space X *Warner bounded* if for every sequence $(U_n)_n$ of non-empty open subsets of X, there exists a compact set $K \subset X$ such that $U_n \cap K \neq \emptyset$ for infinitely many $n \in \mathbb{N}$. In fact, Buchwalter required from $(U_n)_n$ to be disjoint, but due to regularity of X, this condition could be omitted. What already Warner has observed in [636] is that any Warner-bounded space is pseudocompact.

First, we provide the following useful analytic characterization of Warner boundedness; see [349]. In this section, X means always a completely regular Hausdorff topological space.

Theorem 2.6.1 (Kąkol–Saxon–Todd) *$C_c(X)$ does not contain a dense subspace of $\mathbb{R}^{\mathbb{N}}$ if and only if X is Warner bounded.*

For the proof, we need the following:

Lemma 2.6.2 *(a) A lcs E contains a dense subspace G of $\mathbb{R}^{\mathbb{N}}$ if and only if there exists a sequence $(w_n)_n$ of non-zero elements in E such that every continuous seminorm in E vanishes at w_n for almost all $n \in \mathbb{N}$.*

(b) If a lcs E contains a dense subspace G of $\mathbb{R}^{\mathbb{N}}$, then the strong dual $(E', \beta(E', E))$ contains the space φ.

Proof (a) Assume that E contains a sequence as mentioned. For $(w_n)_n$, there exists a biorthogonal sequence $(v_n, u_n)_n$ in $F \times F'$, where $(v_n)_n$ is a subsequence of $(w_n)_n$, F is a linear span of v_n, $n \in \mathbb{N}$, and F' is spanned by u_n, $n \in \mathbb{N}$. Note that F is isomorphic to the linear span G of the unit vectors of $\mathbb{R}^{\mathbb{N}}$. Now assume that $\mathbb{R}^{\mathbb{N}}$ contains a dense subspace G. Let $(p_n)_n$ be a fundamental sequence of continuous seminorms on G. Then each $G_n := \bigcap_{i=1}^{n} p_i^{-1}(0)$ is an infinite-dimensional subspace of G. Therefore, we can find $w_n \in G_n \setminus \{0\}$ for each $n \in \mathbb{N}$ as required.

(b) It is well known that the strong dual of $\mathbb{R}^{\mathbb{N}}$ is the space φ; see, for example, [491]. On the other hand, it is also well known that every bounded set in the completion of $\hat{G} = \mathbb{R}^{\mathbb{N}}$ is contained in the $\hat{G}$-completion of a bounded set in G; see also [491, Observation 8.3.23]. This applies to complete the proof of (b). □

Now we are ready to prove Theorem 2.6.1.

Proof Assume that $C_c(X)$ contains a dense subspace of $\mathbb{R}^{\mathbb{N}}$. Then there exists a sequence $(f_n)_n$ of non-zero elements of $C(X)$ which vanishes for almost $n \in \mathbb{N}$ on any compact subset of X. Then for each $n \in \mathbb{N}$, there exists an open non-zero set U_n in X such that $f_n(y) \neq 0$ for all $y \in U_n$. Therefore, every compact set K in X misses U_n for almost all $n \in \mathbb{N}$. This shows that X is not a Warner-bounded set.

To prove the converse, assume that $(U_n)_n$ is a sequence of non-empty open sets in X such that almost all of them miss each compact set in X. Then we can select a sequence $(f_n)_n$ of non-zero continuous functions on X such that each $f_n(X \setminus U_n) = \{0\}$. Hence, each continuous seminorm on $C_c(X)$ vanishes on almost all elements of the sequence $(f_n)_n$. Now we apply Lemma 2.6.2. □

Next Theorem 2.6.3 (see [348]) looks much more interesting if we have already in mind the previous Theorem 2.6.1.

Theorem 2.6.3 (Kąkol–Saxon–Todd) *The following assertions are equivalent for X.*

(i) *X is pseudocompact.*
(ii) *For each sequence $(\mu_n)_n$ in the weak* dual F of $C_c(X)$, there exists a sequence $(t_n)_n \subset (0, 1]$ such that $(t_n\mu_n)_n$ is bounded in F.*
(iii) *The weak* dual F of $C_c(X)$ is docile, i.e. every infinite-dimensional subspace of F contains an infinite-dimensional-bounded set.*
(iv) *$C_c(X)$ does not contain a copy of $\mathbb{R}^{\mathbb{N}}$.*

Proof (i) $\Rightarrow$ (ii): Take a sequence $(\mu_n)_n$ in F. Set

$$t_n := (\|\mu_n\| + 1)^{-1}$$

for each $n \in \mathbb{N}$. Since X is pseudocompact, $(t_n\mu_n(f))_n$ is bounded for each $f \in C(X)$.

(ii) $\Rightarrow$ (iii) holds for any lcs.

(iii) $\Rightarrow$ (i) is clear.

(iii) $\Rightarrow$ (iv): Assume G is a subspace of $C_c(X)$ isomorphic to $\mathbb{R}^{\mathbb{N}}$. It is clear that the *weak** dual of $\mathbb{R}^{\mathbb{N}}$, and hence $(G', \sigma(G', G))$ is not docile. Using the Hahn–Banach theorem, we extend elements of G' to the whole space $C(X)$ which generate a non-docile subspace of the *weak** dual of $C_c(X)$. This contradicts (iii).

(iv) $\Rightarrow$ (i): Suppose that X is not pseudocompact. Then there exists a sequence $(U_n)_n$ of disjoint open non-empty sets in X, which is locally finite. Choose $x_n \in U_n$ and $f_n \in C(X)$ such that $f_n(x_n) = 1$ and $f_n(X \setminus U_n) = \{0\}$ for each $n \in \mathbb{N}$. Note that because the sequence $(U_n)_n$ is locally finite, the series $\sum_n a_n f_n$ converges in

$C_c(X)$ for any scalar sequence $(a_n)_n$ in $\mathbb{R}^{\mathbb{N}}$. Define a map $T : \mathbb{R}^{\mathbb{N}} \to C_c(X)$ by

$$T((a_n)_n) := \sum_n a_n f_n.$$

The map T is injective. Indeed, since for each evaluation map δ_{x_m} we have

$$\delta_{x_m} \in (C_c'(X), \sigma(C_c(X)', C_c(X))),$$

and

$$\delta_{x_m}(\sum_n a_n f_n) = a_m,$$

then indeed T is an injective open map. We show that T is continuous. Each partial sum map T_N defined by

$$T_N((a_n)_n) := \sum_{n=1}^{N} a_n f_n$$

is continuous, and T is the pointwise limit of the sequence $(T_N)_N$. Since the space $\mathbb{R}^{\mathbb{N}}$ is Baire, the sequence $(T_N)_N$ is equicontinuous, and T is continuous by the classical Banach–Steinhaus theorem. Consequently, the space $C_c(X)$ contains a copy of $\mathbb{R}^{\mathbb{N}}$. □

Many barrelled spaces $C_c(X)$ are not Baire-like, for example, $C_c(\mathbb{Q})$ is not Baire-like (although barrelled by Proposition 2.4.16), since $\mathbb{Q}$ is not locally compact and we apply Proposition 2.4.20. By Theorem 2.4.3, the space $C_c(\mathbb{Q})$ contains φ. It is interesting that owing to Theorem 2.6.3 each space $C_c(X)$, which contains the non-docile space φ, contains also the docile space $\mathbb{R}^{\mathbb{N}}$. So we deduce that every barrelled non-Baire-like space $C_c(X)$ contains both spaces φ and $\mathbb{R}^{\mathbb{N}}$. Note that $C_c(\mathbb{R})$ contains $\mathbb{R}^{\mathbb{N}}$ but not φ.

Theorem 2.6.1 and Lemma 2.6.2 apply to simplify essentially Warner's fundamental [636, Theorem 11].

Theorem 2.6.4 *The following assertions are equivalent:*

(i) X *is Warner bounded.*
(ii) $[X, 1] := \{f \in C(X) : \sup_{x\in X} |f(x)| \leq 1\}$ *absorbs bounded sets in* $C_c(X)$.
(iii) $C_c(X)$ *has a fundamental sequence of bounded sets.*
(iv) Every Cauchy sequence in $C_c(X)$ *is a Cauchy sequence in the space* $C_b(X)$ *of the continuous bounded functions on* X *with the uniform Banach topology.*
(v) X *is pseudocompact and* $C_c(X)$ *is sequentially complete.*
(vi) X *is pseudocompact and* $C_c(X)$ *is locally complete.*

Proof (i) $\Rightarrow$ (ii): Let $B \subset C_c(X)$ be a bounded set. Hence, B is uniformly bounded on each compact set in X. Assume B is not absorbed by $[X, 1]$. Then for each $n \in \mathbb{N}$, there exist $f_n \in B$, and $x_n \in X$ such that $|f_n(x_n)| > n$. Consequently, for each $n \in \mathbb{N}$, there exists an open neighbourhood U_n of x_n such that $|f_n(x)| > n$ for all $x \in U_n$. Since X is Warner bounded, there exists a compact set $K \subset X$ such that $K \cap U_n \neq \emptyset$ for almost all $n \in \mathbb{N}$, a contradiction since $\{f_n : n \in \mathbb{N}\} \subset B$ must be uniformly bounded on K.

(ii) $\Rightarrow$ (iii): The sets $n[X, 1]$ for $n \in \mathbb{N}$ form a fundamental sequence of bounded sets.

(iii) $\Rightarrow$ (i): By the assumption, the strong dual of $C_c(X)$ is metrizable. Now we apply Lemma 2.6.2 (b) and Theorem 2.6.1 to complete the proof.

(i) $\Rightarrow$ (iv): By (i) $\Rightarrow$ (ii) the space X is pseudocompact. To prove the second part, let $(f_n)_n$ be a null sequence in $C_c(X)$.

We show that $(f_n)_n$ is a null sequence in the uniform Banach topology of $C_b(X)$. Assume this fails. Then there exist a subsequence $(h_n)_n$ of $(f_n)_n$, $\epsilon > 0$, and a sequence $(U_n)_n$ of non-zero open sets in X such that $|h_n(x)| > \epsilon$ for all $x \in U_n$. Since $(h_n)_n$ converges to zero uniformly on compacts sets of X, compact sets of X miss U_n for almost all $n \in \mathbb{N}$. This contradicts (i).

(iv) $\Rightarrow$ (v): By (iv), X is pseudocompact and $C_b(X)$ is complete.

(v) $\Rightarrow$ (vi): Any sequentially complete lcs is locally complete.

(vi) $\Rightarrow$ (ii): In a locally complete lcs, barrels absorb bounded sets; this fact is elementary; see, for example, [491, Corollary 5.1.10]. Therefore, the set $[X, 1]$ (which is clearly closed, absolutely convex, and absorbing) absorbs bounded sets. □

It is worth noticing here another interesting characterization of a completely regular Hausdorff space X to be pseudocompact; see [639]: X is pseudocompact if and only if every uniformly bounded pointwise compact set H in the space $C_b(X)$ is weakly compact.

In order to prove the main result of this section, we shall need two additional lemmas.

Lemma 2.6.5 *If every countable subset of X is relatively compact, $[X, 1]$ is bornivorous in $C_c(X)$.*

Proof If a bounded set $A \subset C_c(X)$ is not absorbed by $[X, 1]$, there exist two sequences $(x_n)_n$ in X and $(f_n)_n$ in A, respectively, such that $(f_n(x_n))_n$ is not bounded. Since the closure of the set $\{x_n : n \in \mathbb{N}\}$ is compact in X, $[K, 1]$ is a neighbourhood of zero in $C_c(X)$ which does not absorbs A, a contradiction. □

We need also the following useful fact; see [491] and [537].

Proposition 2.6.6 *A lcs E is locally complete if and only if E is ℓ^1-complete, i.e. for each $(t_n)_n \in \ell^1$ and each bounded sequence $(x_n)_n$ in E, the series $\sum_n t_n x_n$ converges in E.*

Proof Let ξ be the original topology of E. Assume that E is locally complete, and fix arbitrary $(t_n)_n$ in ℓ^1 and a bounded sequence $(x_n)_n$ in E. Then the closed

absolutely convex hull B of $(x_n)_n$ is a Banach disc. Since for $r > s$ we have

$$\sum_{n\le r} t_n x_n - \sum_{n<s} t_n x_n \in (|t_s| + \cdots + |t_r|)B,$$

the series $\sum_n t_n x_n$ converges in the Banach space $(E_B, \|.\|_B)$. Therefore, $\sum_n t_n x_n$ converges in E since $\xi|E_B \le \xi_B$, where ξ_B is a locally convex topology generated by the norm $\|.\|_B$.

To prove the converse, assume that E is ℓ^1-complete, and let B be a closed bounded absolutely convex subset of E.

Let $(y_n)_n$ be a Cauchy sequence in E_B. There exists an increasing sequence $(n_k)_k$ in $\mathbb{N}$ such that $\|y_r - y_s\| \le 2^{-k}$ if $r, s, k \in \mathbb{N}$ and $r, s \ge n_k$. Set

$$t_k := 2^{-k}, \quad x_k := 2^k(y_{n_{k+1}} - y_{n_k}).$$

Then $(x_n)_n \subset B$ is bounded and the series $\sum_n t_n x_n$ converges in E to some $x \in E$. For $r > k$, we have

$$\sum_{n\le r} t_n x_n - \sum_{n<k} t_n x_n \in (2^{-k} + \cdots + 2^{-r}) \subset 2^{-k+1}B.$$

Then $x - \sum_{n<k} t_n x_n$ is the limit of the sequence

$$(\sum_{n\le r} t_n x_n - \sum_{n<k} t_n x_n)_r$$

whose members belong to the set $2^{-k+1}B$. As the last set is closed, then

$$x - \sum_{n<k} t_n x_n \in 2^{-k+1}B$$

for each $k \in \mathbb{N}$. This implies that

$$y_{n_k} = y_{n_1} + \sum_{n<k} t_n x_n \to y_{n_1} + x$$

in the space E_B. Hence $(y_n)_n$ converges in the space E_B. We proved that E is locally complete. □

A lcs E is called *dual locally complete* [613] if $(E', \sigma(E', E))$ is locally complete. The space E is called *dual ℓ^1-complete* if $(E', \sigma(E', E))$ is ℓ^1-complete. Note also a dual version of Proposition 2.6.6; see [537].

Proposition 2.6.7 *For a lcs E, the following assertions are equivalent:*

(i) E is dual locally complete

(ii) *E is dual ℓ^1-complete.*
(iii) *If A is an absorbing and absolutely convex subset of E, f is a linear functional over E, and $f|A$ is continuous on A, then f is continuous on E.*

Proof Because of Proposition 2.6.6, it is enough to prove (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii): Let A be an absolutely convex and absorbing closed subset of E, and let f be a linear functional on E that is continuous on A. We may assume that there exists $x \in E$ such that $f(x) = 1$. For each $n \in \mathbb{N}$, set $A_n := nA$, and

$$C_n := nA \cap \{y \in E : f(y) = 0\}.$$

Then for each $n \in \mathbb{N}$, there exists a continuous linear functional f_n with $f_n(x) = 1$ and $|f_n(y)| < 2^{-n}$ for each $y \in C_n$. Clearly, the sequence $(2^n(f_{n+1} - f_n))_n$ is $\sigma(E', E)$-bounded, so by assumption, we deduce that

$$f = f_1 + \sum_n 2^{-n}[2^n(f_{n+1} - f_n)]$$

is continuous.

(iii) $\Rightarrow$ (ii): Choose an arbitrary sequence $(t_n)_n$ in ℓ^1 and a sequence $(f_n)_n$ in E' which is $\sigma(E', E)$-bounded. Set

$$f(x) = \sum_n t_n f_n(x)$$

for each $x \in E$. We need to show that f is continuous. Because of (iii), it is enough to show that $f|A$ is continuous at zero, where $A := \{f_n : n \in \mathbb{N}\}^\circ$. Fix $\epsilon > 0$ and choose $k \in \mathbb{N}$ with $\sum_{n>k} |t_n| < 2^{-1}\epsilon$. Select a neighbourhood of zero U in E such that

$$|(\sum_{n \le k} t_n f_n)(x)| < 2^{-1}\epsilon$$

for all $x \in U$. Finally, let $x \in A \cap U$. Then

$$|f(x)| \le |(\sum_{n \le k} t_n f_n)(x)| + \sum_{n>k} |t_n| < \epsilon.$$

The proof is completed. □

A subset K of X is the *support* of some continuous linear functional λ on $C_c(X)$ (denoted by supp λ) if it is the intersection of closed sets $M \subset X$ having the property that $\lambda(f) = 0$ if $f \in C(X)$ and $f(M) = \{0\}$. We need the following lemma; see [348].

Lemma 2.6.8 *With $E := C_c(X)$, if one of $(E', \beta(E', E))$, $(E', \sigma(E', E))$, or $(E', n(E', E))$ is both docile and locally complete, then $[X, 1]$ is bornivorous, where*

$(E', n(E', E))$ denotes the dual of E' endowed with the topology generated by the norm $\|\lambda\| := \sup\{|\lambda(f)| : f \in [X.1]\}$.

Proof Assume $[X, 1]$ is not bornivorous. Then there exist a sequence $(x_n)_n$ in X, a bounded sequence $(f_n)_n$ in $C_c(X)$ such that $f_n(x_n) > n$ for each $n \in \mathbb{N}$. Since the sequence $(f_n)_n$ is bounded, the set $\{x_n : n \in \mathbb{N}\}$ is infinite. We may assume that the sequence $(x_n)_n$ is injective, i.e. $x_n \neq x_m$ for $n \neq m$. Hence, the evaluation functionals δ_{x_n} form a Hamel basis for their linear span $D \subset E'$. Clearly, $\gamma = \sum_n b_n \delta_{x_n}$ for $\gamma \in D$, where all but finitely numbers b_n are zero. Then $\operatorname{supp} \gamma := \{x_n : b_n \neq 0\}$ is finite, so there exists

$$\min \gamma := \min\{|b_n| : x_n \in \operatorname{supp} \gamma\}.$$

By the assumption about docility, we deduce that there exists a linearly independent sequence $(\gamma_n)_n$ in D which is bounded in each dual mentioned above. By the local completeness, the series $\sum_n a_n \gamma_n$ converges for each absolutely summable sequence $(a_n)_n$ (by Proposition 2.6.6), and $\sum_n a_n \gamma_n$ converges pointwisely for each $f \in [X, 1]$.

Choose inductively an absolutely summable sequence $(a_n)_n$ of non-zero elements such that $a_n \to 0$ and

$$\sum_{k>n} |a_k \gamma_k| < |a_n| \min \gamma_n$$

for each $n \in \mathbb{N}$. Hence, $\gamma := \sum_n a_n \gamma_n \in E'$. This implies that there exists a compact set $K \subset X$ such that γ is bounded on the neighbourhood of zero $[K, 1]$ in $C_c(X)$. On the other hand, linear independence of $(\gamma_n)_n$ yields that $\bigcup_n \operatorname{supp} \gamma_n$ is an infinite subset of $(x_n)_n$. Then, since $(f_n)_n$ is uniformly bounded, we deduce that $\bigcup_n \operatorname{supp} \gamma_n$ is not a subset of K. Let $p \in \mathbb{N}$ such that $\operatorname{supp} \gamma_p$ is not a subset of K. Fix $y \in \operatorname{supp} \gamma_p \setminus K$. The set

$$A := K \cup \{x \neq y : x \in \operatorname{supp} \gamma_p\}$$

is closed and misses y, and there exists $g \in [X, 1]$ with $g(y) = 1$ and $g(A) = \{0\}$. Since $cg \in [K, 1]$ for each scalar c, we deduce that linear functional γ vanishes on g.

On the other hand, we note that

$$|\gamma(g)| = |\sum_n a_n \gamma_n(g)| = |\sum_{n \leq p} a_n \gamma_n(g)|$$

$$\geq |a_p \gamma_p(g)| - \sum_{k>p} |a_k \gamma_k(g)| \geq |a_p| \min \gamma_p - \sum_{k>p} |a_k| \|\gamma_k\| > 0.$$

This provides a contradiction. □

Buchwalter and Schmets [106, Theorem 4.1] proved

Proposition 2.6.9 *$C_c(X)$ is ℓ^∞-barrelled if and only if the weak dual of $C_c(X)$ is locally complete if and only if a countable union of support sets in X is relatively compact provided it is topologically bounded.*

A lcs E is called a (DF)-*space* if E admits a fundamental sequence of bounded sets and is $\aleph_0$-*quasibarrelled*, i.e. any countable union of equicontinuous sets which is $\beta(E', E)$ bounded is equicontinuous. A lcs E is $\aleph_0$-*barrelled*, if any countable union of equicontinuous sets in E' which is $\sigma(E', E)$-bounded is equicontinuous.

By Warner, [636] (see also [491, Theorem 10.1.22]), we have the following:

Proposition 2.6.10 *$C_c(X)$ is a (DF)-space if and only each countable union of compact sets in X is relatively compact if and only if $C_c(X)$ has a fundamental sequence of bounded sets and is $\aleph_0$-barrelled.*

Recall that a lcs E is called a (df)-*space* if E admits a fundamental sequence of bounded sets and E is c_0-*quasibarrelled*, i.e. every null sequence in $(E', \beta(E', E))$ is equicontinuous. The class of (DF)-spaces and (df)-spaces will be discussed in the next chapters. Clearly, every (DF)-space is a (df)-space. It was a long time unknown if every (df)-space is (DF). In [348] Kąkol, Saxon and Todd provided an example of a (df)-space $C_c(X)$, which is not a (DF)-space; see Example 2.6.13.

To present Example 2.6.13, first we recall some additional concepts and facts.

A lcs E satisfies the *countable neighbourhood property* (cnp), if for every sequence $(U_n)_n$ of neighbourhoods of zero in E, there exists a sequence $(a_n)_n$ of positive scalars such that $\bigcap_n a_n U_n$ is a neighbourhood of zero. By Warner [636], the space $C_c(X)$ is a (DF)-space if and only if $C_c(X)$ satisfies the (cnp). It is known that all support sets satisfy the (cnp); see [322] and [437]. We need also a couple of definitions concerning the *Borel measures*; see [251]. A Borel measure μ on a space X is a σ-additive real-valued finite function on all Borel subsets of X. It is called *regular* if its negative and positive parts satisfy

$$\mu(A) = \sup\{\mu(K) : K \subset A \text{ is compact }\}.$$

It is known that for each regular Borel measure, there exists a smallest closed set M in X such that μ vanishes on each Borel set which misses M, and M is called the support of μ (again we use the notation $\operatorname{supp}\mu$), possibly non-compact. Nevertheless, one has $\operatorname{supp}\lambda = \operatorname{supp}\mu$ if either μ, a non-negative regular Borel measure, or $\lambda \in C_c(X)'$ is given, and the other is appropriately chosen.

We are prepared to formulate and prove the main result of this section; see [348]. Note that the equivalence (ii) $\Leftrightarrow$ (xi) has been already proved by Mazon [434]. Also (iv) $\Leftrightarrow$ (viii) was known to McCoy and Todd [437]. By $C_d(X)$, we denote the space $C(X)$ endowed with the topology of the uniform convergence on supports sets of X.

Theorem 2.6.11 (Kąkol–Saxon–Todd) *The following assertions are equivalent for $E := C_c(X)$:*

(i) E is a (df)-space.
(ii) $(E', \beta(E', E))$ is a Fréchet space.

(iii) $(E', \beta(E', E))$ *is a Banach space and equals* $(E', n(E', E))$.
(iv) $(E', n(E', E))$ *is a Banach space.*
(v) $(E', \beta(E', E))$ *is docile and locally complete.*
(vi) $(E', \sigma(E', E))$ *is docile and locally complete.*
(vii) X *is pseudocompact and* $(E', \sigma(E', E))$ *is locally complete.*
(viii) Every countable union of support sets in X *is relatively compact.*
(ix) For each sequence $(\mu_n)_n$ *in* E'*, there exists a sequence* $(t_n)_n \subset (0, 1]$ *such that* $\{t_n\mu_n)_n$ *is equicontinuous.*
(x) $C_d(X)$ *satisfies the (cnp).*
(xi) Each regular Borel measure on X *has compact support.*

Proof (i) $\Rightarrow$ (ii): It is clear that $(E', \beta(E', E))$ is locally complete (since the strong dual of a c_0-quasibarrelled space is locally complete, [491, Proposition 8.2.23(b)]). But then $(E', \beta(E', E))$ is a Fréchet space since E has a fundamental sequence of bounded sets.

Conditions from (ii) to (vi) are equivalent since any of them implies that $[X, 1]$ is bornivorous owing Lemma 2.6.8. Then $(E', \beta(E', E)) = (E', n(E', E))$, so applying the Banach–Steinhaus theorem, we deduce that $\sigma(E', E)$-bounded sets are $\beta(E', E)$-bounded.

(vi) $\Leftrightarrow$ (vii): This follows from Theorem 2.6.3.

(vii) $\Rightarrow$ (viii): Since X is pseudocompact, every subset of X is topologically bounded. Since by assumption $(E', \sigma(E', E))$ is locally complete, then applying Proposition 2.6.9, we have that (viii) holds.

(viii) $\Rightarrow$ (i): Clearly, every singleton subset of X is a support set, so by Lemma 2.6.5, we note that $\{[X, n] : n \in \mathbb{N}\}$ is a fundamental sequence of bounded sets in in E. Hence, by Proposition 2.6.9, the space E is ℓ^∞-barrelled, hence c_0-quasibarrelled.

(vii) $\Rightarrow$ (ix): Choose $(\mu_n)_n$ in E', and set $t_n := (\|\mu_n\| + 1)^{-1}$ for each $n \in \mathbb{N}$. The sequence $(t_n\mu_n)_n$ is uniformly bounded on the barrel $[X, 1]$, so it is $\sigma(E', E)$-bounded. Now applying again Proposition 2.6.9, we have that $(t_n\mu_n)_n$ is equicontinuous.

(ix) $\Rightarrow$ (x): Let $(U_n)_n$ be a sequence of neighbourhoods of zero in $C_d(X)$. Let $(K_n)_n$ be a sequence of support sets in X, and let $(a_n)_n$ be a sequence of positive scalars such that

$$\{f \in C(X) : \sup_{x \in K_n} |f(x)| \leq a_n\} := [K_n, a_n] \subset U_n$$

for each $n \in \mathbb{N}$. By μ_n, we denote a positive continuous linear functional on $C_c(X)$ whose support is the set K_n. By the assumption, there exists a sequence $(t_n)_n$ of positive scalars such that $(t_n\mu_n)_n$ is equicontinuous. This implies that the series $\sum_n 2^{-n} t_n\mu_n$ has a limit $\mu \in E'$ in the topology $\sigma(E', E)$ with support K. If $f(K) = \{0\}$ for $f \in C(X)$, the positive part f^+ of f and the negative one f^- vanish on K. Therefore,

$$\mu(f) = \mu(f^+) = \mu(f^-) = 0.$$

Consequently, we have

$$\mu_n(f) = \mu_n(f^+) = \mu_n(f^-) = 0$$

for each $n \in \mathbb{N}$, so $K_n \subset K$. This means that

$$[K, 1] \subset [K_n, 1] \subset a_n^{-1} U_n;$$

hence, $\bigcap_n a_n^{-1} U_n$ is a neighbourhood of zero in $C_d(X)$. We proved (x).

(x) $\Rightarrow$ (viii): Let $(K_n)_n$ be a sequence of support sets in X. By the assumption from (x), it follows that there exists a sequence $(a_n)_n$ of positive scalars such that

$$U := \bigcap_n a_n [K_n, 1]$$

is a neighbourhood of zero in $C_d(X)$. Then there exists a compact set K in X such that $[K, \epsilon] \subset U$ for some $\epsilon > 0$. Note that $K_n \subset K$ for each $n \in \mathbb{N}$. Indeed, assume that $x \in X \setminus K_n$. Then there exists $f \in [X, 1]$ such that $f(x) = 0$ and $f(K) = \{0\}$. Hence,

$$(a_n + 1) f \in [K, \epsilon] \subset U \subset [K_n, a_n]$$

for each $n \in \mathbb{N}$. This implies $x \notin K_n$. Since K is compact, the closure of the set $\bigcup_n K_n$ is compact in X. This proves (viii).

To complete the proof, we need to show the equivalence (viii) $\Leftrightarrow$ (xi).

(viii) $\Rightarrow$ (xi): Note that for a regular Borel measure μ on X, there exist sequences $(K_n^+)_n$, $(K_n^-)_n$ of compact subsets of $\operatorname{supp} \mu^+$, and $\operatorname{supp} \mu^-$, respectively, such that

$$\mu^+(X) = \sup_n \mu^+(K_n^+), \quad \mu^-(X) = \sup_n \mu^-(K_n^-).$$

By (viii), we deduce that the closure K^+ of $\bigcup_n K_n^+$ and K^- of $\bigcup_n K_n^-$ are compact sets. This shows that μ^+ and μ^- vanish on the Borel sets, which miss K^+ and K^-, respectively. Hence, $\mu = \mu^+ - \mu^-$ vanishes on the Borel sets, which miss $K^+ \cup K^-$. Consequently, the set $\operatorname{supp} \mu$ is a closed set of the compact set $K^+ \cup K^-$. This proves that $\operatorname{supp} \mu$ is compact.

(xi) $\Rightarrow$ (viii): Assume that $(K_n)_n$ is a sequence of non-empty support sets in X. For each $n \in \mathbb{N}$, choose a non-negative regular Borel measure μ_n on X such that $\mu_n(X) = \{1\}$ and $\operatorname{supp} \mu_n = K_n$. Set

$$\mu(A) := \sum_n 2^{-n} \mu_n(A).$$

It is easy to see that μ is a non-negative regular Borel measure on X whose support contains each K_n. Applying (xi), we deduce that the closure of $\bigcup_n K_n$ is compact. □

Using Theorem 2.6.11, we provide a version to Proposition 2.6.10 about (df)-spaces $C_c(X)$.

Corollary 2.6.12 *The following assertions are equivalent:*

(i) $C_c(X)$ *is a* (df)*-space.*
(ii) $C_c(X)$ *has a fundamental sequence of bounded sets and is* ℓ^∞*-barrelled.*

Proof (i) $\Rightarrow$ (ii): By Theorem 2.6.11, we know that the *weak** dual of $C_c(X)$ is locally complete. Then $C_c(X)$ is ℓ^∞-barrelled.
(ii) $\Rightarrow$ (i): In general, ℓ^∞-barrelled $\Rightarrow$ c_0-barrelled $\Rightarrow$ c_0-quasibarrelled. □

Recall that a topological space X satisfies the *countable chain condition*, if every pairwise disjoint collection of non-empty open subsets of X is countable. We are ready to show the following:

Example 2.6.13 (Kąkol–Saxon–Todd) There exists a (df)-space $C_c(X)$, which is not a (DF)-space.

Proof Recall that a point p in a topological space X is a P-point, if every G_δ set containing p is a neighbourhood of p; see [275, Problems 4L]. By van Mill [630], there exists a non-P-point x_0 in the closed subspace $\beta\mathbb{N} \setminus \mathbb{N}$ such that for a sequence $(K_n)_n$ of closed subsets in $\beta\mathbb{N} \setminus \mathbb{N}$ not containing x_0 (and satisfying the countable chain condition), the closure of $\bigcup_n K_n$ does not contain x_0. Support sets satisfy the countable chain condition. Then a countable union of support sets of $X := (\beta\mathbb{N} \setminus \mathbb{N}) \setminus \{x_0\}$ has the same closure in $\beta\mathbb{N}$ as in X, hence is compact. Applying (viii) of Theorem 2.6.11, we note that $C_c(X)$ is a (df)-space. We prove that $C_c(X)$ is not a (DF)-space. Indeed, since x_0 is not a P-point of $\beta\mathbb{N} \setminus \mathbb{N}$, there exists a sequence $(U_n)_n$ of open neighbourhoods of x_0 in $\beta\mathbb{N} \setminus \mathbb{N}$ such that $\bigcap_n U_n$ is not a neighbourhood of x_0. This implies that x_0 belongs to the closure of the set $X \setminus \bigcap_n U_n = \bigcup_n (X \setminus U_n)$. Since each set

$$K_n := X \setminus U_n = (\beta\mathbb{N} \setminus \mathbb{N}) \setminus U_n$$

is compact whose union $\bigcup_n K_n$ has non-compact closure. By Proposition 2.6.10, the space $C_c(X)$ is not a (DF)-space. □

This concrete space $C_c(X)$ provides an example of a ℓ^∞-barrelled space $C_c(X)$ (by Corollary 2.6.12), which is not $\aleph_0$-quasibarrelled. This space $C_c(X)$ is not a $\aleph_0$-barrelled space. This answers an old question posed by Buchwalter–Schmets [106].

2.7 Sequential Conditions for Locally Convex Baire-Type Spaces

In this section, we collect a few results (mostly from [164] and [352]) about sequential conditions for lcs with some Baire-type assumptions. It is well known

by a classical theorem of Mahowald [426] that a lcs E is barrelled if and only if every closed linear map from E into a Banach space is continuous.

It turns out that for many concrete spaces E (e.g. locally complete bornological spaces E), every linear map with a sequentially closed graph from E into a Banach space is continuous. In [563], Snipes characterized lcs E (called *C-sequential*) (see also [475]), for which every sequentially continuous linear map of E into a Banach space is continuous. De Wilde [155] showed

Proposition 2.7.1 (De Wilde) *If E is the inductive limit of a family of metrizable Baire lcs spaces and F is a webbed space, then every sequentially closed linear map of E into F is continuous.*

In particular, Proposition 2.7.1 applies if F is an (LF)-space. Recall again for a convenience that a lcs E is called *bornological*, if every absolutely convex bornivorous set in E is a neighbourhood of zero if and only if every linear map from E into a Banach space which transforms bounded sets into bounded set is continuous if and only if E is the inductive limit of a family of normed spaces.

In this section, we study lcs E having the following property (s): Every sequentially closed linear map of E into an arbitrary Fréchet space F is continuous.

We shall say that a lcs E is *s-barrelled* [164] if E satisfies the above property (s). Clearly, for metrizable lcs, the barrelledness and s-barrelledness conditions coincide. Inductive limits and Hausdorff quotients of s-barrelled spaces are s-barrelled; in particular inductive limits of metrizable barrelled spaces; hence, ultrabornological (or locally complete bornological) spaces are s-barrelled.

In [352], we introduced and characterized a class of lcs, called *CS-barrelled*, for which the condition (s) holds with *Fréchet space F* replaced by *Banach space F*. We shall say that a lcs E is CS-barrelled if every sequentially closed linear map of E into a Banach space is continuous. Clearly, the class of s-barrelled spaces is included in the class of CS-barrelled spaces. It seems to be unknown if both classes coincide.

A sequence $(U_n)_n$ of absolutely convex subsets of a lcs E such that

$$U_{n+1} + U_{n+1} \subset U_n,$$

$n \in \mathbb{N}$ is said to be *CS-closed* if whenever $n \in \mathbb{N}$, $x_m \in U_m$, $m > n$, and $x = \sum_{m=n+1}^{\infty} x_m$ exists in E, then $x \in U_n$. The sequence $(U_n)_n$ will be called *absorbing* if every U_n is absorbing in E. Set

$$Q = \{(t_n)\ t_n \geq 0, \sum_n t_n = 1\}.$$

By a *convex series* of elements of a subset $A \subset E$, we mean a series of the form $\sum_n t_n x_n$, where $x_n \in A$ and $(t_n) \in Q$, $n \in \mathbb{N}$.

Following Jameson [320, 321], a subset $A \subset E$ is called *CS-closed*, if it contains the sum of every convergent convex series of its elements. Clearly, CS-closed sets are convex. Every open (or sequentially closed) convex set is CS-closed. Every

convex G_δ subset of a Fréchet space is CS-closed; see [252]. Moreover, if $U \subset E$ is CS-closed, the sets $2^{-n}U$ for $n \in \mathbb{N}$ form a CS-closed sequence.

Let $\mathcal{U} = (U_n)_n$ be an absorbing CS-closed sequence in a lcs E. Set $N(\mathcal{U}) = \bigcap_n U_n$, and let

$$Q_{\mathcal{U}} : E \to E/N(\mathcal{U})$$

be the quotient map. Set $E_{\mathcal{U}} = E/N(\mathcal{U})$. Then the sets $Q_{\mathcal{U}}(U_n)$ form a basis of neighbourhoods of zero of a metrizable locally convex topology on $E/N(\mathcal{U})$. Let $\tilde{E}_{\mathcal{U}}$ be the completion of $E_{\mathcal{U}}$. Clearly, $Q_{\mathcal{U}} : E \to E_{\mathcal{U}}$ is continuous if and only if $Q_{\mathcal{U}} : E \to \tilde{E}_{\mathcal{U}}$ is continuous. We start with the following characterization of a lcs to be s-barrelled due to Dierolf and Kąkol; see [164].

Proposition 2.7.2 *(i) For every absorbing CS-closed sequence $\mathcal{U} = (U_n)_n$ in E, the map $Q_{\mathcal{U}}$ as a map from E into $\tilde{E}_{\mathcal{U}}$ is sequentially closed and $U_{n+1} \subset Q_{\mathcal{U}}^{-1}(Q_{\mathcal{U}}(U_{n+1})) \subset U_n$, $n \in \mathbb{N}$. Hence, each U_n absorbs Banach discs of E.*

(ii) For every Fréchet space F and every basis $(V_n)_n$ of absolutely convex closed neighbourhoods of zero in F such that $V_{n+1} + V_{n+1} \subset V_n$, and every sequentially closed linear map $T : E \to F$, the sequence $(T^{-1}(V_n))_n$ is a CS-closed sequence in E.

(iii) E is s-barrelled if and only if every absorbing CS-closed sequence $(U_n)_n$ of E is topological, i.e. each U_n is a neighbourhood of zero.

Proof (i) Let $(x_n)_n$ be a null sequence in E such that $Q_{\mathcal{U}}(x_n) \to y$ in $\tilde{E}_{\mathcal{U}}$. Then there exists a sequence $(n(k))_k$ in $\mathbb{N}$ such that $Q_{\mathcal{U}}(x_{n(k+1)} - x_{n(k+2)}) \in Q_{\mathcal{U}}(U_{k+1})$ for $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, set

$$y_k = x_{n(k+1)} - x_{n(k+2)}.$$

Then $y_k \in U_{k+1} + N(\mathcal{U}) \subset U_k$ for each $k \in \mathbb{N}$. Since $(U_n)_n$ is CS-closed, and

$$x_{n(k+1)} = \sum_{m=k}^{\infty} y_m = \sum_{m=1}^{\infty} y_{m+k-1},$$

then $x_{n(k+1)} \in U_{k-1}$ for each $k \in \mathbb{N}$. This implies that $Q_{\mathcal{U}}(x_{n(k)}) \to 0$, so $y = 0$.

If B is a Banach disc in E, the map

$$Q_{\mathcal{U}} \circ J_B : [B] \to \tilde{E}_{\mathcal{U}}$$

is closed, where $[B]$ denotes the Banach space E_B. Then $Q_{\mathcal{U}} \circ J_B$ is continuous by Proposition 2.7.1. Hence, $Q_{\mathcal{U}}(B)$ is bounded in $E_{\mathcal{U}}$. This shows that B is absorbed by

$$Q_{\mathcal{U}}^{-1}(Q_{\mathcal{U}}(U_{n+1})) \subset U_n.$$

(ii) Fix $m \in \mathbb{N}$, and for all $n \geqslant m+1$, choose $y_n \in T^{-1}(V_n)$ such that $y := \sum_{n=m+1}^{\infty} y_n$ exists in E. Then $(\sum_{n=m+1}^{k} T(y_n))_{k\in\mathbb{N}}$ is a Cauchy sequence in F, hence converges to some $z \in V_m$. Since T is sequentially closed, $z = T(y)$. Consequently, $y \in T^{-1}(V_m)$.

(iii) Is a direct consequence of the previous cases (i) and (ii). □

By Proposition 2.7.2, we have the following:

Corollary 2.7.3 *A lcs E is CS-barrelled if and only if every absorbing CS-closed set in E is a neighbourhood of zero.*

It turns out that the class of s-barrelled spaces is quite far from the class of barrelled spaces. We show that the space $\mathbb{R}^{\mathbb{R}}$ contains a dense Baire and bornological subspace that is not s-barrelled; see Example 2.7.5.

We will need the following technical facts. Let $G = (G, \tau)$ be a lcs, and let L be a sequentially closed linear subspace, and let B be a Banach disc in G. Let $E = L + [B]$, and fix a linear subspace $M \subset [B]$ such that $L \cap M = \{0\}$ and $L + M = E$. Let $q : E \to E/L$ be the quotient map, and set $r = (q|M)^{-1}$. By $j : M \to [B]$ and

$$p : [B] \to [B]/([B] \cap L)$$

we denote the inclusion map and the quotient map, respectively. Then $[B] \cap L$ is closed in the Banach space $[B]$, so $[B]/([B] \cap L)$ is a Banach space.

We need the following:

Lemma 2.7.4 *The linear map*

$$f := p \circ j \circ r \circ q : (E, \tau|E) \to [B]/([B] \cap L)$$

is sequentially closed. If L is not closed in $(E, \tau|E)$, then f is not closed.

Proof First, we show that the map f is sequentially closed. Indeed, let $(x_n)_n$ be a null sequence in $(E, \tau|E)$, and let $y \in [B]/([B] \cap L)$ be such that $f(x_n) \to y$ in $[B]/([B] \cap L)$. Then there exist a sequence $(z_n)_n$ in $[B]$ and an element $z \in [B]$ such that $p(z_n) = f(x_n)$, $p(z) = y$, and $z_n \to z$ in the space $[B]$. This implies that $z_n \to z$ in the space $(E, \tau|E)$.

Moreover, for each $n \in \mathbb{N}$, one has

$$z_n - r(q(x_n)) \in [B] \cap L.$$

On the other hand, $r(q(x_n)) - x_n \in L$ for each $n \in \mathbb{N}$. Consequently,

$$z_n - x_n = (z_n - r(q(x_n))) + (r(q(x_n)) - x_n) \in L.$$

Since $z_n - x_n \to z$ in $(E, \tau|E)$, and L is sequentially closed, $z \in L$. Also $z \in [B]$, so $y = p(z) = 0$.

Note also that the map f is not closed. It is enough to show that the kernel $\ker f$ of f is not closed in E; this is a simple consequence of the fact that $L = \ker f$. □

Now we are ready to present the promised example. Recall that a subspace F of a lcs E is *locally dense* if for every $x \in E$, there exists a Banach disc B in E such that E_B admits a sequence in F, which converges to x in E_B (or, equivalently, if for every $x \in E$, there exists a sequence $(x_n)_n$ in F, and an increasing scalar sequence $(a_n)_n$ with $a_n \to \infty$ such that $a_n(x_n - x) \to 0$). Note that every lcs that contains a locally dense bornological subspace is bornological; see [491, Proposition 6.2.7].

For the following example, we refer to [164].

Example 2.7.5 The space $\mathbb{R}^{\mathbb{R}}$ is s-barrelled and contains a dense subspace, which is Baire bornological but is not CS-barrelled.

Proof Set $G := \mathbb{R}^{\mathbb{R}}$, and let

$$L := \{(x_t)_{t\in\mathbb{R}} \in \mathbb{R}^{\mathbb{R}} : |\{t \in \mathbb{R} : x_t \neq 0\}| \leq \aleph_0\}$$

be endowed with the product topology. Then L is a sequentially closed dense Baire subspace of G. Let C be the absolutely convex hull of the set $\{\varphi_{[a,b)} : a, b \in \mathbb{Q},\ a < b\}$, where $[a, b)$ is the half-closed interval, and $\varphi_{[a,b)}$ is the characteristic function. Then the closure A of C is a Banach disc in G. Next by B, we denote the closure of C in the Banach space $[A]$. Then B is a Banach disc in $[A]$, hence in G. Let $E = L + [B]$ with the topology of G. Now Lemma 2.7.4 applies to deduce that there exists a sequentially closed discontinuous linear map of E into a Banach space $[B]/([B] \cap L)$. This shows that E is not CS-barrelled. The space E contains L as a dense subspace. Hence, E is a Baire space. We prove that E is a bornological space. Indeed, by [165], the space $L + [C]$ is a bornological subspace of G. Since $[C]$ is locally dense in $[B]$ with respect to the norm topology of $[A]$, $[C]$ is locally dense in $[B]$. Therefore, $L + [C]$ is locally dense in $L + [B]$. As every lcs that contains a locally dense bornological subspace is bornological, E is bornological. Finally, since G is the inductive limit of Banach spaces (that means G is ultrabornological), G is s-barrelled by Proposition 2.7.1. □

Note that the space E from Example 2.7.5 is bornological, Baire, which is not the inductive limit of metrizable barrelled spaces. In fact, otherwise E would be CS-barrelled what is impossible.

It is well known that a lcs E is barrelled if and only if every pointwise bounded family of continuous linear maps of E into a Fréchet space is equicontinuous; see [491, Proposition 4.1.3]. One can ask if a similar result for the s-barrelledness holds if the assumption *continuous linear maps* is replaced by *sequentially continuous linear maps*. The answer is negative. In fact, if E is a bornological and barrelled space, then every pointwise bounded family of sequentially continuous linear maps of E into a Fréchet space is equicontinuous. On the other hand, the space E from Example 2.7.5 is bornological and barrelled and is not s-barrelled.

The next example uses the following simple fact, and the proof is obvious.

Proposition 2.7.6 *If F is a s-barrelled locally dense subspace of a lcs E, then E is s-barrelled.*

Example 2.7.7 (Dierolf–Kąkol) There exists a bornological s-barrelled lcs, which is not the inductive limit of metrizable barrelled spaces.

Proof Let F be a (LB)-space of Banach spaces $(F_n)_n$ which admits a bounded set not bounded in any step F_n (i.e. the space F is not *regular*; for such examples, we refer the reader, e.g. to [491, Example 6.4.5]). Let E be the completion of F, and choose $x \in E \setminus F$ for which there exists a sequence $(x_n)_n$ in F which locally converges to x. Since F is s-barrelled, and F is locally dense in $G := F + [x]$, Proposition 2.7.6 applies to deduce that G is s-barrelled. As F is bornological, and F is locally dense in G, then G is bornological. Hence, the quotient space $G/[x]$ is s-barrelled and bornological. Let $q : G \to G/[x]$ be the quotient map. Then the restriction map

$$q|F : F \to G/[x]$$

is continuous and bijective. We prove that $G/[x]$ is not the inductive limit of metrizable barrelled spaces. Indeed, assume that H is a metrizable and barrelled space, and let $j : H \hookrightarrow G/[x]$ be a continuous inclusion. The map

$$(q|F)^{-1} \circ j : H \to F$$

has the closed graph from a metrizable Baire-like space into an (LB)-space, so we apply Theorem 2.4.15 to deduce that this map is continuous. Since $q|F$ is not open, $G/[x]$ is not the inductive limit of metrizable and barrelled spaces. □

We refer also the reader to [164, Example 2.7] to find the proof of the following interesting fact.

Example 2.7.8 There exists a s-barrelled space that is not bornological.

Note that s-barrelled spaces enjoy some properties typical for bornological spaces. For example, we prove the Mackey–Ulam theorem for s-barrelled spaces. Let $\{E_t : t \in T\}$ be a family of lcs. Let E be the product $\prod_t E_t$. For $x \in E$, set $E_x = \prod_t [x_t]$. If $x_t \neq 0$ for every $t \in T$, then E_x is called a *simple subspace* of E; see [156]. Clearly, E_x is isomorphic to $\mathbb{R}^T$. De Wilde [156, Theorem 1] observed that a linear map of E into a lcs F is continuous if and only if its restriction to all factor subspaces and to simple subspaces of E are continuous.

Recall (see [491, Definition 6.2.21]) that a set T satisfies the *Mackey–Ulam condition* if no Ulam measure can be defined on it, i.e. no $(0, 1)$-valued measure $\mathfrak{m}$ on the set 2^T of all the subsets of T with $\mathfrak{m}(T) = 1$ and $\mathfrak{m}((t)) = 0$ for all $t \in T$ can be defined. It is known (see [491, Theorem 6.2.23]) that T satisfies the Mackey–Ulam condition if and only if $\mathbb{R}^T$ is bornological.

We shall need the following fact due to López-Pellicer, [417]; see also [418] and [491].

Lemma 2.7.9 *For a completely regular Hausdorff space X, the space $C_c(X)$ is bornological if and only if every weakly sequentially continuous functional on $C_c(X)$ is continuous.*

Proof Assume that every linear weakly sequentially continuous functional on $C_c(X)$ is continuous. We prove that $C_c(X)$ is bornological. It is enough to show that X is realcompact and apply Proposition 2.4.17.

Fix $x \in \upsilon X$. Define $\xi : C_c(X) \to \mathbb{R}$ by $\xi(f) := f^{\upsilon}(x)$ for each $f \in C_c(X)$. Note that ξ transforms bounded sets into bounded sets. Indeed, let $(f_n)_n$ be a bounded sequence in $C_c(X)$ such that $|\xi(f_n)| \geq n$ for each $n \in \mathbb{N}$. Since there exists $y \in X$ such that $f_n^{\upsilon}(x) = f_n(y)$ for each $n \in \mathbb{N}$, we have $f_n(y) \geq n$ for each $n \in \mathbb{N}$ yielding a contradiction. By the assumption, the map ξ is continuous. Hence, there exists $\epsilon > 0$ and a compact set $K \subset X$ such that

$$|\xi(f)| \leq \epsilon \sup_{z \in K} |f(z)|$$

for each $f \in C(X)$. Then $x \in K$; otherwise, there exists $g \in C(X)$ such that $g|K$ vanishes, and $g^{\upsilon}(x) = \xi(g) = 1$, a contradiction. Hence, $\upsilon X = X$, so X is realcompact. □

Now we prove the following theorem from [164].

Theorem 2.7.10 *Let $\{E_t : t \in T\}$ be a family of s-barrelled spaces.*

(i) The product $\prod_t E_t$ is s-barrelled if and only if $\mathbb{R}^T$ is s-barrelled.
(ii) $\mathbb{R}^T$ is s-barrelled iff T satisfies the Mackey–Ulam condition.

Proof (i) Since $\mathbb{R}^T$ is a minimal space, i.e. $\mathbb{R}^T$ does not admit a weaker Hausdorff vector topology, $\mathbb{R}^T$ is complemented in the product space E; see [491, Corollary 2.6.4]. Hence, since E is s-barrelled, the space $\mathbb{R}^T$ is s-barrelled. The converse follows from De Wilde's [156, Theorem 1] mentioned above.

(ii) : Assume T satisfies the Mackey–Ulam condition. Then $\mathbb{R}^T$ is bornological by the mentioned [491, Theorem 6.2.23]. The space $\mathbb{R}^T$ is complete. Hence, $\mathbb{R}^T$ is the inductive limit of Banach spaces. The space $\mathbb{R}^T$ is s-barrelled by Proposition 2.7.1. If T does not satisfy the Mackey–Ulam condition, $\mathbb{R}^T$ is not bornological. Endow the set T with the discrete topology. Then $\mathbb{R}^T = C_p(T) = C_c(T)$. By Lemma 2.7.9, there exists on $\mathbb{R}^T$ a discontinuous sequentially continuous linear functional ξ. Hence, $\mathbb{R}^T$ is not s-barrelled. □

Lemma 2.7.9 provides also the following:

Corollary 2.7.11 *$C_c(X)$ is s-barrelled if and only if X is realcompact.*

It is known that the completion of a bornological space need not be bornological; see [491, Notes and Remarks, p.196]. If Z is a realcompact space whose associated k_R-space Z_{k_r} is not realcompact (in [87] Blasco provided such examples), then the completion of $C_c(Z)$ is $C_c(Z_{k_r})$. By Corollary 2.7.11, the space $C_c(Z)$ is s-barrelled but its completion is not s-barrelled. Since $C_c(X)$ is barrelled if and only if X is a μ-space (Proposition 2.4.16), and there exist μ-spaces that are not realcompact, Corollary 2.7.11 provides also barrelled spaces $C_c(X)$ that are not s-barrelled.

Chapter 3
K-Analytic and Quasi-Suslin Spaces

Abstract This chapter deals with the K-analyticity of a topological space E and the concept of a *resolution* generated on E (i.e. a family of sets $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that $E = \bigcup_\alpha K_\alpha$ and $K_\alpha \subset K_\beta$ if $\alpha \leq \beta$). Compact resolutions (i.e. resolutions $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ whose members are compact sets) naturally appear in many situations in topology and functional analysis. Any K-analytic space admits a compact resolution, and for many topological spaces X, the existence of such a resolution is enough for X to be K-analytic. Many of the ideas in the book are related to the concept of compact resolution. We gather some results, mostly due to Valdivia, about lcs's admitting resolutions consisting of Banach discs and their relations with the closed graph theorems. We present Hurewicz and Alexandrov's theorems as well as Calbrix–Hurewicz's theorem, which yields that a regular analytic space X is not σ-compact if and only if X contains a closed subset homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

3.1 Elementary Facts

In this part, we recall fundamental properties of K-analytic spaces and provide applications for topological vector spaces.

A subset $A \subset X$ of a topological space X is said to be *relatively countably compact* (*countably compact*) in X if every sequence $(x_k)_k$ in A has a cluster point in X (in A), i.e.

$$\bigcap_n \overline{\{x_k : k \geq n\}} \neq \emptyset, \quad (A \cap [\bigcap_n \overline{\{x_k : k \geq n\}}] \neq \emptyset.)$$

The set A is called *relatively sequentially compact* (*sequentially compact*) if for every sequence in A, there exists a subsequence that converges to a point of X (of A). Clearly, $A \subset X$ is countably compact if and only if each countably open cover of A admits a finite subcover.

The set $\mathbb{N}$ of natural numbers will be endowed with the discrete topology, and the product $\mathbb{N}^{\mathbb{N}}$ is equipped with the product topology. Let S be the set of all finite

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sequences in $\mathbb{N}$. For $s \in S$, set

$$\Sigma s := \{\sigma \in \mathbb{N}^{\mathbb{N}} : s < \sigma\}.$$

Note that the family $\{\Sigma s : s \in S\}$ forms a base of open sets in $\mathbb{N}^{\mathbb{N}}$, and $\Sigma t \subset \Sigma s$ if and only if $s < t$. The space $\mathbb{N}^{\mathbb{N}}$ is known to be homeomorphic with the space of all irrational numbers on the real line. This follows from the following result due to Mazurkiewicz; see [258, Remark, p.409].

Proposition 3.1.1 *Let X be a separable completely metrizable space such that no non-void open set in X is compact and the closed-open sets in X form an open base for X. Then X is homeomorphic with $\mathbb{N}^{\mathbb{N}}$.*

Proof Fix a complete metric on X. First, note that for each $\epsilon > 0$, each non-void open set in X is the infinite countable disjoint union of closed-open sets of a diameter less than ϵ. This fact applies to construct disjoint open covers $\{X_s : s \in S_n\}$ by sets of diameter less than n^{-1} with $X_s \subset X_t$ if $t < s$, where S_n means the set of all elements of S of length $n \in \mathbb{N}$. Define a map $T : \mathbb{N}^{\mathbb{N}} \to X$ that for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ assigns the only point of $\bigcap\{X_s : s < \alpha\}$. Then $T(\Sigma s) = X_s$, which yields that T is a desired homeomorphism. □

A topological space E is called *K-analytic* if E is the image under an upper semicontinuous (usco) compact-valued map T (called a K-analytic map) defined in $\mathbb{N}^{\mathbb{N}}$, i.e. for every $\alpha \in \mathbb{N}^{\mathbb{N}}$, and every open neighbourhood V of $T(\alpha)$ in E, there exists an open neighbourhood U of α such that $T(U) \subset V$.

If T is defined on a non-empty subset $\Omega \subset \mathbb{N}^{\mathbb{N}}$, the space E is called a *Lindelöf Σ-space*. Lindelöf Σ-spaces are often called *K-countably determined* or *countably determined*. K-analytic spaces are also known as K-Suslin spaces; see [433] and [611].

By $\mathcal{K}(E)$ (resp. $\mathcal{P}(E)$), we denote the family of all compact subsets (resp. all parts) of a space E.

We start with the following elementary fact; see [521, 579].

Proposition 3.1.2 *A topological space E is K-analytic if and only if there exists a map T from $\mathbb{N}^{\mathbb{N}}$ into $\mathcal{K}(E)$ satisfying two conditions:*

(i) $\bigcup\{T_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} = E$, *where* $T_\alpha = T(\alpha)$.

(ii) If $(\alpha_n)_n$ is a sequence in $\mathbb{N}^{\mathbb{N}}$ converging to α and $x_n \in T_{\alpha_n}$, for each $n \in \mathbb{N}$, the sequence $(x_n)_n$ has an adherent point $x \in T_\alpha$.

Proof Assume conditions (i) and (ii) hold and E is not K-analytic. Hence, there exist $\alpha \in \mathbb{N}^{\mathbb{N}}$, an open neighbourhood U of $T_\alpha := T(\alpha)$, a sequence $(\alpha_n)_n$ in $\mathbb{N}^{\mathbb{N}}$ converging to α in $\mathbb{N}^{\mathbb{N}}$, and a corresponding sequence $(U_{\alpha_n})_n$ of open neighbourhoods of α_n such that $T(U_{\alpha_n}) \not\subset U$ for each $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, there exists

$$x_n \in T(U_{\alpha_n}) \setminus U.$$

By (ii), the sequence $(x_n)_n$ has an adherent point that belongs to T_α, and this yields a contradiction.

Now assume E is K-analytic. Clearly, (i) holds. Assume that (ii) fails. Then there exist a sequence $(\alpha_n)_n$ in $\mathbb{N}^\mathbb{N}$ which converges to $\alpha \in \mathbb{N}^\mathbb{N}$, and a sequence $x_n \in T(\alpha_n)$ for each $n \in \mathbb{N}$ which does not have an adherent point in $T(\alpha)$. For $y \in T(\alpha)$, there exists an open neighbourhood $V(y)$ of y, and $n(y) \in \mathbb{N}$ such that $x_n \notin V(y)$ for all $n \geq n(y)$. Since $T(\alpha)$ is a compact subset of E, we can find a finite number of points $y_1, y_2, \ldots, y_m$ in $T(\alpha)$ such that

$$T(\alpha) \subset V := V(y_1) \cup V(y_2) \cup \ldots . \cup V(y_m).$$

The space E is K-analytic, so by the definition, there exists a neighbourhood U of α in $\mathbb{N}^\mathbb{N}$ such that $T(U) \subset V$. Then there exists $p \in \mathbb{N}$, where

$$p \geq \max\{n(y_1), n(y_2), \ldots, n(y_m)\}$$

such that $\alpha_p \in U$. This implies

$$x_p \in T(\alpha_p) \subset T(U) \subset V,$$

a contradiction. □

This yields the following [579, Theorem 2.1]:

Corollary 3.1.3 *Let ξ and β be two topologies on X such that ξ and β coincide on all ξ-compact sets. If (X, ξ) is K-analytic, (X, β) is K-analytic.*

The following concept was introduced by Valdivia; see [611]. A topological space E is called *quasi-Suslin* if there exists a map T (called a quasi-Suslin map) from $\mathbb{N}^\mathbb{N}$ into $\mathcal{P}(E)$) satisfying the following:

(a) $\bigcup \{T_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} = E$, where $T_\alpha = T(\alpha)$.

(b) If $(\alpha_n)_n$ is a sequence in $\mathbb{N}^\mathbb{N}$ converging to α and $x_n \in T_{\alpha_n}$, for each $n \in \mathbb{N}$, then the sequence $(x_n)_n$ has an adherent point $x \in T_\alpha$.

Below we collect a few fundamental properties of K-analytic spaces which will be used in the subsequent parts of the book; see [522] or [611] for the proofs.

Proposition 3.1.4 *Closed subspaces of K-analytic spaces are K-analytic. Countable products of K-analytic spaces are K-analytic. Continuous images of K-analytic spaces are K-analytic. If a topological space E is covered by a sequence of K-analytic subspaces, E is K-analytic. Countable intersections of K-analytic subspaces of E are K-analytic.*

A topological space E is called *Lindelöf* if one of the following equivalent conditions holds:

(i) Every open cover of E has a countable subcover.
(ii) For every family $\mathcal{A}$ of closed subsets of E with empty intersection, there is a countable subfamily of $\mathcal{A}$ with empty intersection.
(iii) Every family $\mathcal{A}$ of non-empty closed subsets of E that it is closed under countable intersections verifies $\bigcap \mathcal{A} \neq \emptyset$.

Note the following well-known important facts; see [522] or [117, 579].

Proposition 3.1.5 *The image under a compact-valued (usco) map of a Lindelöf space is a Lindelöf space. Hence, every K-analytic space is Lindelöf.*

Proof Let F be a Lindelöf space and let T be a compact-valued (usco) map from F onto a topological space E. Let $(V_i)_{i\in I}$ be an open cover of E. For $x \in F$, there is a finite subset $I(x)$ of I such that $T(x) \subset \bigcup_{i\in I(x)} V_i$. Since T is (usco), there exists an open neighbourhood $U(x)$ of x such that

$$T(U(x)) \subset \bigcup_{i\in I(x)} V_i.$$

As F is Lindelöf, there exist a sequence $(x_n)_n$ in F, and a sequence $(U(x_n))_n$ of open neighbourhoods of x_n, respectively, such that $F \subset \bigcup_n U(x_n)$. Hence,

$$E = T(F) \subset \bigcup_n T(U(x_n)) \subset \bigcup_n \bigcup_{i\in I(x_n)} V_i,$$

and E is Lindelöf. □

Following De Wilde, a web $\mathcal{W} = \left\{C_{a_1a_2\ldots a_k} : a_i \in \mathbb{N}, 1 \leq i \leq k, k \in \mathbb{N}\right\}$ in a set E is called *ordered* (see [611]) if for integers $k, a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ such that $a_j \leq b_j$, $1 \leq j \leq k$, we have $C_{a_1a_2\ldots a_k} \subset C_{b_1b_2\ldots b_k}$.

Theorem 3.1.6 (Cascales) *The following conditions are equivalent for a topological space E:*

(i) E is a quasi-Suslin space.
(ii) There is a quasi-Suslin ordered map $A : \mathbb{N}^{\mathbb{N}} \longrightarrow \mathcal{P}(E)$, i.e. given $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha \leq \beta$, then $A_\alpha \subset A_\beta$.
(iii) There is an ordered web $\mathcal{W} = \left\{C_{a_1a_2\ldots a_k} : a_i \in \mathbb{N}, 1 \leq i \leq k, k \in \mathbb{N}\right\}$ such that for every $\alpha = (a_k) \in \mathbb{N}^{\mathbb{N}}$, if $x_k \in C_{a_1a_2\ldots a_k}$, $k \in \mathbb{N}$, the sequence $(x_k)_k$ has an adherent point in E which belongs to $\bigcap_k C_{a_1a_2\ldots a_k}$.

Proof (i) $\Rightarrow$ (ii): Let $T : \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(E)$ be a quasi-Suslin map. For $\alpha \in \mathbb{N}^{\mathbb{N}}$, set

$$A_\alpha := \bigcup\{T_\beta : \beta \leq \alpha, \beta \in \mathbb{N}^{\mathbb{N}}\}.$$

Clearly, $A_\alpha \subset A_\beta$ if $\alpha \leq \beta$. In order to show that $A : \alpha \to A_\alpha$ is quasi-Suslin, we need only to check condition (b) for the map $A : \alpha \mapsto A_\alpha$.

Let (α_j), with $\alpha_j := (a_n^j)$, be a sequence in $\mathbb{N}^{\mathbb{N}}$ which converges to $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$, and assume that $x_j \in A_{\alpha_j}$ for each $j \in \mathbb{N}$. By the definition of the sets A_α, it follows that for each $j \in \mathbb{N}$, there exists $\beta_j \leq \alpha_j$ such that $x_j \in T_{\beta_j}$. Since $\mathbb{N}$ is a discrete space, there exists an increasing sequence $(j_k)_k$ of elements of $\mathbb{N}$ such that $a_n^{j_k} = a_n$ for all $k, n \in \mathbb{N}$. The subsequence $(\beta_{j_k})_k$ of $(\beta_j)_j$ is relatively compact in $\mathbb{N}^{\mathbb{N}}$, so $(\beta_{j_k})_k$ contains a subsequence $(\beta_{j_{k_l}})_l$, which converges to $\beta \in \mathbb{N}^{\mathbb{N}}$ and $\beta \leq \alpha$. As the map T is a quasi-Suslin map, condition (b) is satisfied, so the subsequence $(x_{j_{k_l}})_l$ of $(x_j)_j$ has an adherent point $x \in T_\beta$. This implies also that the sequence $(x_j)_j$ has an adherent point belonging to A_α. The condition (b) for the map $A : \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(E)$ is proved.

(ii) $\Rightarrow$ (iii): For $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, set

$$C_{n_1 n_2 \ldots . n_k} := \bigcup \{A_\alpha : \alpha = (a_k) \in \mathbb{N}^{\mathbb{N}}, a_j = n_j, 1 \leq j \leq k\}.$$

Then $\{C_{n_1 n_2 \ldots . n_k}\}_k$ is an ordered web in E. We claim that $\{C_{n_1 n_2 \ldots . n_k}\}_k$ satisfies the conditions claimed in (iii). Indeed, take $\alpha = (a_k) \in \mathbb{N}^{\mathbb{N}}$, and select $x_k \in C_{n_1 n_2 \ldots . n_k}$ for each $k \in \mathbb{N}$. By the definition of $C_{n_1 n_2 \ldots . n_k}$, one gets $\alpha_k = (a_n^k) \in \mathbb{N}^{\mathbb{N}}$ such that $a_j^k = a_j$, and $x_k \in A_{\alpha_k}$ for $j = 1, 2, \ldots k$. Then the sequence $(\alpha_k)_k$ converges to α. Consequently, the sequence $(x_k)_k$ has an adherent point in E which belongs to

$$A_\alpha \subset \bigcap_k C_{n_1 n_2 \ldots . n_k}.$$

(iii) $\Rightarrow$ (i): Define the map $T : \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(E)$ by $T(\alpha) := T_\alpha$, where

$$T_\alpha := \bigcap_k C_{n_1 n_2 \ldots . n_k}$$

for $\alpha = (a_k) \in \mathbb{N}^{\mathbb{N}}$. This is a quasi-Suslin map. Indeed, let (α_k) be a sequence in $\mathbb{N}^{\mathbb{N}}$ with $\alpha_k = (a_n^k)$, which converges to $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$, and let $x_j \in T_{\alpha_j}$ for all $j \in \mathbb{N}$. There exists a subsequence $(x_{j_n})_n$ of $(x_j)_j$ with an adherent point x in E which belongs to T_α by the assumption (iii). Since x is also an adherent point of $(x_j)_j$, the conclusion holds. □

A similar argument applies to prove the following result of Talagrand [579]; see also [521] and [117].

Theorem 3.1.7 (Talagrand–Rogers) *The following conditions are equivalent for a topological space E:*

(i) E is K-analytic.
(ii) There is a K-analytic ordered map $A : \mathbb{N}^{\mathbb{N}} \longrightarrow \mathcal{K}(E)$.
(iii) There is an ordered web $\mathcal{W} = \left\{C_{a_1 a_2 \ldots a_k} : a_i \in \mathbb{N}, 1 \leq i \leq k, k \in \mathbb{N}\right\}$ in E such that for every $\alpha = (a_k) \in \mathbb{N}^{\mathbb{N}}$, the set $\bigcap_k C_{a_1 a_2 \ldots a_k}$ is compact, and if $x_k \in C_{a_1 a_2 \ldots a_k}$, for $k \in \mathbb{N}$, the sequence $(x_k)_k$ has an adherent point in E belonging to $\bigcap_k C_{a_1 a_2 \ldots a_k}$.

We complete this section with some additional information about Lindelöf Σ-spaces. We recall another approach to define the class of Lindelöf Σ-spaces, due to Nagami [460]. A topological space X is a Lindelöf Σ-space if X is a Σ-space (in sense of [460]) and Lindelöf; see [29] and [386]. Proposition 3.1.8 gathers several known characterizations of Lindelöf Σ-spaces; see also [28, 386, 522], and [477]. We refer the reader to recent papers [594] and [130] about Lindelöf Σ-spaces.

A family ω in a topological space X is a *network* of X if for every $x \in X$, and every open neighbourhood U of x, there exists $V \in \mathcal{N}$ with $x \in V \subset U$. Recall that a family $\mathcal{N}$ of subsets of X is called a *network with respect to a cover* $\mathcal{C}$ of X if for every set $C \in \mathcal{C}$, and every open neighbourhood U of C, there exists $N \in \mathcal{N}$ such that $C \subset N \subset U$.

Proposition 3.1.8 *Let X be a completely regular Hausdorff space. The following assertions are equivalent:*

(i) *X is a Lindelöf Σ-space.*
(ii) *There exist a compact cover $\mathcal{C}$ and a countable network $\mathcal{N}$ with respect to $\mathcal{C}$.*
(iii) *There exist a metrizable and separable space M and a compact-valued (usco) map T such that $T(M) = X$.*
(iv) *There exist a compact cover $\mathcal{C}$ and a countable network $\mathcal{G}$ with respect to $\mathcal{C}$ consisting of closed sets.*
(v) *There is a compact-valued (usco) map from a subset of $\mathbb{N}^{\mathbb{N}}$.*
(vi) *There exists a countable family $\mathfrak{F}$ of compact subsets of βX such that $\mathfrak{F}$ separates X, i.e. for any $x \in X$ and $y \in \beta X \setminus X$, there exists $F \in \mathfrak{F}$ such that $x \in F$ and $y \notin F$.*
(vii) *There exist a metrizable separable space M, a space L, and maps $g : L \to M$ and $f : L \to X$ such that g is perfect, and f is surjective and continuous.*
(viii) *There are a metrizable and separable space M, a compact space K, a closed subspace F of $M \times K$, and a continuous map $f : F \to X$ such that $f(F) = X$.*

Proof We prove only the equivalences (ii) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi). For the remaining part of the proof, we refer to [386] and [522].

(ii) $\Rightarrow$ (iii): Let $\mathcal{N} = \{N_i\}_i$. For each $C \in \mathcal{C}$, determine a sequence $(c_n)_n$ such that $C = \bigcap_n N_{c_n}$. Let $M := \{(c_n)_n : C \in \mathcal{C}\}$. Then the map defined in M by

$$T((c_n)_n) := \bigcap_n N_{c_n}$$

is compact-valued, (usco), and $T(M) = X$.

(iii) $\Rightarrow$ (ii): Let $T : M \to X$ be a compact-valued (usc) map onto X. Let $\mathcal{B}$ be a countable basis for M. Then $\{T(B) : B \in \mathcal{B}\}$ is a countable network with respect to the compact cover $\{T(t) : t \in M\}$ of X.

(v) $\Rightarrow$ (ii): Let $\varphi : \Sigma \to X$ be a (usco) compact-valued map from a subset Σ of $\mathbb{N}^{\mathbb{N}}$ onto X. Let $\mathcal{B}$ be a countable base in Σ which is closed under the finite unions. Set $\mathcal{N} := \{\varphi(B) : B \in \mathcal{B}\}$. Then $\mathcal{C} := \{\varphi(x) : x \in \Sigma\}$ is a compact cover of X, and $\mathcal{N}$ is a countable network with respect to $\mathcal{C}$.

(ii) $\Rightarrow$ (iv): Let $\mathcal{C}$ and $\mathcal{N}$ be as in (ii). Set

$$\mathcal{G} := \{\overline{N} : N \in \mathcal{N}\}.$$

It is enough to show that $\mathcal{G}$ is a network with respect to $\mathcal{C}$. Fix $C \in \mathcal{C}$, and let U be an open neighbourhood of C. By the regularity of X for $x \in C$, there exists an open neighbourhood V_x of x such that $\overline{V_x} \subset U$. Since C is compact, there exists finite $D \subset C$ such that

$$C \subset V := \bigcup_{x \in D} V_x.$$

Take $N \in \mathcal{N}$ with $C \subset N \subset V$. Then

$$C \subset \overline{N} \subset \overline{V} \subset U.$$

This provides (iv).

(iv) $\Rightarrow$ (vi): Let $\mathcal{C}$ and $\mathcal{G}$ be as in (iv). Then

$$\mathfrak{F} := \{\overline{N} : N \in \mathcal{G}\}$$

is countable, where the closure is taken in βX. Let $x \in X$ and $y \in \beta X \setminus X$. There exists $C \in \mathcal{C}$ with $x \in C$. Let W be an open neighbourhood of C in βX with $y \notin \overline{W}$. Then there exists $N \in \mathcal{G}$ such that

$$C \subset N \subset W \cap X.$$

Consequently,

$$x \in C \subset \overline{N} \subset \overline{W \cap X} \subset \overline{W} \subset \beta X \setminus \{y\}.$$

Hence, $F := \overline{N} \in \mathfrak{F}$ and $x \in F$ and $y \notin F$.

(vi) $\Rightarrow$ (v): Let $\mathfrak{F} = (K_n)_n$ be as in (vi). Set

$$\Sigma := \{\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}} : \bigcap_n K_{a_n} \subset X\}.$$

For each $\alpha \in \Sigma$, the set $\varphi(\alpha) := \bigcap_n K_{a_n}$ is compact and $\varphi : \Sigma \to X$ is a compact-valued map. Let $x \in X$. Choose $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$ such that

Since $\mathfrak{F}$ separates X from $\beta X \setminus X$, so $F := \bigcap_n K_{a_n} \subset X$. Hence,

$$\alpha := (a_n) \in \Sigma,\ x \in F = \varphi(\alpha),\ \varphi(\Sigma) = X.$$

We need to show that φ is a (usco) map. Fix arbitrary $\alpha \in \Sigma$, and let U be an open neighbourhood of the compact set $\varphi(\alpha)$. Let V be an open set in βX with

$V \cap X = U$. There exists $m \in \mathbb{N}$ such that $\bigcap_{n \le m} K_{a_n} \subset V$. Then

$$G := \{\beta = (b_n) \in \Sigma : b_n = a_n, n \le m\}$$

is open in Σ and $\varphi(G) \subset U$. □

The first consequence of the previous result (part (v)) is the following:

Corollary 3.1.9 *Any σ-compact space X, i.e. X covered by a sequence of compact sets, or, more generally, every K-analytic space, is a Lindelöf Σ-space.*

Corollary 3.1.10 *Any topological space X with a countable network is a Lindelöf Σ-space.*

Proof A countable network is a network with respect to the compact cover $\{\{x\} : x \in X\}$. Now apply Proposition 3.1.8 (ii). □

We note also another interesting consequence of Theorem 3.1.8; see [594, Theorem 2.14] or [32, Proposition IV.6.15].

Corollary 3.1.11 *If for a Lindelöf Σ-space X every compact subset of X is finite, X is countable.*

Proof By Theorem 3.1.8 (ii), there exist a compact cover $\mathcal{C}$ and a countable network $\mathcal{N}$ with respect to $\mathcal{C}$ which is closed under the finite intersections. Set $\mathcal{N}_0 := \{N \in \mathcal{N} : N \text{ is finite}\}$. Assume X is uncountable. Since $M := \bigcup_{N \in \mathcal{N}_0} N$ is countable, there exists $x \in X \setminus M$. Fix $C \in \mathcal{C}$ with $x \in C$. There exists a countable decreasing sequence $(P_n)_n$ in $\mathcal{N}$ such that, for each open neighbourhood U of C, there exists $n \in \mathbb{N}$ such that $C \subset P_n \subset U$. As $x \notin M$, every P_n is infinite. Since C is finite (by the assumption), we can find an injective sequence $(x_n)_n$ with

$$x_n \in P_n \setminus C$$

for each $n \in \mathbb{N}$. Set $D := \{x_n : n \in \mathbb{N}\}$. Since $D \setminus U$ is finite for each open neighbourhood of C, $C \cup D$ is an infinite compact subset of X, a contradiction. We proved that $X = M$. □

We note also the following:

Proposition 3.1.12 *Let X be a K-analytic space endowed with a web $\mathcal{N} := \{C_{n_1,n_2,\ldots,n_k} : n_k \in \mathbb{N},\ k \in \mathbb{N}\}$ as in Theorem 3.1.7. Set $U_\alpha := \bigcap_k C_{n_1,n_2,\ldots,n_k}$ for $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$. Then $\mathcal{C} := \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact cover of X, and $\mathcal{N}$ is a countable network with respect to $\mathcal{C}$.*

Proof Let U be an open set in X, and assume that $U_\alpha \subset U$. Since $U_\alpha \subset C_{n_1,n_2,\ldots,n_k}$ for each $k \in \mathbb{N}$, it is enough to show that there exists $k \in \mathbb{N}$ such that $C_{n_1,n_2,\ldots,n_k} \subset U$. Assume that for each $k \in \mathbb{N}$, there exists $x_k \in C_{n_1,n_2,\ldots,n_k}$ such that $x_k \notin U$. Then $(x_k)_k$ has an adherent point $x \in \bigcap_k C_{n_1,n_2,\ldots,n_k} = U_\alpha$. This provides a contradiction, since $U_\alpha \subset U$. □

We shall need the following result due to Talagrand [579, Theorem 2.4].

Lemma 3.1.13 (Talagrand) *Let (X, τ) be a regular space that admits a finer topology ξ such that (X, ξ) is a Lindelöf Σ-space. Then* $\operatorname{dens}(X, \xi) \leq \omega(X, \tau)$, *where* $\operatorname{dens}(X)$ *and* $\omega(X)$ *denote the density and weight of X, respectively.*

Proof Let $\{\omega_i : i \in I\}$ be a basis of open sets in τ of cardinality $\aleph$. Since (X, ξ) is a Lindelöf Σ-space, there exists a non-empty subset Σ of $\mathbb{N}^{\mathbb{N}}$ and a compact-valued (usco) map T from Σ covering (X, ξ). Set $T(\alpha) := K_\alpha$, and as usual

$$C_{n_1,\ldots,n_k} := \bigcup \{K_\beta : \beta = (m_j) \in \Sigma, m_j = n_j, 1 \leq j \leq k\},$$

for $\alpha \in \Sigma$. For every non-empty set $\omega_i \cap C_{n_1,\ldots n_k}$, where $i \in I$, $k, n_k \in \mathbb{N}$, and $(n_k)_k \in \Sigma$, choose a single point, and let H be the set of such points. Then $|H| \leq \aleph$. Observe that H is dense in (X, ξ). Indeed, let U be an open set in ξ and fix $x \in U$. There exist $\alpha = (n_k) \in \Sigma$ such that $x \in K_\alpha$, and $i \in I$ such that

$$x \in \omega_i \cap K_\alpha \subset \overline{\omega_i} \cap K_\alpha \subset U,$$

where the closure is taken in τ. Moreover, there exists $k \in \mathbb{N}$ such that

$$x \in \overline{\omega_i} \cap C_{n_1,\ldots,n_k} \subset U.$$

Indeed, otherwise there exists a sequence

$$x_k \in \overline{\omega_i} \cap C_{n_1,\ldots,n_k} \setminus U$$

for $k \in \mathbb{N}$. Since T is (usco), $(x_k)_k$ has an accumulation point

$$y \in \overline{\omega_i} \cap K_\alpha \setminus U,$$

a contradiction. This proves that $H \cap U$ is non-empty, so H is dense in (X, ξ). □

Note also that the class of Lindelöf Σ-spaces is the minimal class which contains all second countable spaces, all compact spaces, and is closed with respect to the finite products, closed subspaces, and continuous images; see [32].

3.2 Resolutions and K-Analyticity

Theorem 3.1.7 shows also that every K-analytic space E admits a covering $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact subsets of E such that $A_\alpha \subset A_\beta$ for $\alpha \leq \beta$. Such an ordered covering will be called a *compact resolution* of E.

If the sets A_α are countably compact and relatively countably compact the resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ will also be called countably compact and relatively countably compact, respectively.

Let us start with the following simple observation concerning resolutions used in the subsequent parts of the book. The argument in the proof is essentially due to Talagrand; see [579, Proposition 6.13] and also Corollary 13.1.8.

Proposition 3.2.1 *Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a resolution on an uncountable set A. Then for some $\gamma \in \mathbb{N}^{\mathbb{N}}$, the set A_γ is infinite.*

Proof We can choose a sequence $(k_n)_n$ of positive integers such that the set

$$B_n = \bigcup \{A_\alpha : \alpha = (a_k) \in \mathbb{N}^{\mathbb{N}},\ a_i = k_i \ \ 1 \le i \le n\}$$

is uncountable for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, choose

$$x_n \in B_n \setminus \{x_i : 1 \le i < n\}.$$

Then for every $n \in \mathbb{N}$, there exists $\beta_n = (\beta_{n,i}) \in \mathbb{N}^{\mathbb{N}}$ such that $x_n \in A_{\beta_n}$, and $\beta_{n,i} = k_i$ for $1 \le i \le n$. Let $\gamma_i = \sup_n \beta_{n,i}$ for $i \in \mathbb{N}$. Then $\gamma = (\gamma_i) \in \mathbb{N}^{\mathbb{N}}$, and $\beta_n \le \gamma$ for $n \in \mathbb{N}$. Hence, $\{x_n : n \in \mathbb{N}\} \subset A_\gamma$, and A_γ is infinite. □

Proposition 3.2.1 can be also deduced from Corollary 3.2.8, and compare also Corollary 3.1.11. Proposition 3.2.1 yields the following:

Proposition 3.2.2 *Let E be an uncountable-dimensional vector space, and let E^* be an algebraic dual of E. Then neither $(E, \sigma(E, E^*))$ nor $(E^*, \sigma(E^*, E))$ is quasi-Suslin.*

Proof Let $\{x_k : k \in \omega_1\}$ be a linearly independent set in E, where ω_1 is the set of all countable ordinals. For each $k \in \omega_1$, where $k \ge \omega_0$, choose a bijection

$$T_k : \{l \in \omega_1 : l \le k\} \to \mathbb{N}.$$

Since $\{x_k : k \in \omega_1\}$ is linearly independent, for each $l \in \omega_1$, there exists a linear functional $u_l \in E^*$ such that

$$u_l(x_k) = T_k(l), \quad \text{if} \quad k \ge \max(\omega_0, l).$$

If S is a countable subset of ω_1, and $k = \sup S$, the set

$$\{u_l(x_k) = T_k(l) : l \in S\}$$

consists of distinct points in $\mathbb{N}$ and is unbounded in $\mathbb{N}$. This implies that $H := \{u_l : l \in \omega_1\}$ consists of $\aleph_1$-distinct points, and no infinite subset of H is $\sigma(E^*, E)$-bounded. Therefore, no infinite subset of H is contained in a $\sigma(E^*, E)$-countable compact set.

Assume $(E^*, \sigma(E^*, E))$ is quasi-Suslin. Then $(E^*, \sigma(E^*, E))$ has a resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of countably compact sets. By Proposition 3.2.1, there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $H \cap K_\alpha$ is infinite, a contradiction.

For the other case, for each infinite $k \in \omega_1$, there exists a linear functional $v_k \in E^*$ such that

$$v_k(x_l) = T_k(l), \quad \text{if} \quad k \geq \max(\omega_0, l).$$

Using the above argument, we note that $(E, \sigma(E, E^*))$ is not a quasi-Suslin space as well. □

One cannot prove or disprove in the usual set theory a similar statement that a set with a resolution consisting of countable sets is countable; see Proposition 3.2.3. This also refers to the question if a Banach space that has a resolution of separable sets is separable. Indeed, note that under CH, the non-separable Banach space $c_0[0, \omega_1)$ has a resolution consisting of closed separable subspaces of the form $c_0[0, \mu)$ with $\mu < \omega_1$; see also [180].

For $\alpha = (n_k)$ and $\beta = (m_k) \in \mathbb{N}^{\mathbb{N}}$, the relation $\alpha \leq^* \beta$ means that there exists $m \in \mathbb{N}$ such that $n_k \leq m_k$ for all $k \geq m$. We note the following fact from [592, Theorem 3.6].

Proposition 3.2.3 *Under CH, the space ω_1 has a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that every compact set in ω_1 is contained in some K_α, and under MA + $\neg$CH, the space ω_1 does not admit a compact resolution.*

Proof As each countable subset in $(\mathbb{N}^{\mathbb{N}} \leq^*)$ has an upper bound, by the transfinite induction, we determine an injective map Ψ from $([0, \omega_1), \leq)$ into a well-ordered subset M of $(\mathbb{N}^{\mathbb{N}} \leq^*)$ that preserves the order.

If we assume CH, there exists a bijection φ between $[0, \omega_1)$ and $\mathbb{N}^{\mathbb{N}}$. Then, we may assume that $\varphi(t) \leq \Psi(t)$, for each $t \in [0, \omega_1)$, which implies that M is a cofinal subset in $(\mathbb{N}^{\mathbb{N}} \leq^*)$. For $\alpha \in \mathbb{N}^{\mathbb{N}}$, we denote by t_α the first element in $[0, \omega_1)$ such that $\alpha \leq^* \Psi(t_\alpha)$ and let $K_\alpha := [0, t_\alpha]$. Clearly, $K_{\Psi(\delta)} = [0, \delta]$.

The fact that $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution swallowing compact sets follows from that for a compact subset K in $[0, \omega_1)$, there exists $\delta \subset [0, \omega_1)$ such that $K \subset [0, \delta] = K_{\Psi(\delta)}$. Since ω_1 is non-compact but countably compact, it is not Lindelöf, hence is not K-analytic.

Now observe that under MA + $\neg$ CH, the space $[0, \omega_1)$ does not have a compact resolution. Indeed, assume that $[0, \omega_1)$ admits a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. If $\alpha < \omega_1$, there exists $t_\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha \in K_{t_\alpha}$. By the assumption MA + $\neg$ CH, one gets $t \in \mathbb{N}^{\mathbb{N}}$ with $t_\alpha \leq^* t$ for all $\alpha < \omega_1$. Set

$$W_t := \{q \in \mathbb{N}^{\mathbb{N}} : \exists m \in \mathbb{N} : q(n) = t(n), n \geq m\}.$$

Then W_t is countable. Also for each $\alpha < \omega_1$, there exists $q \in W_t$ such that $t_\alpha \leq q$. This proves that $\{K_q : q \in W_t\}$ covers ω_1, and hence some set K must be uncountable. Since all compact sets in ω_1 are countable, we reached a contradiction. □

It is easy to see that any separable metric and complete space E has a compact resolution that swallows compact sets. Indeed, for $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, and for a countable and dense sequence $(x_n)_n$ in E, set

$$K_\alpha := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{n_k} B(x_j, k^{-1}),$$

where $B(x_j, k^{-1})$ is the closed ball in E with the centre at point x_j, and radius k^{-1}, $j, k \in \mathbb{N}$. Then $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution, and for every compact set K in E, there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $K \subset K_\alpha$.

The following observation is due to Tkachuk [592].

Proposition 3.2.4 *(i) Any K-analytic space has a compact resolution.*
(ii) Any continuous image of a compact resolution is a compact resolution.
(iii) A closed subset of a space with a compact resolution has a compact resolution.
(iv) A countable union of subspaces of a topological space with a compact resolution has a compact resolution.
(v) A countable product of spaces with a compact resolution has a compact resolution.
(vi) A countable intersection of subspaces of a topological space with a compact resolution has a compact resolution.

Proof (i) follows from Theorem 3.1.7. Conditions (ii) and (iii) are clear.

(iv) Let $(E_n)_n$ be a sequence of subspaces of a space E covering E, and assume that each E_n has a compact resolution $\{K^n_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in E_n. For $\alpha = (a_k) \in \mathbb{N}^{\mathbb{N}}$, the set $A_\alpha := \bigcup_{1 \le n \le a_1} K^n_\alpha$ is compact, and $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution on E.

(v) Let $E = \prod_n E_n$ be the countable product of topological spaces E_n endowed with the product topology, and for each $n \in \mathbb{N}$, let $\{A^n_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a compact resolution in E_n. Let $(D_n)_n$ be a division of the space $\mathbb{N}$ in disjoint infinite sets such that $\mathbb{N} = \bigcup_n D_n$. If $\xi_n : \mathbb{N} \to D_n$ is a bijection, the map

$$\xi_n^* : \mathbb{N}^{D_n} \to \mathbb{N}^{\mathbb{N}}$$

defined by

$$\xi_n^*(f) := f \circ \xi_n$$

for each $n \in \mathbb{N}$ is a bijection and such that $\xi_n^*(f) \le \xi_n^*(g)$ if and only if $f(n) \le g(n)$ for $f, g \in \mathbb{N}^{D_n}$. If $\alpha|n := (a_k)_{k \in D_n}$, where $\alpha = (a_k) \in \mathbb{N}^{\mathbb{N}}$, the set

$$K_\alpha := \prod_n A^n_{\xi_n^*(\alpha|n)}$$

is compact, and $\{K_\alpha :\in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution in E.

(vi) The intersection is homeomorphic to the diagonal of the product of the subspaces. □

Since every Polish space has a compact resolution swallowing compact sets, Proposition 3.2.4 yields that every Čech-complete Lindelöf space E admits a compact resolution swallowing compact sets. Indeed, E is homeomorphic to a closed subset of the product $M \times K$ for some Polish space M and a compact Hausdorff space K.

Note that there exist (even locally convex) spaces with a compact resolution which are not K-analytic; see [117, 220, 579], and [221]. We provide such examples in the next parts below.

A subset A of a space E is called *full*, if A contains the adherent points of the sequences in A. Note the following simple fact from [220]; see also [117].

Proposition 3.2.5 *Assume that E admits a resolution $\left\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ consisting of countably compact full sets. Then E is quasi-Suslin.*

Proof For $\alpha = (a_n)_n \in \mathbb{N}^{\mathbb{N}}$, set

$$C_{a_1 a_2 \ldots a_k} = \bigcup \left\{ A_\beta : \beta \in \mathbb{N}^{\mathbb{N}}, \beta|_k = \alpha|_k \right\}$$

and $B_\alpha = \bigcap_{k\in\mathbb{N}} C_{a_1 a_2 \ldots a_k}$, where as usual, we denote $\alpha|_k := (a_1, a_2, \ldots, a_k)$ for each $k \in \mathbb{N}$. We prove that the map T, defined by $T(\alpha) = B_\alpha$, for $\alpha \in \mathbb{N}^{\mathbb{N}}$, is a quasi-Suslin map. Indeed, if $\alpha = (a_k) \in \mathbb{N}^{\mathbb{N}}$, and $\beta_k \in \mathbb{N}^{\mathbb{N}}$, $\beta_k|_k = \alpha|_k$ and $x_k \in B_{\beta_k}$ for each $k \in \mathbb{N}$, then, by the order condition, there exists $\gamma_k \in \mathbb{N}^{\mathbb{N}}$ such that $\{x_n : n \geq k\} \subset A_{\gamma_k}$ and $\gamma_k|_k = \alpha|_k$ for each $k \in \mathbb{N}$. The sequence $(x_n)_n$ has an adherent point $x \in A_{\gamma_1} \subset C_{a_1}$. As x is also an adherent point of $(x_n)_{n\geq k}$, and $(x_n)_{n\geq k}$ is contained in the full set A_{γ_k}, we have that $x \in A_{\gamma_k} \subset C_{a_1 a_2 \ldots a_k}$, for $k \geq 1$. Then the sequence $(x_n)_n$ has an adherent point $x \in \bigcap_k C_{a_1 a_2 \ldots a_k} = B_\alpha$, implying that T is a quasi-Suslin map. □

We shall need also the following well-known property for Lindelöf spaces; see [195].

Proposition 3.2.6 *Every Lindelöf space E is realcompact.*

Proof If E is not realcompact, there exists $y \in \beta E \backslash E$ such that $Z_f := f^{-1}(0) \cap E$ is a closed non-void subset of E for each $f \in \mathcal{F} := \{f \in C(\beta E) : f(y) = 0,\ f(\beta E) \subset [0, 1]\}$; see Proposition 2.5.1. By the continuity of $\sum_n 2^{-n} f_n$, for $f_n \in \mathcal{F}$, we know that the family $\{Z_f : f \in \mathcal{F}\}$ satisfies the countably intersection property. Since $\bigcap\{Z_f : f \in \mathcal{F}\} = \emptyset$, we deduce E is not Lindelöf. □

We provide another useful characterization of a space to be K-analytic; see [275].

Proposition 3.2.7 *For a regular topological space E, the following assertions are equivalent:*

(i) *E is K-analytic.*
(ii) *E is quasi-Suslin and Lindelöf.*
(iii) *E is Lindelöf and admits a compact resolution.*
(iv) *E is Dieudonné complete, i.e. E is a closed subspace of a product of metrizable spaces, and E admits a compact resolution.*
(v) *Every relatively countably compact set in E is relatively compact, and E admits a compact resolution.*

Proof (i) $\Rightarrow$ (ii): Clearly, any K-analytic space is quasi-Suslin. By Proposition 3.1.5, every K-analytic space is Lindelöf.

(ii) $\Rightarrow$ (i): If T is a quasi-Suslin map, each set $T(\alpha)$ is compact.

(i) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (iv): Every Lindelöf space is realcompact, so homeomorphic to a closed subspace of a product of the real lines.

(iv) $\Rightarrow$ (v): Every relatively countably compact set in a Dieudonné space is relatively compact.

(v) $\Rightarrow$ (i): Let $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a compact resolution in E, and set $B_\alpha := \bigcap_k C_{n_1,n_2,\ldots,n_k}$ for $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$. Since B_α is countably compact, $\overline{B_\alpha}$ is compact. Now it is clear that the map $K(\alpha) := B_\alpha$ is (usco). Hence, E is K-analytic. □

Corollary 3.2.8 follows from [592].

Corollary 3.2.8 (Tkachuk) *Every topological space E having a compact resolution has a countable extent, i.e. every closed discrete subset of X is countable.*

Proof Let D be a closed and discrete subset of E. Then D with the induced topology has a compact resolution. Proposition 3.2.7(v) applies to deduce that D is K-analytic, hence Lindelöf, so D is countable. □

The following concept is due to Fremlin; see [240]. A Hausdorff topological space E is called *angelic* if every relatively countably compact set K in E is relatively compact, and for every $x \in \overline{K}$, there exists a sequence in K converging to x. In angelic spaces, the compactness and relative compactness coincide; the (relatively) countably compact, (relatively) compact, and (relatively) sequentially compact sets are the same; see [240, Theorem 3.3] and also [281], where a proper subclass of angelic spaces has been studied.

Classical examples of angelic spaces include spaces $C_p(X)$ with compact X (see, e.g. [285] and [364]), Banach spaces E with the weak topology $\sigma(E, E')$, and spaces of first Baire class functions on any Polish space P with the topology of the pointwise topology on P (see [101]). Below we prove a general result due to Orihuela providing much larger class of angelic spaces $C_p(X)$ and applications.

If a topological space E admits a compact resolution, E is quasi-Suslin by Proposition 3.2.5. Nevertheless, E need not be K-analytic, since the countably compact sets B_α, obtained in the proof of Proposition 3.2.5, need not be compact. For angelic spaces, countably compact sets are compact, so we note the following facts; see [117].

Corollary 3.2.9 *For an angelic space E, the following statements are equivalent:*

(i) *E admits a compact resolution.*
(ii) *E is a quasi-Suslin space.*
(iii) *E is a K-analytic space.*

Proof (i) $\Rightarrow$ (ii): This follows from Proposition 3.2.5.
(ii) $\Rightarrow$ (iii): Since every countable compact set is compact in any angelic space.
(iii) $\Rightarrow$ (i): Since every K-analytic space admits a compact resolution. □

From Proposition 3.2.5, we deduce also the following:

Corollary 3.2.10 *Let $\mathcal{W} = \left\{D_{n_1 n_2 \ldots n_k}, n_i \in \mathbb{N}, 1 \leq i \leq k,\ k \in \mathbb{N}\right\}$ be an ordered web in a topological space E. If for every $\alpha = (a_k)_k$ in $\mathbb{N}^{\mathbb{N}}$ the set $A_\alpha = \bigcap_k \overline{D_{a_1 a_2 \ldots a_k}}$ is countably compact (compact), E is quasi-Suslin (K-analytic).*

Proof From Proposition 3.2.5, it follows that E is quasi-Suslin. As usual, define

$$C_{a_1 a_2 \ldots a_k} := \bigcup \left\{A_\beta : \beta \in \mathbb{N}^{\mathbb{N}}, \beta|_k = \alpha|_k\right\}$$

for each $\alpha = (a_k)_k \in \mathbb{N}^{\mathbb{N}}$, and let $B_\alpha := \bigcap_k C_{a_1 a_2 \ldots a_k}$. Then the map T defined by $T(\alpha) := B_\alpha$ for $\alpha \in \mathbb{N}^{\mathbb{N}}$ is a quasi-Suslin map. Obviously, $A_\alpha \subset C_{a_1 a_2 \ldots a_k}$ (see Proposition 3.2.5 and the proof), and, if $\beta \in \mathbb{N}^{\mathbb{N}}$, then $A_\beta \subset \overline{D_{\beta_1 \beta_2 \ldots \beta_k}}$. Therefore, if $\beta|_k = \alpha|_k$, then $A_\beta \subset \overline{D_{a_1 a_2 \ldots a_k}}$. This implies

$$C_{a_1 a_2 \ldots a_k} \subset \overline{D_{a_1 a_2 \ldots a_k}}.$$

Since

$$A_\alpha \subset C_{a_1 a_2 \ldots a_k} \subset \overline{D_{a_1 a_2 \ldots a_k}},$$

for each $k \in \mathbb{N}$, we have

$$A_\alpha \subset B_\alpha \subset \bigcap_k \overline{D_{a_1 a_2 \ldots a_k}} = A_\alpha.$$

Since $A_\alpha = B_\alpha$, the map T is a K-analytic map in E if each A_α is compact. Consequently, E is K-analytic. □

For a locally convex space E by the *Mackey topology*, we mean the finest locally convex topology $\mu(E, E')$ on E having the same continuous linear functionals as the original topology of E. By the Mackey–Arens theorem (see [322, Theorem 8.5.5]), the topology $\mu(E, E')$ is the topology of the uniform convergence on absolutely convex $\sigma(E', E)$-compact subsets of E'.

A lcs is called quasi-complete if every closed bounded set in E is complete, [322]. For the following application of Proposition 3.2.5, we refer to [117].

Corollary 3.2.11 *Let E be a lcs that is quasi-complete for the Mackey topology $\mu\left(E, E'\right)$. Then the following statements are equivalent:*

(i) *E admits a compact resolution.*
(ii) *E is a quasi-Suslin space.*
(iii) *E is a K-analytic space.*

Proof From Proposition 3.2.5, it follows (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii): Assume E is quasi-Suslin, and let T_1 be an ordered quasi-Suslin map defined by $T_1(\alpha) := B_\alpha$, where $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a countably compact resolution. For $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, set

$$D_{n_1 \ldots n_k} = \bigcup \left\{ B_\beta : \beta \in \mathbb{N}^{\mathbb{N}}, \beta|_k = (n_1 \ldots n_k) \right\},$$

and $A_\alpha := \bigcap_{k\in\mathbb{N}} D_{n_1 n_2 \ldots n_k}$. It is clear that the map T defined by $T(\alpha) := A_\alpha$ is a quasi-Suslin map. The regularity and the accumulation property imply

$$\overline{A_\alpha} = \bigcap_k \overline{D_{n_1 n_2 \ldots n_k}}.$$

By the assumption on E, the closure $\overline{A_\alpha}$ of the countable compact set A_α is compact (see [374, Theorem 24.2.1]), and then Corollary 3.2.10 applies. (iii) $\Rightarrow$ (i) is obvious. □

We note another application of Corollary 3.2.10; see [117].

Corollary 3.2.12 *Let E be a semireflexive lcs. The following assertions are equivalent:*

(i) *E admits a bounded resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, i.e. the sets A_α are bounded in the locally convex space E.*
(ii) *$(E, \sigma(E, E'))$ is K-analytic.*
(iii) *$(E, \sigma(E, E'))$ is quasi-Suslin.*

Proof (i) $\Rightarrow$ (ii): Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a bounded resolution. Let $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$. Then for every closed absolutely convex neighbourhood of zero U in E, there exists $k \in \mathbb{N}$ such that

$$C_{a_1 a_2 \ldots . a_k} \subset 2^k U.$$

Indeed, otherwise, there exists a neighbourhood of zero U in E, such that for every $k \in \mathbb{N}$, there exists $x_k \in C_{a_1 a_2 \ldots . a_k}$ such that $2^{-k} x_k \notin U$. Since $x_k \in C_{a_1 a_2 \ldots . a_k}$ for every $k \in \mathbb{N}$, there exists $\beta_k = (b_n^k)_n \in \mathbb{N}^{\mathbb{N}}$ such that $x_k \in A_{\beta_k}$, and $b_j^k = a_j$ for $j = 1, 2, \ldots, k$. Set

$$d_n = \max\{b_n^k : k \in \mathbb{N}\}$$

for $n \in \mathbb{N}$. Set $\gamma = (d_n)$. Since $\gamma \geq \beta_k$ for every $k \in \mathbb{N}$, then $A_{\beta_k} \subset A_\gamma$, so $x_k \in A_\gamma$ for all $k \in \mathbb{N}$. As A_γ is bounded, $2^{-k}x_k \to 0$ in E, a contradiction. This implies that the set $W := \bigcap_k \overline{C_{a_1 a_2 \ldots . a_k}}$ is bounded and closed in E, where the closure is taken in $\sigma(E, E')$. Since E is semireflexive, the set W is compact in $\sigma(E, E')$. Now Corollary 3.2.10 applies to get that $(E, \sigma(E, E'))$ is K-analytic.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i): Since every quasi-Suslin space admits a resolution of countably compact sets and every such set is bounded, the conclusion follows. □

The following proposition for Banach spaces E is due to Talagrand; see [579, Théoréme 3.6]. The same statement for a Fréchet space E was proved by Canela [114, Proposition 4]. The argument for the proof of Proposition 3.2.13 is adopted from [117, Proposition 7]. Proposition 3.2.13 fails in general: $\mathbb{R}^{\mathbb{R}}$ is not K-analytic and $\mathbb{R}^{\mathbb{R}}$, as a separable space, contains a dense countable-dimensional K-analytic subspace.

Proposition 3.2.13 *Let E be a Fréchet space, and let F be a dense subspace of E such that $(F, \sigma(F, F'))$ is K-analytic. Then $(E, \sigma(E, E'))$ is K-analytic.*

Proof By Corollary 3.2.11, it is enough to show that $(E, \sigma(E, E')$ has a compact resolution. Assume $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution in $(F, \sigma(F, F'))$. Let $(U_n)_n$ be a decreasing basis of closed absolutely convex neighbourhoods of zero in E. For $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, set $B_\alpha := \bigcap_k n_k U_k$. Then $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution consisting of bounded complete sets, and for each bounded set $B \subset E$, there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $B \subset B_\alpha$. Set

$$C_{n_1 \ldots n_k} = \bigcup \left\{ B_\beta : \beta \in \mathbb{N}^{\mathbb{N}}, \beta|_k = (n_1 \ldots n_k) \right\}.$$

For $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, set

$$D_\alpha := (A_\alpha + \overline{C_{n_1}}) \cap (A_{(n_2, n_3, \ldots)} + 2^{-1}\overline{C_{n_1 n_2}}) \cap \cdots \cap$$

$$(A_{(n_k, n_{k+1}, \ldots)} + k^{-1}\overline{C_{n_1 n_2 \ldots n_k}}) \cap \ldots,$$

where the closure is taken in $(E'', \sigma(E'', E'))$. We show that every D_α is contained in E being $\sigma(E, E')$-compact and that $E = \bigcup\{D_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Fix $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$. Let U be a $\sigma(E'', E')$-closed neighbourhood of zero in the strong topology $\beta(E'', E')$ of E''. Note that there exists $k \in \mathbb{N}$ such that $k^{-1}\overline{C_{n_1 n_2 \ldots n_k}} \subset U$. Moreover,

$$D_\alpha \subset \bigcap \{E + U\}$$

for any neighbourhood of zero U in $(E'', \beta(E'', E'))$. Since E is complete, we have

$$D_\alpha \subset \bigcap\{E + U\} = E,$$

if U run over a basis of neighbourhoods of zero in $(E'', \beta(E'', E'))$. This shows that

$$D_\alpha = (A_\alpha + \overline{C_{n_1}} \cap E) \cap (A_{(n_2,n_3,\dots)} + 2^{-1}\overline{C_{n_1 n_2}} \cap E) \cap \dots \cap$$

$$(A_{(n_k,n_{k+1},\dots)} + k^{-1}\overline{C_{n_1 n_2 \dots . n_k}} \cap E) \cap \dots .$$

Consequently, every set D_α is bounded in E. Since every D_α is an intersection of $\sigma(E'', E')$-closed subsets of E'', we have that D_α is $\sigma(E'', E')$-closed, so it is a $\sigma(E, E')$-compact subset of E. Now fix any $x \in E$. Then there exists a sequence $(x_n)_n$ in F such that $n(x_n - x) \to 0$. Choose $\alpha = (a_k) \in \mathbb{N}^\mathbb{N}$ such that $\{n(x_n - x) : n \in \mathbb{N}\} \subset B_\alpha$. Since $B_\alpha \subset \bigcap_k C_{a_1 \dots . a_k}$, we have

$$x - x_k \in k^{-1} C_{a_1 \dots . a_k}$$

for all $k \in \mathbb{N}$. Let $\beta = (b_j) \in \mathbb{N}^\mathbb{N}$ be such that $x_k \in A_{(b_k,b_{k+1},\dots)}$ for each $k \in \mathbb{N}$. Let $c_j := \max\{a_j, b_j\}$ for all $j \in \mathbb{N}$. Then $x \in D_\alpha$ if $\alpha = (c_j)$. We proved that $\{D_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution on $(E, \sigma(E, E'))$. □

We conclude this section with the following result from [118] generating special (usco) maps.

Theorem 3.2.14 (Cascales–Kąkol–Saxon) *Let X be a first-countable topological space. Let Y be a regular topological space for which the relatively countably compact subsets are relatively compact. Let $\varphi : X \to 2^Y$ be a set-valued map such $\bigcup_{n\in\mathbb{N}} \varphi(x_n)$ is relatively compact for each convergent sequence $(x_n)_n$ in X. If for each $x \in X$ we set*

$$\psi(x) := \bigcap\{\overline{\varphi(V)} : V \text{ neighbourhood of } x \text{ in } X\}, \tag{3.1}$$

the map $\psi : X \to 2^Y$ is (usco) and satisfies $\varphi(x) \subset \psi(x)$ for every $x \in X$.

Proof For $x \in X$, set

$$C(x) := \{y \in Y : \exists x_n \to x,\ \forall n\, \exists\ y_n \in \varphi(x_n),\ y \text{ cluster of } (y_n)_n\}. \tag{3.2}$$

Fix $(V_n^x)_n$, a decreasing basis of open neighbourhoods of x.

Claim 1. $C(x)$ is countably compact, and thus $\overline{C(x)}$ is compact. Indeed, we prove that every sequence in $C(x)$ has a cluster point in $C(x)$. Take $(y_j)_j$ in $C(x)$, and assume that $(x_n^j)_n$ converges to x for every $j \in \mathbb{N}$, and let $y_n^j \in \varphi(x_n^j)$ be such that y_j is a cluster point of $(y_n^j)_n$. There exist natural numbers n_i^j, for $i, j \in \mathbb{N}$, such that

$$1 \le n_1^j < n_2^j < \dots < n_i^j < \dots , \quad j \in \mathbb{N},$$

and

$$x_n^j \in V_k^x, \;\; n_k^j \leq n < n_{k+1}^j,$$

for $k \in \mathbb{N}$ and $j \in \mathbb{N}$. The sequence $(x_n)_n$ defined by

$$\{x_1^1, x_2^1, \ldots, x_{n_2^1-1}^1, x_{n_2^1}^1, \ldots, x_{n_3^1-1}^1, x_{n_2^2}^2, \ldots, x_{n_3^2-1}^2, x_{n_3^1}^1, \ldots, x_{n_4^1-1}^1, \tag{3.3}$$

$$x_{n_3^2}^2, \ldots, x_{n_4^2-1}^2, x_{n_3^3}^3, \ldots, x_{n_4^3-1}^3, x_{n_4^1}^1, \ldots\} \tag{3.4}$$

converges to x, and also we have

$$y_j \in \overline{\bigcup_{n\in\mathbb{N}} \varphi(x_n)}$$

for every $j \in \mathbb{N}$. Then $(y_j)_j$ has a cluster point $y \in Y$. Note that y belongs to $C(x)$ because, if we consider the sequence $(z_n)_n$ corresponding to (3.3), and defined by

$$\{y_1^1, y_2^1, \ldots, y_{n_2^1-1}^1, y_{n_2^1}^1, \ldots, y_{n_3^1-1}^1, y_{n_2^2}^2, \ldots, y_{n_3^2-1}^2, y_{n_3^1}^1, \ldots, y_{n_4^1-1}^1, \tag{3.5}$$

$$y_{n_3^2}^2, \ldots, y_{n_4^2-1}^2, y_{n_3^3}^3, \ldots, y_{n_4^3-1}^3, y_{n_4^1}^1, \ldots\}, \tag{3.6}$$

then $z_n \in \varphi(x_n)$, and y is a cluster point of $(z_n)_n$. Claim 1 is proved.

Claim 2. If G is an open set in Y such that $C(x) \subset G$, there is an open neighbourhood V of x such that $\varphi(V) \subset G$. Indeed, assume the claim fails. Then for every $n \in \mathbb{N}$, there is $x_n \in V_n^x$ such that $\varphi(x_n) \not\subset G$. Hence, we can find $y_n \in \varphi(x_n)$ such that $y_n \in Y \setminus G$. Observe that $x_n \to x$, and therefore $(y_n)_n$ has a cluster point $y \in Y \setminus G$ by the assumption. A contradiction, since by the definition, $y \in C(x) \subset G$.

Claim 3. If G is an open set in Y such that $\overline{C(x)} \subset G$, there is an open neighbourhood V of x such that $\overline{\varphi(V)} \subset G$. Indeed, if $O \subset Y$ is open such that $\overline{C(x)} \subset O \subset \overline{O} \subset G$, then applying Claim 2 to $C(x)$ and O, one gets an open neighbourhood V of x such that $\varphi(V) \subset O$. Finally, $\overline{\varphi(V)} \subset \overline{O} \subset G$.

Claim 4. $\psi(x) = \overline{C(x)}$, and thus $\psi(x)$ is compact-valued. Indeed, the inclusion $C(x) \subset \psi(x)$ is a consequence of the definitions of the sets $C(x)$, $\psi(x)$, and the definition of the cluster point of a sequence. The inclusion $\overline{C(x)} \subset \psi(x)$ follows from the fact that $\psi(x)$ is closed. To prove the converse, fix $z \in \psi(x)$. Note that $z \in \overline{C(x)}$. Indeed, for any closed neighbourhood U of z in Y, and for every $n \in \mathbb{N}$, there is $y_n \in U \cap \varphi(V_n^x)$. Choose $x_n \in V_n^x$ and $y_n \in \varphi(x_n) \cap U$. Then $x_n \to x$, and therefore $(y_n)_n$ has a cluster point y in Y because of the hypothesis. By the definition $y \in C(x)$, and since U is closed, $y \in U$. Hence, $z \in \overline{C(x)}$. We proved Claim 4.

Claim 5. $\psi : X \to 2^Y$ is (usco). Indeed, we show that for every open set $G \supset \psi(x)$, there is an open neighbourhood V of x such that $\psi(V) \subset G$. Take G as above. Since $\psi(x) = \overline{C(x)}$, we apply Claim 3 to get an open neighbourhood V of x such that $\overline{\varphi(V)} \subset G$. Now V is also an open neighbourhood of any $y \in V$, and by the definition, we have

$$\psi(y) \subset \overline{\varphi(V)} \subset G.$$

Hence, $\psi(V) \subset G$. We proved the upper semi-continuity of ψ. Finally, observe that by the definition, $\varphi(x) \subset \psi(x)$ for every $x \in X$. The claims together yield the result. □

Theorem 3.2.14 applies for many topological (vector) spaces Y such as Lindelöf spaces, realcompact spaces, angelic spaces, and Banach spaces with the weak topology. Theorem 3.2.14 can be used also to prove the following corollary; see [125, Corollary 4.2].

Corollary 3.2.15 *Let E be a K-analytic space. Let $f : E \to Y$ be a map from E onto a regular topological space whose relatively countably compact subsets are relatively compact. Assume that if $(x_n)_n$ has a cluster point in E, the sequence $(f(x_n))_n$ has a cluster point in Y. Then Y is K-analytic.*

Proof Let $\theta : \mathbb{N}^{\mathbb{N}} \to 2^E$ be an (usco) map such that $E = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \theta(\alpha)$. Define a map $\varphi : \mathbb{N}^{\mathbb{N}} \to 2^Y$ by $\varphi(\alpha) := f(\theta(\alpha))$. The map φ satisfies the condition in Theorem 3.2.14. Indeed, if $(\alpha_n)_n$ is a sequence in $\mathbb{N}^{\mathbb{N}}$ which converges to $\alpha \in \mathbb{N}^{\mathbb{N}}$, then

$$\bigcup_n \varphi(\alpha_n) = \bigcup_n f(\theta(\alpha_n)) = f(\bigcup_n \theta(\alpha_n)),$$

and $\bigcup_n \theta(\alpha_n)$ is relatively countably compact. Now we use the condition about the map f, and Theorem 3.2.14 applies to get an (usco) map as required. This proves that Y is K-analytic. □

We complete this section with a brief note on possible new directions of research on K-analytical spaces proposed in [480]. The analytic subsets of the real line (or the irrationals) can be generalized to the sigma-projective ones by closing under countable unions and complements and continuous real-valued images. Assuming the extra set-theoretic axiom of sigma-projective determinacy, which is consistent with ZFC and follows from large cardinals, these sets have many of the nice properties of the analytic sets, e.g. measurability. One can then generalize the K-analytic spaces to get the K-sigma-projective spaces by taking upper semicontinuous compact-valued images of sigma-projective sets and again prove consistent analogues of classical theorems. For details, see recent [480].

3.3 Quasi-(*LB*)-Spaces

In this section, we discuss a class of lcs E such that E is covered by a resolution consisting of Banach discs. Most of the results presented here are due to Valdivia [616].

Grothendieck conjectured in [288] that the closed graph theorem for a linear map $T : E \to F$ holds when the domain space E is ultrabornological i.e. E is the inductive limit of a family of Banach spaces, and the range space F belonging to a class of lcs that contains the Banach spaces is stable under countable topological products, countable topological direct sums, separated quotients, and closed subspaces. For example, the space of the test functions $\mathfrak{D}(\Omega)$ and the space of distributions $\mathfrak{D}'(\Omega)$ belong to this class. Clearly, Grothendieck's conjecture reduces to the case when E is a Banach space.

In [562], Slowikowski defined a class of range spaces F that contain $\mathfrak{D}(\Omega)$, $\mathfrak{D}'(\Omega)$, and verify the closed graph theorem in case when the domain space is a Banach space. Also, Raikov presented a solution for Grothendieck's conjecture in [514].

In [614], Valdivia proved that the range class introduced by Raikov coincides with the class given by Slowikowski. De Wilde provided in [155] a positive answer for Grothendieck's conjecture for *webbed* spaces F; see also [157].

Valdivia presented in [616] another solution for Grothendieck's conjecture with the class of quasi-(LB)-spaces F, that is, a subclass of the class considered by Slowikowski. In fact a quasi-(LB)-space is a strict webbed space, and a strict webbed space is a Slowikowski space. It turns out that for lcs, the classes of quasi-(LB)-spaces and ordered strict webbed spaces are the same [616].

A lcs E is a *quasi-(LB)-space* if E admits a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ consisting of *Banach discs* [616] (called a *quasi-(LB)-representation* of E). Quasi-(LB)-spaces enjoy good hereditary properties. We gather some of them.

Proposition 3.3.1 *If G is a closed subspace of a quasi-(LB)-space E, the space G is a quasi-(LB)-space.*

Proof Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a quasi-(LB)-representation of E. The normed space $F_{A_\alpha \cap G}$ is a normed subspace of the Banach space F_{A_α}. Therefore, if $(x_n)_n$ is a Cauchy sequence in $F_{A_\alpha \cap G}$, and if x is the limit of $(x_n)_n$ in F_{A_α}, then $x \in G$ by the closedness, and therefore $(x_n)_n$ converges to x in the Banach space $F_{A_\alpha \cap G}$. Therefore, $\{A_\alpha \cap G : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a quasi-(LB)-representation of G. □

Proposition 3.3.2 *Let G be a continuous image of a quasi-(LB)-space F under a linear map T. Then G is a quasi-(LB)-space. In particular, a Hausdorff quotient of a quasi-(LB)-space is a quasi-(LB)-space.*

Proof Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a quasi-(LB)-representation of F. It is easy to prove that $V_\alpha := F_{A_\alpha} \cap T^{-1}(0)$ is a closed subspace of the Banach space F_{A_α}, and by the construction, the normed space $G_{T(A_\alpha)}$ is norm isomorphic to the quotient space

F_{A_α}/V_α. Then $G_{T(A_\alpha)}$ is a Banach space, and $\{T(A_\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a quasi-(LB)-representation of G. □

Proposition 3.3.3 *A countable product of quasi-(LB)-spaces is a quasi-(LB)-space.*

Proof Let J be a countable set, and let $(F_j)_{j\in J}$ be a sequence of quasi-(LB)-spaces. Let $\{A_\alpha^j : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a quasi-(LB)-representation of F_j for each $j \in J$. Let $t : \mathbb{N} \to J \times \mathbb{N}$ be a bijection. For $\alpha_j = (a_{j,n}) \in \mathbb{N}^\mathbb{N}$, $j \in J$, set $a_{t(n)} := b_n$. Define a bijection $T : (\mathbb{N}^\mathbb{N})^J \to \mathbb{N}^\mathbb{N}$ by $T(\{\alpha_j : j \in J\}) = (b_n)$. Let $\alpha \in \mathbb{N}^\mathbb{N}$ and $\{\alpha_j : j \in J\} = T^{-1}(\alpha)$. Define $A_\alpha := \prod_j A_{\alpha_j}^j$. Then $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a quasi-(LB)-representation in the product $\prod_i F_i$. □

Proposition 3.3.4 *A topological direct sum of a countable family of quasi-(LB)-spaces is a quasi-(LB)-space.*

Proof Let $E := \bigoplus_{i\in\mathbb{N}} F_i$ be the topological direct sum of quasi-(LB)-spaces F_i. Let $\{A_\alpha^i : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a quasi-(LB)-representation of F_i, $i \in \mathbb{N}$. It is well known that the finite topological product $\prod_{i=1}^n F_i$ and the finite topological direct sum $\bigoplus_{i=1}^n F_i$ are isomorphic. Therefore, for each $\alpha = (a_n)_n$, we note by Proposition 3.3.3 that

$$A_\alpha := A_\alpha^1 + A_\alpha^2 + \cdots + A_\alpha^{a_1}$$

is a Banach disc in $F := \bigoplus_{i=1}^n F_i$. Clearly, $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a quasi-(LB)-representation of E. □

1. *Every (LF)-space is a quasi-(LB)-space*. First, note that each Fréchet space E is a quasi-(LB)-space. Indeed, if $(U_n)_n$ is a decreasing basis of absolutely convex closed neighbourhoods of zero in E, the sets $A_\alpha := \bigcap_n a_n U_n$ for $\alpha = (a_n) \in \mathbb{N}^\mathbb{N}$ form a quasi-(LB)-representation of E. Next, since every (LF)-space is isomorphic to a quotient of a countable topological direct sum of Fréchet spaces, it is enough to see that Hausdorff quotients and countable topological direct sums of quasi-(LB)-spaces are quasi-(LB). This we showed above.
2. *The strong dual of each (LF)-space is a quasi-(LB)-space*. Indeed, we know already that the countable topological product of quasi-(LB)-spaces is a quasi-(LB)-space. Next, observe that, if E is a Fréchet space, the strong dual $(E', \beta(E', E))$ of E is a quasi-(LB)-space. Indeed, let $(U_n)_n$ be a countable basis of absolutely convex closed neighbourhoods of zero in E. Set $A_\alpha := U_{a_1}^\circ$ for $\alpha = (a_n) \in \mathbb{N}^\mathbb{N}$, where U° means the polar of U. The family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a quasi-(LB)-representation of $(E', \beta(E', E))$. Finally, assume that E is an (LF)-space with a defining sequence $(E_n)_n$ of Fréchet spaces. Note that $(E', \sigma(E', E))$ is isomorphic to a closed subspace of the product $\prod_n (E_n', \sigma(E_n', E_n))$. Then $(E', \sigma(E', E))$ is a quasi-(LB)-space. Since $(E', \sigma(E', E))$ and $(E', \beta(E', E))$ have the same Banach discs, the conclusion follows.

Proposition 3.3.5 *An infinite-dimensional separable Banach space E admits a quasi-(LB)-representation $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of E consisting of absolutely convex compact sets. Hence, the unit ball in E cannot be included in any A_α.*

Proof By Valdivia [611, p.221], the space E is isomorphic to a quotient λ_1/F of a Montel echelon space $\lambda_1 := \lambda_1(\mathbb{N}, A)$; we refer the reader to Chaps. 6 and 16 of this book for more information about spaces $\lambda_1 := \lambda_1(\mathbb{N}, A)$. Let T be a continuous linear map from λ_1/F onto E. Let $(U_n)_n$ be a countable basis of absolutely convex closed neighbourhoods of zero in λ_1. Then (as it is easily seen) the sets

$$A_\alpha := \bigcap_n a_n U_n$$

for $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$ form a resolution consisting of absolutely convex bounded and closed sets. Since λ_1 is a Montel space, i.e. every bounded and closed set in λ_1 is compact, $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution in λ_1 of absolutely convex sets. Then $\{q(A_\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution consisting of absolutely convex sets in λ_1/F, where $q : \lambda_1 \to \lambda_1/F$ is the quotient map. Consequently,

$$\{T(q(A_\alpha)) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$$

is a compact resolution in E consisting of absolutely convex elements. Since E is infinite-dimensional, the unit closed ball in E is not contained in any set $T(q(A_\alpha))$. □

If $\alpha = (a_n)_n \in \mathbb{N}^{\mathbb{N}}$, $k \in \mathbb{N}$, then, as usual, we set $\alpha|_k := (a_1, a_2, \ldots, a_k)$. If $m_1, m_2, \ldots, m_k$ are natural numbers, and $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a quasi-(LB)-representation F, we set

$$C_{m_1,m_2,\ldots,m_k} := \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}, \alpha|_k = (m_1, m_2, \ldots, m_k)\}. \tag{3.7}$$

For a linear map $T : E \to F$, we set

$$U_{m_1,m_2,\ldots,m_k} := T^{-1}(C_{m_1,m_2,\ldots,m_k}).$$

Clearly

$$E = \bigcup\{U_{m_1} : m_1 \in \mathbb{N}\}$$

and

$$U_{m_1,m_2,\ldots,m_k} = \bigcup\{U_{m_1,m_2,\ldots,m_{k+1}} : m_{k+1} \in \mathbb{N}\}$$

for each $k \in \mathbb{N}$. If E is a Baire lcs, there exists a sequence $(r_n)_n$ in $\mathbb{N}$ such that each $\overline{U}_{r_1,r_2,\ldots,r_k}$ is a neighbourhood of zero in E.

Also it is known (see [374, 16.4(7)]) that if C is an absolutely convex neighbourhood of zero in a lcs E, and p_C is the Minkowski functional of C, the closure $\overline{C}$ and the interior C^0 of C are described by

$$\overline{C} = \{x \in E : p_C(x) \leq 1\}, \quad C^0 = \{x \in E : p_C(x) < 1\}.$$

The argument of the following technical lemma follows from [616, Proposition 10].

Lemma 3.3.6 *Let E be a lcs. Let $(U_k)_k$ be a decreasing sequence of absolutely convex sets such that each $\overline{U}_k$ is a neighbourhood of zero. For each $k \in \mathbb{N}$, let p_k be the Minkowski functional of the set $\overline{U}_k$. Let τ be a locally convex pseudometrizable topology on E defined by the family of seminorms $(p_n)_n$. If $x \in (\overline{U}_1)^0$, then x is the limit in τ of a series $\sum_k \alpha_k x_k$ with $x_k \in U_k$ for $k \in \mathbb{N}$, and $\sum_k |\alpha_k| < 1$.*

Proof Clearly, there exists $\lambda > 1$ such that $\lambda x \in \overline{U}_1$. Therefore, there exists $x_1 \in U_1$ and

$$y_2 \in \frac{\lambda - 1}{2^2}\overline{U}_2 \tag{3.8}$$

such that

$$\lambda x = x_1 + y_2. \tag{3.9}$$

From (3.8), it follows that there exist

$$x_2 \in U_2, \quad y_3 \in \frac{\lambda - 1}{2^3}\overline{U}_3 \tag{3.10}$$

such that

$$y_2 = \frac{\lambda - 1}{2^2}x_2 + y_3.$$

Then

$$\lambda x = x_1 + \frac{\lambda - 1}{2^2}x_2 + y_3. \tag{3.11}$$

From (3.10), we note that there exist

$$x_3 \in U_3, \quad y_4 \in \frac{\lambda - 1}{2^4}\overline{U_4}$$

such that

$$y_3 = \frac{\lambda - 1}{2^3}x_3 + y_4.$$

Hence,

$$\lambda x = x_1 + \frac{\lambda - 1}{2^2} x_2 + \frac{\lambda - 1}{2^3} x_3 + y_4.$$

By an obvious induction, we construct two sequences $(x_n)_n$, $(y_{n+1})_n$, such that for each $n \in \mathbb{N}$

$$x_n \in U_n, \;\; y_{n+1} \in \frac{\lambda - 1}{2^{n+1}} \overline{U_{n+1}}, \tag{3.12}$$

and

$$\lambda x = x_1 + \frac{\lambda - 1}{2^2} x_2 + \frac{\lambda - 1}{2^3} x_3 + \cdots + \frac{\lambda - 1}{2^n} x_n + y_{n+1}.$$

From

$$x = \frac{1}{\lambda} x_1 + \frac{\lambda - 1}{\lambda 2^2} x_2 + \frac{\lambda - 1}{\lambda 2^3} x_3 + \cdots + \frac{\lambda - 1}{\lambda 2^n} x_n + \frac{1}{\lambda} y_{n+1}$$

we have the lemma with $\alpha_1 = \lambda^{-1}$ and $\alpha_k = (\lambda - 1)\lambda^{-1} 2^{-k}$ for $k \geq 2$. Indeed, by (3.12), we have

$$\lim_n y_{n+1} = 0$$

(in τ), and

$$\sum_k |\alpha_k| = \frac{1}{\lambda} + \frac{\lambda - 1}{\lambda} \left(\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} + \cdots \right) = \frac{1}{\lambda} + \frac{\lambda - 1}{2\lambda} < \frac{1}{\lambda} + \frac{\lambda - 1}{\lambda} = 1.$$

□

We shall say that the series $\sum_k \alpha_k x_k$ constructed in Lemma 3.3.6 is a τ-*representation* of the point $x \in (\overline{U_1})^0$ by the elements of the sequence $(U_k)_k$.

If the sequences $\alpha(n) = (\alpha(n)_k)_k \in \mathbb{N}^{\mathbb{N}}$, $n \in \mathbb{N}$, verify $\alpha(k)|_k = \alpha(k+s)|_k$ for each $k, s \in \mathbb{N}$, the supremum $b_k = \sup\{\alpha(n)_k : n = 1, 2, \dots\}$ is finite for $k \in \mathbb{N}$. Therefore, $\beta := (b_k)_k \in \mathbb{N}^{\mathbb{N}}$ and $\beta = \sup\{\alpha(n) : n \in \mathbb{N}\}$. This applies to prove the following; see [616, Proposition 9].

Proposition 3.3.7 *Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a quasi-(LB)-representation of a lcs E. For a sequence $(m_n)_n$ in $\mathbb{N}$ and a neighbourhood V of zero in E, there exists $k \in \mathbb{N}$ such that*

$$k^{-1} C_{m_1, m_2, \dots, m_k} \subset V.$$

Proof Assume the conclusion fails. Then there exists $\alpha(k) \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha(k)|_k = (m_1, m_2, \ldots, m_k)$ and $k^{-1}A_{\alpha(k)} \not\subseteq V$ for $k \in \mathbb{N}$. If $\beta = \sup\{\alpha(k) : k \in \mathbb{N}\}$, then $A_{\alpha(k)} \subset A_\beta$. Hence, for each $k \in \mathbb{N}$, we have $k^{-1}A_\beta \not\subseteq V$, contradicting the boundedness of the Banach disc A_β. □

We supplement Lemma 3.3.6, where $U_{m_1 \ldots m_k} := T^{-1}(C_{m_1 m_2 \ldots m_k})$.

Proposition 3.3.8 (Valdivia) *Let E and F be lcs. Let $T : E \to F$ be a continuous linear map. Assume that F admits a quasi-(LB)-representation $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and that there exists a sequence $(r_n)_n$ in $\mathbb{N}$ such that each $\overline{U}_{r_1,r_2,\ldots,r_k}$ is a neighbourhood of the origin in E. Then*

$$(\overline{U}_{r_1,r_2,\ldots,r_k})^0 \subset U_{r_1,r_2,\ldots,r_k},$$

for each $k \in \mathbb{N}$.

Proof If V is a closed neighbourhood of zero in F, Proposition 3.3.7 implies that there exists $s \in \mathbb{N}$ such that $s^{-1}C_{r_1,r_2,\ldots,r_s} \subset V$. By the continuity of T, we know that for the set $U_{r_1,r_2,\ldots,r_s} = T^{-1}(C_{r_1,r_2,\ldots,r_s})$ we have

$$s^{-1}\overline{U}_{r_1,r_2,\ldots,r_s} \subset T^{-1}(V). \tag{3.13}$$

Let τ be a locally convex pseudometrizable topology generated by the Minkowski functionals of the sets $\overline{U}_{r_1,r_2,\ldots,r_k}$ for $k \in \mathbb{N}$. From (3.13), it follows that the linear map $T : (E, \tau) \to F$ is also continuous. By Lemma 3.3.6 for $k \in \mathbb{N}$, each

$$x \in (\overline{U}_{r_1,r_2,\ldots,r_k})^0 \tag{3.14}$$

admits a τ-absolutely convex representation

$$\sum_{s=0}^{\infty} \alpha_{k+s} x_{k+s}$$

with $x_{k+s} \in U_{r_1,r_2,\ldots,r_{k+s}}$ and $\Sigma\,|\alpha_{k+s}| < 1$, and the point x is a limit in τ of the series $\Sigma_{s=0}^{\infty}\alpha_{k+s}x_{k+s}$. The continuity of T yields

$$T(x) = \sum_{s=0}^{\infty} \alpha_{k+s} T(x_{k+s}). \tag{3.15}$$

From

$$T(x_{k+s}) \in T(U_{r_1,r_2,\ldots,r_{k+s}}) = C_{r_1,r_2,\ldots,r_{k+s}},$$

for $s = 0, 1, \ldots$, we note that for each s, there exists $\beta(s) \in \mathbb{N}^{\mathbb{N}}$ with

$$\beta(s)|_{k+s} = (r_1, r_2, \ldots, r_{k+s})$$

and $T(x_{k+s}) \in A_{\beta(s)}$. For $\beta := \sup\{\beta(s) : s \in \mathbb{N}\}$, we have

$$T(x_{k+s}) \in A_\beta$$

for each $s \in \mathbb{N}$. As A_β is a Banach disc, and $\sum |\alpha_{k+s}| < 1$, it follows that in the Banach space generated by A_β

$$\sum_{s=0}^{\infty} \alpha_{k+s} T(x_{k+s}) = y \in A_\beta \subset C_{r_1, r_2, \ldots, r_k}. \tag{3.16}$$

From (3.15) and (3.16), we note $y = T(x)$. Therefore

$$x \in T^{-1}(y) \subset T^{-1}(C_{r_1, r_2, \ldots, r_k}) = U_{r_1, r_2, \ldots, r_k}, \tag{3.17}$$

and using (3.14), (3.17), we conclude

$$(\overline{U}_{r_1, r_2, \ldots, r_k})^0 \subset U_{r_1, r_2, \ldots, r_k} \tag{3.18}$$

for each $k \in \mathbb{N}$. □

We shall need also the following useful fact.

Lemma 3.3.9 *A linear map $T : E \to F$ between tvs E and F has closed graph if and only if there exists on F a weaker Hausdorff topology ξ such that $T : E \to (F, \xi)$ is continuous.*

Proof Assume the map $T : E \to F$ has closed graph. Let $\mathcal{V}$ and $\mathcal{W}$ be fundamental systems of balanced neighbourhood of zero in E and F, respectively. The family of balanced sets

$$\{T(V) + W : V \in \mathcal{V}, W \in \mathcal{W}\}$$

is a basis of balanced neighbourhoods of zero for a vector topology ξ on F, weaker than the initial one of F. If

$$y \in \bigcap \{T(V) + W : V \in \mathcal{V}, W \in \mathcal{W}\}, \tag{3.19}$$

for each pair $(V, W) \in \mathcal{V} \times \mathcal{W}$, there exist $x_v \in V$ and $y_w \in W$ such that

$$y = T(x_v) + y_w. \tag{3.20}$$

With the inclusion $\supset$ as preorder in $\mathcal{V} \times \mathcal{W}$ for the nets $\{x_v : (V, W) \in \mathcal{V} \times \mathcal{W}, \supset\}$ and $\{y_w : (V, W) \in \mathcal{V} \times \mathcal{W}, \supset\}$ we have

$$\lim_{(V,W)\in\mathcal{V}\times\mathcal{W}} x_v = 0, \quad \lim_{(V,W)\in\mathcal{V}\times\mathcal{W}} y_w = 0. \tag{3.21}$$

By (3.20) it follows

$$\lim_{(V,W)\in\mathcal{V}\times\mathcal{W}} T(x_v) = y, \tag{3.22}$$

and by (3.21), (3.22)

$$\lim_{(V,W)\in\mathcal{V}\times\mathcal{W}} (x_v, T(x_v)) = (0, y). \tag{3.23}$$

Since the graph of T is closed, $y = T(0)$ by (3.23). Therefore, using (3.19), and since $T(0) = 0$, we have

$$\bigcap \{T(V) + W : V \in \mathcal{V},\ W \in \mathcal{W}\} = \{0\}.$$

This proves that ξ is Hausdorff. The converse is clear. □

Now we are ready to prove the following interesting closed graph theorem due to Valdivia; see [616, Proposition 11].

Theorem 3.3.10 (Valdivia) *Let E and F be lcs. Let $T : E \to F$ be a linear map with closed graph. Suppose that F admits a quasi-(LB)-representation $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and that there exists a sequence $(r_n)_n$ in $\mathbb{N}$ such that each $\overline{U}_{r_1,r_2,\ldots,r_k}$, $k \in \mathbb{N}$, is a neighbourhood of zero in E. Then T is continuous.*

Proof Let V be a neighbourhood of zero in F. By Proposition 3.3.7, there exists a positive integer s such that

$$s^{-1} C_{r_1,r_2,\ldots,r_s} \subset V.$$

By Lemma 3.3.9, there exists on F a weaker Hausdorff topology ξ such that $T : E \to (F, \xi)$ is continuous. Clearly, $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is also a quasi-(LB)-representation for (F, ξ). By Proposition 3.3.8,

$$U_{r_1,r_2,\ldots,r_s} = T^{-1}(C_{r_1,r_2,\ldots,r_s})$$

is a neighbourhood of zero in E. By

$$s^{-1} U_{r_1,r_2,\ldots,r_s} = s^{-1} T^{-1}(C_{r_1,r_2,\ldots,r_s}) \subset T^{-1}(V),$$

and Proposition 3.3.8, we obtain the continuity of $T : E \to F$. □

This easily yields the classical Banach closed graph theorem.

Corollary 3.3.11 *If E is a Baire lcs, F is a metrizable and complete lcs. If $T : E \to F$ is a linear map with closed graph, T is continuous.*

We provide another interesting property for quasi-(LB)-spaces; see [616, Proposition 22].

Theorem 3.3.12 (Valdivia) *If E is a quasi-(LB)-space, there exists a quasi-(LB)-representation $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that every Banach disc in E is contained in some K_α.*

Proof Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a quasi-(LB)-representation of E. Define $C_{a_1,a_2,\ldots,a_n}$ for $\alpha = (a_k) \in \mathbb{N}^{\mathbb{N}}$. Let $F_{a_1,a_2,\ldots,a_n}$ be the linear span of $C_{a_1,a_2,\ldots,a_n}$ for all $n \in \mathbb{N}$. Set

$$F_\alpha := \bigcap_n F_{a_1,a_2,\ldots,a_n}.$$

Equip the space F_α with the topology ξ_α having a basis of neighbourhoods of zero of the form

$$(F_\alpha \cap n^{-1} C_{a_1,a_2,\ldots,a_n})_n.$$

Clearly, ξ_α is stronger than the original topology induced from E. Moreover, (F_α, ξ_α) is a Fréchet space; see [616, Proposition 21] for details. This procedure provides a family $\{F_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of Fréchet spaces. For $\alpha := (a_n) \in \mathbb{N}^{\mathbb{N}}$, set

$$\alpha_1 := (a_{2n-1}), \quad K_\alpha := \bigcap_n a_{2n}(F_{\alpha_1} \cap C_{a_1,a_3,\ldots,a_{2n-1}}).$$

Then $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a quasi-(LB)-representation of E. Let A be a Banach disc in E, and let F_A be the Banach space associated with A whose Banach topology is generated by the Minkowski functional norm. Consider the natural continuous inclusion $J : F_A \to E$. We show that there exists $\beta = (b_n) \in \mathbb{N}^{\mathbb{N}}$ such that $J : F_A \to F_\beta$ is continuous and $J(A)$ is bounded in F_β. This will provide $\gamma \in \mathbb{N}^{\mathbb{N}}$ with $A \subset K_\gamma$. Set

$$U_{a_1,a_2,\ldots,a_n} := J^{-1}(C_{a_1,a_2,\ldots,a_n})$$

for each $n \in \mathbb{N}$ and all $a_1, a_2, \ldots, a_n \in \mathbb{N}$. Since

$$F_A = \bigcup_{a_1} U_{a_1}, \quad U_{a_1,a_2,\ldots,a_n} = \bigcup_{a_{n+1}} U_{a_1,a_2,\ldots,a_n,a_{n+1}}$$

for each $n \in \mathbb{N}$ and F_A is a Banach space, there exists $\beta = (b_n) \in \mathbb{N}^{\mathbb{N}}$ such that $\overline{U}_{b_1,b_2,\ldots,b_n}$ is a neighbourhood of zero in F_A for each $n \in \mathbb{N}$. By Proposition 3.3.8

$$(\overline{U}_{b_1,b_2,\ldots,b_n})^0 \subset U_{b_1,b_2,\ldots,b_n}.$$

For $x \in F_A$ and $k \in \mathbb{N}$, there exists $t > 0$ such that $tx \in U_{b_1,b_2,\ldots,b_k}$. Hence, $tJ(x) \in C_{b_1,b_2,\ldots,b_k}$, so $J(x) \in F_{b_1,b_2,\ldots,b_k}$. This proves that $J(A) \subset F_\beta$.

Since $J : F_A \to E$ is continuous, $J : F_A \to F_\beta$ has closed graph, hence is continuous by Corollary 3.3.11. Finally, choose a sequence $(r_n)_n$ in $\mathbb{N}$ such that

$$A \subset \bigcap_k r_k(F_\beta \cap C_{b_1,b_2,\ldots,b_k}).$$

Set $\gamma = (w_n)$, where

$$w_{2n-1} := b_n, \;\; w_{2n} := r_n$$

for each $n \in \mathbb{N}$. We showed that $A \subset K_\gamma$. □

We have the following corollary from [616, Corollary 1.6].

Corollary 3.3.13 (Valdivia) *Every Baire lcs that is a quasi-(LB)-space is a Fréchet space.*

3.4 Suslin Schemes

This section deals with a very applicable concept, called a *Suslin scheme*, which provides a powerful tool to obtain several structure theorems for separable metric spaces; see [522] as a good source for this and the next section.

The main result of this section, due to Hurewicz, states that every analytic metrizable topological space that is not a σ-compact space contains a closed subspace that is homeomorphic to the space $\mathbb{N}^{\mathbb{N}}$.

Set

$$\mathbb{N}^{(\mathbb{N})} := \mathbb{N}^0 \cup \bigcup_n \mathbb{N}^n,$$

where $\mathbb{N}^0 := \emptyset$. For $\sigma = (\sigma_n) \in \mathbb{N}^{\mathbb{N}}$ as usual, we set

$$\sigma|0 = \emptyset, \quad \sigma|n := (\sigma_1, \sigma_2, \ldots, \sigma_n).$$

Assume X is an arbitrary set. A *Suslin scheme* on X is a map

$$A(.) : \mathbb{N}^{(\mathbb{N})} \to 2^X.$$

If X is a topological space, the Suslin scheme $A(.)$ is said to be *open* (*closed*) if all values $A(\sigma|n)$ are open (closed) sets for $\sigma|n \in \mathbb{N}^{\mathbb{N}}$.

If (X, d) is a metric space, we say that a Suslin scheme $A(.)$ satisfies the *diameter condition*, if for each $\sigma \in \mathbb{N}^{\mathbb{N}}$, one has

$$\lim_{n\to\infty} \operatorname{diam} A(\sigma|n) = 0.$$

If a Suslin scheme satisfies the diameter condition, $\bigcap_n A(\sigma|n)$ is either the empty set or contains at most one point. Thus, we may define a function $\varphi : Z \to X$ by the formula

$$\varphi(\sigma) = \bigcap_n A(\sigma|n),$$

where

$$Z = \{\sigma \in \mathbb{N}^{\mathbb{N}} : \bigcap_n A(\sigma|n) \neq \emptyset\}.$$

The map φ is called the *map associated* with the Suslin scheme $A(.)$. The first result collects some fundamental properties about Suslin schemes.

Proposition 3.4.1 *The map $\varphi : Z \to (X, d)$ associated with a Suslin scheme $A(.)$ satisfying the diameter condition has the following properties:*

(a) φ is continuous.
(b) If (X, d) is complete and $A(.)$ is closed, Z is a closed subset of $\mathbb{N}^{\mathbb{N}}$.
(c) If for each $\sigma \in \mathbb{N}^{\mathbb{N}}$, $k \in \mathbb{N}$, we have $\bigcap\{A(\sigma|n) : n = 0, 1, \ldots, k\} \neq \emptyset$ and $A(\sigma|k) \subset \bigcup\{A(\sigma|k, q) : q \in \mathbb{N}\}$, then Z is a dense subset of $\mathbb{N}^{\mathbb{N}}$.
(d) If $A(.)$ is open, the map $\varphi : Z \to \varphi(Z)$ is open.
(e) If $[\bigcap_n A(\sigma|n)] \cap [\bigcap_n A(\tau|n)] = \emptyset$ for all $\sigma \neq \tau$, then φ is injective.
(f) If $A(.)$ is open and for each $\sigma|n \in \mathbb{N}^{\mathbb{N}}$ the family $\{A(\sigma|n, q) : q \in \mathbb{N}\}$ consists of pairwise disjoint subsets of $A(\sigma|n)$, then φ is a homeomorphism of Z onto $\varphi(Z)$.
(g) If $A(\emptyset) = X$ and $\{A(\sigma|n, q) : q \in \mathbb{N}\}$ is a covering of $A(\sigma|n)$ for each $\sigma|n \in \mathbb{N}^{\mathbb{N}}$, then $\varphi(Z) = X$.

Proof (a) If σ and τ belong to Z, and $\sigma|n = \tau|n$, then $d(\varphi(\sigma), \varphi(\tau)) \leq diam\ A(\sigma|n)$. This combined with the diameter condition implies the continuity of φ.

(b) Let $(\tau[n])_n$ be a sequence in Z converging to $\sigma \in \mathbb{N}^{\mathbb{N}}$. We may assume that $\tau[n]|m = \sigma|m$ for each $n > m$ in $\mathbb{N}$. Therefore,

$$\varphi(\tau[n]) \in A(\sigma|m)$$

for each $n > m$. By the diameter condition, we deduce that the sequence $(\varphi(\tau[n]))_n$ is a Cauchy sequence. The completeness and closedness imply

$$\lim_{n\to\infty} \varphi(\tau[n])$$

belongs to $A(\sigma|m)$ for each $m \in \mathbb{N}$. Hence, $\sigma \in Z$. Therefore, Z is a closed subset of $\mathbb{N}^{\mathbb{N}}$.

(c) The assumptions from (c) imply that for $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $k \geq 1$, there exists $\tau \in Z$ such that $\sigma|k = \tau|k$. Therefore, Z is a dense subset of $\mathbb{N}^{\mathbb{N}}$.
(d) Let $\sigma \in Z$. The sets

$$C_n = \{\tau \in Z : \tau|n = \sigma|n\},$$

for $n \geq 1$, determine a neighbourhood basis of σ in Z. Since

$$\varphi(C_n) = \left[\bigcap\{A(\sigma|k) : 0 \leq k \leq n\}\right] \cap \varphi(Z),$$

we deduce that φ is open onto the range.

The property (e) is clear, and (f) follows directly from the properties (a), (d), and (e).

If $A(.)$ verifies the property (g), for $x \in X$, there exists $\sigma = (\sigma_n) \in \mathbb{N}^{\mathbb{N}}$ such that $x \in A(\sigma|n)$ for each $n \in \mathbb{N}$. Therefore, $x = \varphi(\sigma)$. □

3.5 Applications of Suslin Schemes to Separable Metrizable Spaces

We consider a couple of concrete cases; see [522] for results of this section.

Case 1. Structure Theorems for Zero-Dimensional Metric Spaces
As usual, $|X|$ denotes the cardinality of a set X. A topological space X is called *zero-dimensional* if X admits a basis of clopen sets, i.e. sets that are closed and open.

It is well known that every non-empty regular topological space X such that $|X| \leq \aleph_0$ is zero-dimensional. Indeed, since X is Lindelöf, it is normal; see Lemma 6.1.3. Then, if $x \in V \subset X$, where V is open, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(X\backslash V) = \{1\}$. From $|X| \leq \aleph_0$, it follows that there exists $r \in I\backslash f(X)$, where $I := [0, 1]$. Obviously,

$$U = f^{-1}([0, r[) = f^{-1}([0, r])$$

is a clopen set that contains x and is contained in V.

Using the concept of the Suslin schemes, we show some structure theorems for separable metrizable zero-dimensional spaces. In the following, d means a metric compatible with the topology of the space X.

Theorem 3.5.1 *If X is a zero-dimensional separable metrizable space, the space X is homeomorphic to a subset Z of N^N. If additionally, the space (X, d) is complete, then X is homeomorphic to a closed subset Z of N^N.*

Proof It is easy to construct a Suslin scheme $A(.)$ on X satisfying the following conditions:

(i) $A(\emptyset) = X$ and, for each $n \geq 1$ and $\sigma \in \mathbb{N}^{\mathbb{N}}$, the set $A(\sigma|n)$ is a clopen set with diam $A(\sigma|n) \leq 2^{-n-1}$.
(ii) $\{A(\sigma|n, q) : q \in \mathbb{N}\}$ is a partition of $A(\sigma|n)$ for each $\sigma|n \in \mathbb{N}^{\mathbb{N}}$.

Then the map φ associated with $A(.)$ is a homeomorphism of Z onto X (compare (f) and (g) from Proposition 3.4.1).

Moreover, by the property (b) of Proposition 3.4.1, if (X, d) is complete, Z is a closed subset of $\mathbb{N}^{\mathbb{N}}$. □

If all the sets $A(\sigma|n)$ from the Suslin scheme $A(.)$ are non-void, the set Z in Theorem 3.5.1 is dense in $\mathbb{N}^{\mathbb{N}}$. This is the case for the next two theorems.

Theorem 3.5.2 is due to Sierpiński; see [522].

Theorem 3.5.2 (Sierpiński) *If X is a countable metric space without isolated points, then X is homeomorphic to the space of rational numbers $\mathbb{Q}$.*

Proof By the above remark, the space X is zero-dimensional. If A is a non-void clopen subset of X, the set A is infinite because X has no isolated points. Set $A := \{a_1, a_2, \ldots, \}$. For $\varepsilon > 0$, let A_1 be a clopen neighbourhood of a_1 such that $A_1 \subset A$ and diam $A_1 \leq \varepsilon$, $A_1 \neq A$. Now set

$$n = \min\{j \in \mathbb{N} : a_j \notin A_1\},$$

and pick a clopen neighbourhood A_2 of a_2 such that $A_2 \subset A \backslash A_1$, and diam $A_2 \leq \varepsilon$, $A_2 \neq A \backslash A_1$. Continuing this way, we construct a sequence of disjoint clopen non-empty sets $(A_n)_n$ such that $A = \bigcup_j A_j$, diam $A_n \leq \varepsilon$ for each $n \in \mathbb{N}$.

Therefore, similarly as in the preceding case, we can construct a Suslin scheme $A(.)$ on X satisfying both properties considered above and, additionally, with each set $A(\sigma|n) \neq \emptyset$ such that:

(i) $A(\emptyset) = X$, and for each $n \geq 1$ and $\sigma \in \mathbb{N}^{\mathbb{N}}$, the set $A(\sigma|n)$ is a non-void clopen set with diam $A(\sigma|n) \leq 2^{-n-1}$.
(ii) $\{A(\sigma|n, q) : q \in \mathbb{N}\}$ is a partition of $A(\sigma|n)$ for each $\sigma|n \in \mathbb{N}^{\mathbb{N}}$.

Then, by Proposition 3.4.1, the map φ associated with $A(.)$ is a homeomorphism of Z onto X (see the properties (f) and (g)), and Z is a dense subset of $\mathbb{N}^{\mathbb{N}}$ by the property (c) of Proposition 3.4.1. This proves that X is homeomorphic to a dense countable subset of $\mathbb{N}^{\mathbb{N}}$, and since $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to the space of the irrational numbers, X is homeomorphic to a countable dense subset of $\mathbb{R}$. Then X is homeomorphic to $\mathbb{Q}$. □

Lemma 3.5.3 leads to a Suslin scheme that will be used in Theorem 3.5.4.

Lemma 3.5.3 *Let (X, d) be a complete metric space such that each compact subset of X has an empty interior. Then for each non-empty open set A, there exists $\varepsilon_A > 0$ such that, if $A \subset \bigcup_q B_q$ and* diam $B_q < \varepsilon_A$, *then $B_q \neq \emptyset$ for infinitely many $q \in \mathbb{N}$.*

Proof If (X, d) is complete, a subset $M \subset X$ is relatively compact if and only if it is totally bounded, that is, for every $\varepsilon > 0$, there is a finite cover of M by sets of diam $M \leq \varepsilon$. By the assumption on A, we deduce that A is not totally bounded, and the conclusion follows. □

The following result is due to Alexandrov and Urysohn; see [522].

Theorem 3.5.4 (Alexandrov–Urysohn) *If (X, d) is a separable, complete, and zero-dimensional metric space such that each compact subset of X has an empty interior, then X is homeomorphic to $\mathbb{N}^{\mathbb{N}}$.*

Proof Lemma 3.5.3 can be used to construct on X a Suslin scheme $A(.)$ such that:

(i) $A(\emptyset) = X$, and for each $n \geq 1$, $\sigma \in \mathbb{N}^{\mathbb{N}}$, the set $A(\sigma|n)$ is a non-void clopen set with diam $A(\sigma|n) \leq 2^{-n-1}$.
(ii) $\{A(\sigma|n, q) : q \in \mathbb{N}\}$ is a partition of $A(\sigma|n)$ for each $\sigma|n \in \mathbb{N}^{(\mathbb{N})}$.

Using the properties (b), (c), and (f) of Proposition 3.4.1, we can see that the map φ associated with $A(.)$ is a homeomorphism from $\mathbb{N}^{\mathbb{N}}$ onto X. □

Case 2. Structure Theorems for Separable Metric Spaces

Theorem 3.5.5 *Let (X, d) be a separable metric space. Then X is a continuous open image of a dense subset Z of $\mathbb{N}^{\mathbb{N}}$. If (X, d) is complete, the space X is a continuous open image of $\mathbb{N}^{\mathbb{N}}$. Hence, every Polish space X is a continuous open image of $\mathbb{N}^{\mathbb{N}}$.*

Proof It is straightforward to construct on (X, d) an open Suslin scheme $B(.)$ such that:

(i) $B(\emptyset) = X$, and for $n \geq 1$ and $\sigma \in \mathbb{N}^{\mathbb{N}}$, one has $B(\sigma|n) \neq \emptyset$ and diam $B(\sigma|n) \leq 2^{-n-1}$.
(ii) $B(\sigma|n) = \bigcup_q B(\sigma|n, q) = \bigcup_q \overline{B(\sigma|n, q)}$ for each $\sigma|n \in \mathbb{N}^{\mathbb{N}}$.

Clearly, the map φ associated with $B(.)$ coincides with the one that is associated with the Suslin scheme $\overline{B(.)}$. By the conditions (a), (c), (d), and (g) of Proposition 3.4.1, we deduce that φ is continuous and open from Z onto X, where Z is a dense subset of $\mathbb{N}^{\mathbb{N}}$.

Finally, if (X, d) is complete, then $Z = \mathbb{N}^{\mathbb{N}}$ by applying (b) of Proposition 3.4.1 to the Suslin scheme $\overline{B(.)}$ for which the associated map is φ. □

For another result of this type, we need the following lemma.

Lemma 3.5.6 *If C is a $\mathcal{F}_\sigma$-set in a separable metric space $X := (X, d)$, and $\varepsilon > 0$, there exists a sequence $(C_k)_k$ of pairwise disjoint $\mathcal{F}_\sigma$-sets such that $\overline{C_k} \subset C$, diam $C_k < \varepsilon$, and $C = \bigcup_k C_k$.*

Proof By the assumption, there exists a sequence $(G_n)_n$ of closed sets in X such that $C = \bigcup_n G_n$. Then

$$C = G_1 \cup \left(\bigcup_n (G_{n+1} \setminus \bigcup_{m \le n} G_m) \right).$$

Now the conclusion follows from the following two general observations:

(a) If D is a closed subset of X, then D and $X \setminus D$ are covered by two countable families $\{D_n : n \in \mathbb{N}\}$ and $\{E_n : n \in \mathbb{N}\}$, respectively, of closed subsets of X of diameter less than ε. Then

$$D = \bigcup_n (D_n \setminus \bigcup_{m<n} D_m),$$

and

$$X \setminus D = \bigcup_n (E_n \setminus \bigcup_{m<n} E_m)$$

are the union of the sequences of $\mathcal{F}_\sigma$-sets.

(b) If F is a closed set, we have

$$F \setminus D = \bigcup_n (F \cap E_n \setminus \bigcup_{m<n} E_m).$$

□

Theorem 3.5.7 *Let (X, d) be a separable metric space. Then X is a continuous injective image of a subset of Z of $\mathbb{N}^{\mathbb{N}}$. If (X, d) is complete, then X is a continuous injective image of a closed subset of $\mathbb{N}^{\mathbb{N}}$. Hence, every Polish space X is a continuous injective image of a closed subset of $\mathbb{N}^{\mathbb{N}}$.*

Proof Lemma 3.5.6 applies to get a Suslin scheme $C(.)$ such that:

(i) $C(\emptyset) = X$, and $\operatorname{diam} C(\sigma|n) \le 2^{-n-1}$ for $n \ge 1$ and $\sigma \in \mathbb{N}^{\mathbb{N}}$.
(ii) For $\sigma|n$ and $\tau|n$ in $\mathbb{N}^{\mathbb{N}}$ with $\sigma|n \ne \tau|n$, we have $C(\sigma|n) \cap C(\tau|n) = \emptyset$.
(iii) $C(\sigma|n) = \bigcup_q C(\sigma|n, q)) = \bigcup_q \overline{C(\sigma|n, q))}$ for each $\sigma|n \in \mathbb{N}^{\mathbb{N}}$.

The map φ associated with $C(.)$ coincides with the map associated with the Suslin scheme $\overline{C(.)}$. By the conditions (a), (e), and (g) of Proposition 3.4.1, we conclude that φ is an injective and continuous map from Z onto X, where Z is a subset of $\mathbb{N}^{\mathbb{N}}$. If (X, d) is complete, Z is closed by (b) of Proposition 3.4.1 applied to the Suslin scheme $\overline{C(.)}$. □

A sequence $(A_n)_n$ of subsets of a topological space X *converges* to an element $a \in X$ if the filter base $\bigcup\{A_p : p \ge n\}$ converges to a; we will write $a = \lim_{n\to\infty} A_n$. We need a couple of additional lemmas; see [522].

Lemma 3.5.8 *Let $H(.)$ be an open Suslin scheme on a complete metric space (X, d) such that for each $(\sigma|n) \in \mathbb{N}^{\mathbb{N}}$*

$$\operatorname{diam} H(\sigma|n+1) \leq 2^{-n-1}, \tag{3.24}$$

$$\overline{H(\sigma|n+1)} \subset H(\sigma|n), \tag{3.25}$$

and $H(.)$ verifies the condition

$$x(\sigma|n) := \lim_{q\to\infty} H(\sigma|n,q) \in H(\sigma|n) \setminus \bigcup_q \overline{H(\sigma|n,q)}, \tag{3.26}$$

and if $\tau|n \neq \sigma|n$, then

$$\overline{H(\sigma|n)} \cap \overline{H(\tau|n)} = \emptyset. \tag{3.27}$$

Then the set $D := \{x(\sigma|n)\dot{} : (\sigma|n) \in \mathbb{N}^{(\mathbb{N})}\}$ is homeomorphic to the topological space $\mathbb{Q}$ of the rational numbers, and $\overline{D}\setminus D$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

Proof From (3.26), it follows that $x(\sigma|n) = \lim_{q\to\infty} x(\sigma|n,q)$ and $x(\sigma|n) \neq x(\sigma|n,q)$. Theorem 3.5.2 applies to deduce that D is homeomorphic to $\mathbb{Q}$.

From (3.24), (3.25), (3.26), and (3.27), we deduce that for each $\sigma \in \mathbb{N}^{\mathbb{N}}$, the sequence $(x(\sigma|n))_n$ is a Cauchy sequence whose limit $x(\sigma)$ verifies

$$\{x(\sigma)\} = \{\lim_{n\to\infty} x(\sigma|n)\} = \bigcap_n H(\sigma|n) = \bigcap_n \overline{H(\sigma|n)}, \tag{3.28}$$

and if $\tau \in \mathbb{N}^{\mathbb{N}} \setminus \{\sigma\}$, then $x(\sigma) \neq x(\tau)$. Therefore, the equality $x(\sigma) = \lim_{n\to\infty} x(\sigma|n)$ defines a bijection $x(.)$ from $\mathbb{N}^{\mathbb{N}}$ onto $\{x(\sigma) : \sigma \in \mathbb{N}^{\mathbb{N}}\}$.

From (3.24) and (3.25), we derive that the map $x(.)$ is continuous. The formulas (3.25) and (3.27) imply that $x(\tau) \in H(\sigma|n)$ holds for a point $\tau \in \mathbb{N}^{\mathbb{N}}$ if and only if $\tau|n = \sigma|n$. Therefore, the map $x(.)$ is open, because if

$$A_n := \{\tau \in \mathbb{N}^{\mathbb{N}} : \tau|n = \sigma|n\},$$

then $\{A_n : n \in \mathbb{N}\}$ is a neighbourhood basis of σ, and also $\{H(\sigma|n) : n \in \mathbb{N}\}$ is a neighbourhood basis of $x(\sigma)$.

Therefore, $x(.)$ is a homeomorphism from $\mathbb{N}^{\mathbb{N}}$ onto $\{x(\sigma) : \sigma \in \mathbb{N}^{\mathbb{N}}\}$. If $t \in \overline{D}\setminus D$, there exist two sequences $(\sigma(m))_m$, $(n(m))_m$, in $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}$, respectively, such that $\lim_{m\to\infty} x(\sigma(m)|n(m)) = t$. Let

$$P_q : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$$

be the projection defined by

$$P_q((a_1, a_2, \ldots, a_n, \ldots)) = a_q.$$

From (3.26), it follows that the set $\{n(m), m \in \mathbb{N}\}$ is unbounded, and each set

$$\{P_q(\sigma(m)|n(m)), m \in \mathbb{N}, n(m) \geq q\}$$

is bounded. The diagonal method, taking an adequate sequence if necessary, enables us to assume that

$$P_q(\sigma(m)) = \beta_q$$

if $m \geq q$. If $\beta = (\beta_1, \beta_2, \dots)$, from (3.24) and (3.25) we conclude that

$$x(\beta) = \lim_{m\to\infty} x(\beta|m) = \lim_{m\to\infty} x(\sigma(m)|n(m)) = t.$$

This proves that $\overline{D}\backslash D \subset x(\mathbb{N}^{\mathbb{N}})$. Conversely, if $\sigma \in \mathbb{N}^{\mathbb{N}}$, then by (3.28), we have $x(\sigma) \in \overline{D}$. Since

$$\{x(\sigma)\} = \bigcap_n H(\sigma|n),$$

(3.26) and (3.27), ensure that for each $n \in \mathbb{N}$ and $\tau \in \mathbb{N}^{\mathbb{N}}\backslash\{\sigma\}$, one has

$$x(\sigma|n) \neq x(\sigma) \neq x(\tau|n).$$

This proves that $x(\mathbb{N}^{\mathbb{N}}) \subset \overline{D}\backslash D$. □

A topological space X is called *analytic* if X is a continuous image of a Polish space (or equivalently, of the space $\mathbb{N}^{\mathbb{N}}$). Clearly, every analytic space is K-analytic.

Lemma 3.5.9 *Let X be a topological space. Let Y be an analytic subspace of X which is not a $\mathcal{F}_\sigma$-subset of X. Then there exist a non-empty Polish space T, a continuous map $\theta : T \to Y$ such that $\overline{\theta(G)}^X \backslash Y \neq \emptyset$ for each non-empty open subset G of T.*

Proof Since Y is analytic, there exists a Polish space S and a continuous surjection θ_0 of S onto Y. Let $\mathcal{B}$ be the family of all open subsets of S such that $B \in \mathcal{B}$ if there exists in X an $\mathcal{F}_\sigma$-subset A such that

$$\theta_0(B) \subset A \subset Y.$$

Let G_0 be the union of the sets of the family $\mathcal{B}$. Clearly, G_0 is the union of a countable family of elements of $\mathcal{B}$. This implies that $G_0 \in \mathcal{B}$, and then there exists in X a $\mathcal{F}_\sigma$-subset A_0 such that

$$\theta_0(G_0) \subset A_0 \subset Y.$$

The assumption on Y ensures $A_0 \neq Y$. The set $T := S\backslash\theta_0^{-1}(A_0)$ is a non-void G_δ-subset of S, and then T is a Polish space. If for a non-void open subset G of T one has $\overline{\theta_0(G)}^X \subset Y$, then $G_0 \cup G \in \mathcal{B}$, a contradiction with the definition of G_0. Hence,

$$\overline{\theta_0(G)}^X \backslash Y \neq \emptyset.$$

To complete the proof, we define θ as the map θ_0 restricted to T. □

We need also the following:

Lemma 3.5.10 *Let X be a metric space. Let Y be an analytic subspace of X that is not a $\mathcal{F}_\sigma$-subset. Let T and θ be chosen as in Lemma 3.5.9. Then there exist two open Suslin schemes $G(.)$ in T and $H(.)$ in X, respectively, such that for each $\sigma \in \mathbb{N}^{\mathbb{N}}$ and each $n \in \mathbb{N}$*

$$\operatorname{diam} G(\sigma|n+1) \leq 2^{-n-1}, \tag{3.29}$$

$$\operatorname{diam}\ H(\sigma|n+1) \leq 2^{-n-1}, \tag{3.30}$$

$$\overline{G(\sigma|n+1)} \subset G(\sigma|n), \tag{3.31}$$

$$\overline{H(\sigma|n+1)} \subset H(\sigma|n), \tag{3.32}$$

$$\overline{\theta(G(\sigma|n))} \subset H(\sigma|n). \tag{3.33}$$

Furthermore, $H(.)$ verifies the following convergence condition

$$x(\sigma|n) := \lim_{q\to\infty} H(\sigma|n,q) \in \{\overline{\theta(G(\sigma|n))}\backslash Y\}\backslash \bigcup_q \overline{H(\sigma|n,q)}, \tag{3.34}$$

and if $\sigma|n \neq \tau|n$, then

$$\overline{H(\sigma|n)} \cap \overline{H(\tau|n)} = \emptyset. \tag{3.35}$$

Proof The construction will be provided by the induction.

Step 1. By Lemma 3.5.9, for the sets $G(\emptyset) := T$ and $H(\emptyset) := X$, there exists a point $x(\emptyset)$ such that

$$x(\emptyset) \in \overline{\theta(G(\emptyset))}\backslash Y. \tag{3.36}$$

From (3.36), there exists a sequence $(z(n))_n$ in $\theta(G(\emptyset))$ such that

$$x(\emptyset) = \lim_{n\to\infty} z(n).$$

Then there exists a sequence $(H(n))_n$ of open neighbourhoods of the points $z(n)$, $n \in \mathbb{N}$, with pairwise disjoint closures contained in

$$X \backslash \{x(\emptyset)\} = H(\emptyset) \backslash \{x(\emptyset)\},$$

and such that for each $n \in \mathbb{N}$

$$\operatorname{diam} H(n) \leq 2^{-1}, \tag{3.37}$$

and the sequence $(H(n))_n$ converges to $x(\emptyset)$, that is

$$x(\emptyset) = \lim_{n \to \infty} H(n). \tag{3.38}$$

Then clearly

$$x(\emptyset) \in \{\overline{\theta(G(\emptyset))} \backslash Y\} \backslash \bigcup_q \overline{H(q)}. \tag{3.39}$$

The continuity of θ enables us to determine a sequence $(G(n))_n$ of open subsets of T such that for each $n \in \mathbb{N}$

$$\operatorname{diam} G(n) \leq 2^{-1}, \tag{3.40}$$

$$\overline{\theta(G(n))} \subset H(n). \tag{3.41}$$

Lemma 3.5.9 implies that $\overline{\theta(G(n))} \backslash Y \neq \emptyset$, and therefore we may choose

$$x(n) \in \overline{\theta(G(n))} \backslash Y \tag{3.42}$$

for each $n \in \mathbb{N}$. Then from (3.38) and (3.41), it follows that

$$x(\emptyset) = \lim_{n \to \infty} x(n). \tag{3.43}$$

Step 2. Assume that for each $n \in \mathbb{N}$ we already constructed the sets $G(n)$ and $H(n)$ and the point $x(n)$ verifying (3.41) and (3.42). Therefore, there exists an injective sequence $(z(n, p))_p$ in $\theta(G(n))$ such that $x(n) = \lim_{p \to \infty} z(n, p)$. We may determine a sequence $(H(n, p))_p$ of open neighbourhoods of the points $z(n, p)$, $p \in \mathbb{N}$, with the pairwise disjoint closures contained in $H(n) \backslash \{x(n)\}$, and such that for each $p \in \mathbb{N}$

$$\operatorname{diam} H(n, p) \leq 2^{-2}, \tag{3.44}$$

and the sequence $(H(n, p))_p$ verifies the following convergence condition:

$$x(n) = \lim_{p \to \infty} H(n, p). \tag{3.45}$$

It is obvious that

$$x(n) \in \{\overline{\theta(G(n))} \backslash Y\} \backslash \bigcup_q \overline{H(n,p)}. \tag{3.46}$$

The continuity of θ provides a sequence $(G(n,p))_p$ of open subsets of $G(n)$ such that for each $p \in \mathbb{N}$

$$\operatorname{diam} G(n,p) \leq 2^{-2}, \tag{3.47}$$

$$\overline{\theta(G(n,p))} \subset H(n,p). \tag{3.48}$$

Lemma 3.5.9 implies that $\overline{\theta(G(n,p))} \backslash Y \neq \emptyset$, and therefore we may choose

$$x(n,p) \in \overline{\theta(G(n))} \backslash Y \tag{3.49}$$

for each $n, p \in \mathbb{N}$. Then from (3.45), (3.48), and 3.49, it follows that

$$x(n) = \lim_{p\to\infty} x(n,p), \tag{3.50}$$

and this completes the second step of the induction process. Now it is obvious how to complete the inductive process. □

We are ready to prove the following:

Theorem 3.5.11 *Let X be a complete metric space. Let Y be an analytic subspace of X that is not a $\mathcal{F}_\sigma$-subset. Then there exists a closed set $F \subset X$ such that:*

(a) $P = F \cap Y$ *is homeomorphic to* $\mathbb{N}^{\mathbb{N}}$.
(b) $Q = F \backslash Y$ *is homeomorphic to* $\mathbb{Q}$.
(c) $\overline{Q} = F$.

Proof Let T and θ be chosen as in Lemma 3.5.9. Let $G(.)$ and $H(.)$ be the Suslin schemes provided in Lemma 3.5.10, and let $x(\sigma|n) = \lim_{q\to\infty} H(\sigma|n,q)$; see (3.34). From (3.43), (3.50), and Theorem 3.5.2, it follows that the set $Q := \{x(\sigma|n) : n \in \mathbb{N}^{(\mathbb{N})}\}$ is homeomorphic to the space $\mathbb{Q}$ of rational numbers. By (3.36) and (3.42), we have $Q \subset X \backslash Y$.

Lemma 3.5.8 implies that the map $x(.) : \mathbb{N}^{\mathbb{N}} \to \overline{Q} \backslash Q$ given by $x(\sigma) = \lim_{n\to\infty} x(\sigma|n)$ is a homeomorphism. Now we prove that $\overline{Q} \backslash Q \subset Y$. This will show that

$$\overline{Q} \cap Y = \overline{Q} \backslash Q,$$

and the theorem will be proved. If $y \in \overline{Q} \backslash Q = \varphi(\mathbb{N}^{\mathbb{N}})$, there exists $\sigma \in \mathbb{N}^{\mathbb{N}}$ with

$$y = \varphi(\sigma) = \lim_{n\to\infty} x(\sigma|n).$$

By (3.34) we note that

$$x(\sigma|n) \in \overline{\theta(G(\sigma|n)}.$$

Therefore, there exists $t_n \in G(\sigma|n)$ such that

$$d(x(\sigma|n), \theta(t_n)) \leq 2^{-n},$$

and we have

$$y = \lim_{n\to\infty} \theta(t_n).$$

Clearly, the sequence $(t_n)_n$ verifies the Cauchy condition (see (3.29) and (3.31)). By the assumption, the space T is Polish, so there exists $t \in T$ such that $t = \lim_{n\to\infty} t_n$. By the continuity,

$$y = \lim_{n\to\infty} \theta(t_n) = \theta(t) \in \theta(T) \subset Y.$$

□

We complete this section with the following result of Hurewicz.

Theorem 3.5.12 (Hurewicz) *If Y is an analytic metrizable space that is not σ-compact, then Y contains $\mathbb{N}^{\mathbb{N}}$ as a closed subset.*

Proof Clearly, Y has a countable base. Hence, Y is a normal topological space. Therefore, Y may be embedded in a topological space X homeomorphic to a product of a countable family of intervals $[0, 1]$. Since Y is not a σ-compact space, Y is not a $\mathcal{F}_\sigma$-space in X, and the conclusion follows from Theorem 3.5.11. □

3.6 Calbrix–Hurewicz's Theorem

The aim of this section is to present an extension of the Hurewicz theorem 3.5.12. We show a theorem, due to Calbrix [113], which in particular case, states that, if X is a regular analytic topological space that is not σ-compact, then X contains a closed subset that is homeomorphic with the space $\mathbb{N}^{\mathbb{N}}$. We start with a couple of additional lemmas; see [522]. The first one is similar to Lemma 3.5.9. Recall that a topological space X is hereditary Lindelöf if each subspace of X is Lindelöf.

Lemma 3.6.1 *Let X be a hereditarily Lindelöf subspace of a topological space Y such that X is not contained in a countable union of compacts subsets of Y. Then X has a non-void closed subset F such that each non-void open subset of F is not contained in a countable union of compacts subsets of Y.*

Proof Let B be the union of all open subsets of X such that each of them is contained in a countable union of compacts subsets of Y. Since B is a Lindelöf space, B is the maximal open subset of X contained in a countable union of compacts subsets of Y. Then $F := X \backslash B$ is non-void and closed and verifies the lemma. □

For a K-analytic space X, let $K : \mathbb{N}^{\mathbb{N}} \to 2^X$ be an (usco) compact-valued map. We recall again our notations: For $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n, \dots) \in \mathbb{N}^{\mathbb{N}}$, set $\sigma|0 := \emptyset$ and $\sigma|n := (\sigma_1, \sigma_2, \dots, \sigma_n)$ for $n \in \mathbb{N}$. Define

$$I(\sigma|n) = \{\tau \in \mathbb{N}^{\mathbb{N}} : \tau|n = \sigma|n\}, \tag{3.51}$$

where

$$I(\sigma|0) = I(\emptyset) = \mathbb{N}^{\mathbb{N}}. \tag{3.52}$$

The sets $I(\sigma|n)$ compose a basis of neighbourhoods of σ in $\mathbb{N}^{\mathbb{N}}$. The map $M(.) : \mathbb{N}^{\mathbb{N}} \to 2^X$, called *the Suslin scheme associated with the (usco)* map K, is defined by $M(\sigma|n) := K(I(\sigma|n))$. Clearly,

$$M(\sigma|0) = M(\emptyset) = X \tag{3.53}$$

and

$$M(\sigma|n) = \bigcup\{K(\tau) : \tau \in \mathbb{N}^{\mathbb{N}}, \tau|n = \sigma|n\}. \tag{3.54}$$

From the upper semicontinuity of K, we deduce that $K(\sigma) = \bigcap_n M(\sigma|n)$. If moreover X is regular, then $K(\sigma)$ is the intersection of its closed neighbourhoods. Therefore, if X is a regular K-analytic space,

$$K(\sigma) = \cap\{\overline{M(\sigma|n)} : n \in \mathbb{N}\}. \tag{3.55}$$

We also need the following lemma.

Lemma 3.6.2 *Let X be a K-analytic space. Let $M(.)$ be a Suslin scheme associated with an (usco) map $K : \mathbb{N}^{\mathbb{N}} \to 2^X$. Let $\mathcal{B}$ be a family of subsets of X satisfying the non-void finite intersection property. If there exists $\sigma \in \mathbb{N}^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$ there exists $B_n \in \mathcal{B}$ with $B_n \subset M(\sigma|n)$, then*

$$\left[\bigcap\{\overline{B}^X : B \in \mathcal{B}\}\right] \cap K(\sigma) \neq \emptyset. \tag{3.56}$$

Moreover, if X is a topological subspace of Y, we have

$$\bigcap\{\overline{B}^X : B \in \mathcal{B}\} = \bigcap\{\overline{B}^Y : B \in \mathcal{B}\}. \tag{3.57}$$

Proof If $\left[\bigcap\{\overline{B}^X : B \in \mathcal{B}\}\right] \cap K(\sigma) = \emptyset$, the compactness of $K(\sigma)$ yields a finite family $\mathcal{B}_0 \subset \mathcal{B}$ such that $K(\sigma) \subset \bigcup\{X\backslash\overline{B}^X : B \in \mathcal{B}_0\}$. By the upper semicontinuity of K, there exists $n \in \mathbb{N}$ such that

$$K(I(\sigma|n)) = M(\sigma|n) \subset \bigcup\{X\backslash\overline{B}^X : B \in \mathcal{B}_0\}.$$

Since $B_n \subset M(\sigma|n)$, we have $B_n \subset \bigcup\{X\backslash\overline{B}^X : B \in \mathcal{B}_0\}$, contradicting the finite intersection property of $\mathcal{B}$ This proves (3.56).

To prove (3.57), let $y \in \bigcap\{\overline{B}^Y : B \in \mathcal{B}\}$, and let $\mathcal{C}$ be the family of all intersections of neighbourhoods of y in Y with sets of $\mathcal{B}$. Applying (3.56) to the family $\mathcal{C}$, we deduce that $\left[\bigcap\{\overline{C}^X : C \in \mathcal{C}\}\right] \cap K(\sigma) \neq \emptyset$. Since $\bigcap\{\overline{C}^Y : C \in \mathcal{C}\} = \{y\}$, $y \in K(\sigma) \subset X$. This shows (3.57) □

Lemma 3.6.3 *Let $K : \mathbb{N}^{\mathbb{N}} \to 2^X$ be an (usco) map of a regular K-analytic space X, and let $M(.)$ be a Suslin scheme associated with K. Assume that for each $\sigma \in \mathbb{N}^{\mathbb{N}}$, there exists a sequence $(A(\sigma|n))_n$ of subsets of X such that*

$$A(\sigma|n) \subset M(\sigma|n) \tag{3.58}$$

and

$$A(\sigma|n+1) \subset A(\sigma|n). \tag{3.59}$$

Then $A(\sigma) := \bigcap_n \overline{A(\sigma|n)}$ is compact, and if U is a neighbourhood of $A(\sigma)$, there exists $m \in \mathbb{N}$ such that

$$\overline{A(\sigma|m)} \subset U. \tag{3.60}$$

Therefore, if for each $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$ the sets $A(\sigma|n)$ satisfy (3.58) and (3.59), then

$$A := \bigcup\{A(\sigma) : \sigma \in \mathbb{N}^{\mathbb{N}}\} \tag{3.61}$$

is K-analytic. The map $P : \mathbb{N}^{\mathbb{N}} \to 2^A$ defined by $P(\sigma) := A(\sigma)$ is (usco), $P(\mathbb{N}^{\mathbb{N}}) = A$, and $P(\sigma) \subset K(\sigma)$ for each $\sigma \in \mathbb{N}^{\mathbb{N}}$.

Proof From (3.55), it follows the compactness of $A(\sigma)$. If U is an open set that contains $A(\sigma)$ and $\overline{A(\sigma|n)} \cap (X\backslash U) \neq \emptyset$ for each $n \in \mathbb{N}$, then applying (3.56) from Lemma 3.6.2, we note

$$\left[\bigcap_n \overline{A(\sigma|n)} \cap (X\backslash U)^X\right] \cap K(\sigma) \neq \emptyset.$$

On the other hand,

$$\left[\bigcap_n \overline{A(\sigma|n)} \cap (X\backslash U)^X\right] \cap K(\sigma) \subset A(\sigma) \cap (X\backslash U) = \emptyset.$$

This contradiction yields $m \in \mathbb{N}$ such that (3.60) holds. Then the K-analyticity of A and the desired properties of the map P are clear. □

Next the lemma is related with Lemma 3.6.1. The proof is given for the sake of completeness.

Lemma 3.6.4 *Let X be a hereditarily Lindelöf subspace of Y such that $\overline{X}$ is not a σ-compact subset of Y. Then X has a non-void closed subset F such that for each non-void open subset A of F, the set $\overline{A}$ is not a σ-compact subset of Y.*

Proof Let $\mathcal{B}$ be the family of all open subsets of X such that $C \in \mathcal{B}$ if and only if $\overline{C}$ is a σ-compact subset of Y. Set $B := \bigcup\{C : C \in \mathcal{B}\}$. As B is Lindelöf, then $\overline{B}$ is a σ-compact subset of Y. Therefore, B is the maximal open subset of X such that $\overline{B}$ is a σ-compact subset of Y. Clearly, the non-void closed subset $F = X\backslash B$ is as required. □

Lemma 3.6.5 *Let $K : \mathbb{N}^{\mathbb{N}} \to 2^X$ be an (usco) map for a K-analytic regular hereditarily Lindelöf space X, and let $M(.)$ be a Suslin scheme associated with K. Assume that X is a subspace of the hereditarily paracompact topological space Y, and let A be a subset of X such that $\overline{A}^Y$ (the closure in Y) is not a σ-compact subset of Y. If $s \in \mathbb{N}^{\mathbb{N}}$ such that $A \subset M(s)$ and, G is an open subset of X such that $\overline{A}^X \subset G$, then there exist three sequences $(A_n)_n$, $(G_n)_n$ and $(k_n)_n$, such that each $k_n \in \mathbb{N}$, and for each $n \in \mathbb{N}$*

(1) $A_n \subset A$ such that $\overline{A_n}^Y$ is not a σ-compact subset of Y,
(2) $G_n \subset G$ is open,
(3) $A_n \subset M((s, k_n))$,
(4) $\overline{A_n}^X \subset G_n$,
(5) $\overline{G_n}^Y \cap \overline{G_m}^Y = \emptyset$, if $n \neq m$
(6) $\{\overline{G_n}^Y : n \in \mathbb{N}\}$ is a locally finite family in Y.

Proof Changing A by a suitable subset if necessary, we can use Lemma 3.6.4 to assume that for each non-void open subset B of A, the set $\overline{B}^Y$ is not a σ-compact subset of Y. Hence, $\overline{A}^Y$ is not a compact subspace of Y. Therefore, there exists an open covering $\{L'_j : j \in J\}$ of $\overline{A}^Y$ such that no finite subfamily of $\{L'_j : j \in J\}$ covers $\overline{A}^Y$. Set $L := \bigcup\{L'_i : j \in J\}$. Then, since L is paracompact, there exists a locally finite open covering $\{L_i : i \in I\}$ of L that refines $\{L'_j : j \in J\}$, i.e. for each L_i, $i \in I$, there exists $j \in J$ such that $L_i \subset L'_j$. Hence, no finite subfamily of $\{L_i : i \in I\}$ covers $\overline{A}^Y$. If $x_1 \in A$, there exists $i_1 \in I$ such that $x_1 \in L_{i_1}$, and we

choose an open neighbourhood G_1 of x_1 in X such that $G_1 \subset G$ and $\overline{G_1}^Y \subset L_{i_1}$. As no finite subfamily of $\{L_i : i \in I\}$ covers $\overline{A}^Y$, there exists $z_2 \in \overline{A}^Y \backslash L_{i_1}$, and therefore, we may find $i_2 \in I\backslash\{i_1\}$ such that $z_2 \in \overline{A}^Y \cap L_{i_2}$. Since

$$z_2 \in \overline{A}^Y \cap [L_{i_2}\backslash\overline{G_1}^Y],$$

there exists

$$x_2 \in A \cap [L_{i_2}\backslash\overline{G_1}^Y].$$

Then there exists an open neighbourhood G_2 of x_2 in X such that $G_2 \subset G$ and $\overline{G_2}^Y \subset L_{i_2}\backslash\overline{G_1}^Y$. Choose

$$z_3 \in \overline{A}^Y \backslash \{L_{i_1} \cup L_{i_2}\}$$

and $i_3 \in I\backslash\{i_1, i_2\}$ such that $z_3 \in \overline{A}^Y \cap L_{i_3}$. Since

$$z_3 \in \overline{A}^Y \cap \left[L_{i_3}\backslash\{\overline{G_1}^Y \cup \overline{G_2}^Y\}\right],$$

there exists

$$x_3 \in A \cap \left[L_{i_3}\backslash\{\overline{G_1}^Y \cup \overline{G_2}^Y\}\right].$$

Therefore, there exists an open neighbourhood G_3 of x_3 in X such that $G_3 \subset G$ and

$$\overline{G_3}^Y \subset L_{i_3}\backslash\{\overline{G_1}^Y \cup \overline{G_2}^Y\}.$$

Since $\overline{A}^Y \backslash \{L_{i_1} \cup L_{i_2} \cup L_{i_3}\} \neq \emptyset$, we continue this process to determine a sequence $(x_n)_n$ in A, a sequence $(G_n)_n$ such that each G_n is an open neighbourhood of x_n in X contained in G, and a sequence $(L_{i_n})_n$ such that for $m \neq n$

$$L_{i_m} \neq L_{i_n} \tag{3.62}$$

and

$$\overline{G_n}^Y \subset L_{i_n}\backslash\{\overline{G_1}^Y \cup \overline{G_2}^Y \cup \cdots \cup \overline{G_{n-1}}^Y\}.$$

By (3.62), and since $\overline{G_n}^Y \subset L_{i_n}$ for each $n \in \mathbb{N}$, we deduce that the family $\{\overline{G_n}^Y : n = 1, 2, \dots\}$ is locally finite.

For each $n \in \mathbb{N}$, let H_n be a neighbourhood of x_n in X such that $\overline{H_n}^X \subset G_n$. The assumption from the beginning of the proof implies that $\overline{H_n \cap A}^Y$ is not a σ-compact

subset of Y. Since $A \subset M(s) := \bigcup_k M(s,k)$ and $H_n \cap A = \bigcup_k [H_n \cap A \cap M(s,k)]$, there exists k_n such that the closure (in Y) of the set $A_n := H_n \cap A \cap M(s,k_n)$ is not a σ-compact subset of Y. We constructed the sequences $(A_n)_n$, $(G_n)_n$, and $(k_n)_n$ as required. □

Lemma 3.6.5 enables us to construct special K-analytic subspaces in certain K-analytic spaces.

Lemma 3.6.6 *Let X be a regular hereditarily Lindelöf K-analytic space that is a subspace of a hereditarily paracompact space Y. If $\overline{X}$ is not a σ-compact subset of Y, there exists a K-analytic subspace $A \subset X$ closed in Y, and a continuous map $f : A \to \mathbb{N}^{\mathbb{N}}$ such that the map $f^{-1} : \mathbb{N}^{\mathbb{N}} \to 2^A$ is an (usco) map, and $f^{-1}(\mathbb{N}^{\mathbb{N}}) = A$. If additionally X is analytic, f is a homeomorphism.*

Proof Let K be an (usco) map defining the K-analyticity of X. Let $M(.)$ a Suslin scheme associated with K.

Applying Lemma 3.6.5 for $A = G = M(\emptyset) = X$, we obtain:

$(1')$ A sequence $(A(\sigma_1))_{\sigma_1}$ such that each $A(\sigma_1)$ is a subset of A with $\overline{A(\sigma_1)}^Y$ not σ-compact subset of Y.

$(2')$ A sequence $(G(\sigma_1))_{\sigma_1}$ such that each $G(\sigma_1)$ is open in $G := X$.

$(3')$ A sequence $(k_{\sigma_1})_{\sigma_1}$ of natural numbers such that $A(\sigma_1) \subset M(k_{\sigma_1})$, $\overline{A(\sigma_1)}^X \subset G(\sigma_1)$ for each $\sigma_1 \in \mathbb{N}$, and the family $\{\overline{G_{\sigma_1}}^Y, \sigma_1 = 1, 2, \dots\}$ is locally finite in Y of pairwise disjoint subsets.

Applying again Lemma 3.6.5 with $A = A(\sigma_1)$, $G = G(\sigma_1)$ and $M(s) = M(k_{\sigma_1})$ we obtain:

$(1'')$ A sequence $(A(\sigma_1, \sigma_2))_{\sigma_2}$ such that each $A(\sigma_1, \sigma_2)$ is a subset of $A(\sigma_1)$ with $\overline{A(\sigma_{11}, \sigma_2))}^Y$ and not σ-compact in Y.

$(2'')$ A sequence $(G(\sigma_1, \sigma_2))_{\sigma_2}$ such that each $G(\sigma_1, \sigma_2)$ is an open subset of $G(\sigma_1)$.

$(3'')$ A sequence of natural numbers $(k_{\sigma_1\sigma_2})_{\sigma_2}$ with $A(\sigma_1, \sigma_2) \subset M(k_{\sigma_1}, k_{\sigma_1\sigma_2})$ for each $\sigma_2 \in \mathbb{N}$, and

$$\overline{A(\sigma_1, \sigma_2)}^X \subset G(\sigma_1, \sigma_2), \tag{3.63}$$

and such that

$$\{\overline{G(\sigma_1, \sigma_2)}^Y, \sigma_2 = 1, 2, \dots\} \tag{3.64}$$

is a locally finite family in Y of pairwise disjoint subsets. Then the family $\{\overline{G(\sigma_1, \sigma_2)}^Y, (\sigma_1, \sigma_2) \in \mathbb{N}^2\}$ is also locally finite in Y and the sets are pairwise disjoint.

Continuing this way, we obtain on X:

(1) A Suslin scheme $A(.)$, where for each $s \in \mathbb{N}^{\mathbb{N}}$, $p \in \mathbb{N}$, the set $A(s, p)$ is a subset of $A(s)$ such that $\overline{A(s, p)}$ is not a σ-compact subset of Y.
(2) A Suslin scheme $G(.)$, where each $G(s, p)$ is an open subset of $G(s)$.

(3) A Suslin scheme $k(.)$, where $k(\sigma_1\sigma_2\ldots\sigma_n) := \{k_{\sigma_1\sigma_2\ldots\sigma_n}\}$, $k(\emptyset) = \mathbb{N}$, and such that for each $\sigma = (\sigma_i) \in \mathbb{N}^{\mathbb{N}}$, $n \in \mathbb{N}$, the sets

$$A(\sigma|n) = A(\sigma_1\sigma_2\ldots\sigma_n),$$

the open sets

$$G(\sigma|n) = G(\sigma_1\sigma_2\ldots\sigma_n),$$

and the sets of the Suslin scheme $M(.)$

$$M(k(\sigma|1), k(\sigma|2), \ldots, k(\sigma|n)) = M(k_{\sigma_1}, k_{\sigma_1\sigma_2}, \ldots, k_{\sigma_1\sigma_2\ldots\sigma_n})$$

satisfy

$$A(\sigma|n) \subset M(k(\sigma|1), k(\sigma|2), \ldots, k(\sigma|n)), \tag{3.65}$$

$$\overline{A(\sigma|n)}^X \subset G(\sigma|n), \tag{3.66}$$

and for each $n \in \mathbb{N}$ the family

$$\{\overline{G(\sigma|n)}^Y, \sigma \in \mathbb{N}^{\mathbb{N}}\} \tag{3.67}$$

is locally finite in Y with pairwise disjoint sets.

For each $\sigma \in \mathbb{N}^{\mathbb{N}}$ set

$$A(\sigma) := \bigcap_n \overline{A(\sigma|n)}^X$$

and

$$A := \bigcup\{A(\sigma) : \sigma \in \mathbb{N}^{\mathbb{N}}\}.$$

From (3.65) and Lemma 3.6.3, we deduce that A is K-analytic under the mapping $P(.) : \mathbb{N}^{\mathbb{N}} \to A$ defined by

$$P(\sigma) = A(\sigma).$$

The formulas (3.66) and (3.67) imply $A(\sigma) \cap A(\tau) = \emptyset$ for $\sigma \neq \tau$. Therefore, we may define a map $f : A \to \mathbb{N}^{\mathbb{N}}$ such that $f(x) = \sigma$ if $x \in P(\sigma)$.

The map f is continuous, since from (3.66) and (3.67), we note

$$f(A \cap G(\sigma_1\sigma_2\ldots\sigma_n)) \subset \{\tau \in \mathbb{N}^{\mathbb{N}} : \tau|n = \sigma|n\}.$$

Indeed, if $x \in A \cap G(\sigma_1\sigma_2 \dots \sigma_n)$ and $f(x) = \tau$, then

$$x \in P(\tau) = A(\tau) \subset \overline{A(\tau|n)}^X \subset G(\tau_1\tau_2 \dots \tau_n).$$

On the other hand, $G(\tau_1\tau_2 \dots \tau_n) \cap G(\sigma_1\sigma_2 \dots \sigma_n) \neq \emptyset$ if and only if $\tau|n = \sigma|n$. This proves $f(x) \in \{\tau \in \mathbb{N}^{\mathbb{N}} : \tau|n = \sigma|n\}$. Clearly $P = f^{-1}$. Finally, we prove that A is a closed subset of Y. From (3.57) in Lemma 3.6.2 it follows

$$A(\sigma) = \bigcap_n \overline{A(\sigma|n)}^X = \bigcap_n \overline{A(\sigma|n)}^Y.$$

Then $A = \bigcup\{A(\sigma) : \sigma \in \mathbb{N}^{\mathbb{N}}\}$ is closed in Y, since from (3.66) and (3.67), it follows that the family $\{A(\sigma) : \sigma \in \mathbb{N}^{\mathbb{N}}\}$ is locally finite in Y and each $A(\sigma) = \bigcap_n \overline{A(\sigma|n)}^Y$ is closed in Y. If X is analytic, there exists a compact-valued (usco) map $K : \mathbb{N}^{\mathbb{N}} \to 2^X$ such that each image $K(\sigma)$, for $\sigma \in \mathbb{N}^{\mathbb{N}}$, is a unitary set. Consequently, the non-void set $P(\sigma)$ is unitary. Now the map $f : A \to \mathbb{N}^{\mathbb{N}}$ such that $f(x) = \sigma$ if $\{x\} = P(\sigma)$ is a homeomorphism. □

Definition 3.6.7 Let A and B be two topological spaces. B is a proper image of A if there exists a continuous map $f : A \to B$ such that:

(1) $f(A) = B$,
(2) For each $y \in B$, the set $f^{-1}(y)$ is a compact subset of A,
(3) For each $\{y, y'\} \subset B$ with $y \neq y'$, we have $f^{-1}(y) \cap f^{-1}(y') = \emptyset$,
(4) The map $P : B \to 2^A$ defined by $P(y) = f^{-1}(y)$ is (usco).

Now we are ready to prove the main result; see [522].

Theorem 3.6.8 (Calbrix–Hurewicz) *Let X be a regular hereditarily Lindelöf K-analytic space that is a subspace of the hereditarily paracompact topological space Y. Then $\overline{X}$ is not a σ-compact subset of Y if and only if there exists in X a subspace A that is closed in Y and such that $\mathbb{N}^{\mathbb{N}}$ is a proper image of A. In particular, if X is a regular and analytic subspace of a hereditarily paracompact topological space Y, then $\overline{X}$ is not a σ-compact subset of Y if and only if there exists in X a subspace A, closed in Y, such that $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to A.*

Proof One part of the proof follows directly from Lemma 3.6.6. The converse follows from the known fact stating that $\mathbb{N}^{\mathbb{N}}$ is not a σ-compact space. □

It is well known that a regular and (hereditarily) Lindelöf topological space is (hereditarily) paracompact; see [195]. By Theorem 3.6.8, we note the following:

Corollary 3.6.9 *A regular analytic space X is not a σ-compact if and only if there exists in X a closed subspace homeomorphic to $\mathbb{N}^{\mathbb{N}}$.*

Chapter 4
Web-Compact Spaces and Angelic Theorems

Abstract This chapter deals with the class of angelic spaces, introduced by Fremlin, for which several variants of compactness coincide. A remarkable paper of Orihuela introduces a large class of topological spaces X (under the name web-compact) for which the space $C_p(X)$ is angelic. Orihuela's theorem covers many already known partial results providing Eberlein–Šmulian-type results. Following Orihuela, we show that $C_p(X)$ is angelic if X is web-compact. This yields, in particular, Talagrand's result stating that for a compact space X, the space $C_p(X)$ is K-analytic if and only if $C(X)$ is weakly K-analytic. We present some quantitative versions of Grothendieck's characterization of the weak compactness for spaces $C(X)$ (for compact Hausdorff spaces X) and quantitative versions of the classical Eberlein–Grothendieck and Krein–Šmulian theorems.

4.1 Angelic Lemma and Angelicity

In 1940, Šmulian proved that weakly relatively compact sets in a Banach space E are weakly relatively sequentially compact, and if $(E', \sigma(E', E))$ is separable, then $A \subset E$ is weakly relatively countably compact if and only if A is weakly relatively sequentially compact. Dieudonné and Schwartz extended this result to locally convex spaces admitting a weaker metrizable topology. Eberlein showed that weakly relatively countably compact sets are weakly relatively compact for a Banach space E. This result has been generalized by Grothendieck for lcs that are quasi-complete in the Mackey topology $\mu(E, E')$; see [240]. Finally, Fremlin's approach to angelic spaces led to present all mentioned results in a much easier form; see [240, 481], and [127]. For the definition of an angelic space, we refer to Chap. 3.

In this section, we recall elementary facts about angelic spaces, Fremlin's angelic lemma, and some applications.

The following result, due to Fremlin (called also *angelic lemma*), will be used several times in the book; see [240].

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Developments in Mathematics 24, https://doi.org/10.1007/978-3-031-76062-4_4

Theorem 4.1.1 *Let X and Y be topological spaces, where X regular, and let $\Phi : X \to Y$ be an injective and continuous map. If $A \subset X$ is relatively countably compact, and if for each $B \subset \Phi(A)$ and $y \in \overline{B}$ there exists a sequence $(y_n)_n$ in B converging to y, then $\Phi(\overline{A})$ is closed and $\Phi|\overline{A}$ is a homeomorphism.*

Proof If $D \subset A$ and $y \in \overline{\Phi(D)}$, there exists a sequence $(d_n)_n$ in D such that $(\Phi(d_n))_n$ converges to y. Let x be a cluster point of $(d_n)_n$. Since $y = \Phi(x)$, $y \in \Phi(\overline{D})$. Hence, $\Phi(\overline{D}) = \overline{\Phi(D)}$. This shows also that $\Phi(\overline{A})$ is closed in Y. Let C be a closed subset of $\overline{A}$. If $x \in \overline{A} \setminus C$, and if V is an open neighbourhood of x such that

$$C \subset B_x := X \setminus \overline{V},$$

then $C \subset \overline{D_x}$ and $x \notin \overline{D_x}$, where $D_x := B_x \cap A$. Therefore,

$$C = \bigcap \{\overline{D_x} : x \in \overline{A} \setminus C\},$$

and

$$\Phi(C) = \bigcap \{\Phi(\overline{D_x}) : x \in \overline{A} \setminus C\} = \bigcap \{\overline{\Phi(D_x)} : x \in \overline{A} \setminus C\}.$$

Hence, $\Phi(C)$ is closed and $\Phi|\overline{A}$ is a homeomorphism. □

Corollary 4.1.2 *Let $\Phi : X \to Y$ be a continuous injective map defined on a regular topological space X with values in a compact space Y. If for each relatively countably compact subset A of X and for each $b \in \overline{\Phi(A)}$ there exists a sequence $(a_n)_n$ in A such that $b = \lim_n \Phi(a_n)$, the space X is angelic.*

Theorem 4.1.1 yields the following useful property for angelic spaces; see [240].

Corollary 4.1.3 *If E is an angelic space, then E endowed with any stronger regular topology than the original one is also angelic.*

Theorem 4.1.1 applies to show the following result of Šmulian; see [240, Theorem 3.2].

Theorem 4.1.4 *Let E be a lcs such that $(E', \sigma(E', E))$ is separable. Then for the space $(E, \sigma(E, E'))$, the following conditions hold:*

(i) (Relatively) compact, (relatively) countably compact, and (relatively) sequentially compact sets are the same.
(ii) If A is relatively (countably) compact and $x \in \overline{A}$, there exists a sequence in A converging to x.
(iii) Every relatively countably compact set that is sequentially closed is compact.

We will need also the following fact due to Fremlin; see [240, Theorem 3.5].

Theorem 4.1.5 *If $C_p(X)$ is angelic, then $C_p(X, Z)$ is angelic for any metric space Z.*

4.2 Orihuela's Angelic Theorem

Let Z be a topological space. Let X be a set, and let $A \subset Z^X$ be a set of functions. X and A are said to have the *interchangeable double limit property* (in Z) if

$$\lim_{m\to\infty}\lim_{n\to\infty} f_m(x_n) = \lim_{n\to\infty}\lim_{m\to\infty} f_m(x_n),$$

for each sequence $(x_n)_n$ in X, each sequence $(f_m)_n$ in A, if all the limits involved exist.

Let Σ be a subset of $\mathbb{N}^{\mathbb{N}}$. Assume that there exists a family $\{A_\alpha : \alpha \in \Sigma\}$ of non-void subsets of a set Y covering Y. For $\alpha = (a_k) \in \Sigma$ as usual set

$$C_{a_1,a_2,\ldots,a_k} := \bigcup\{A_\beta : \beta = (b_k) \in \Sigma,\ b_j = a_j,\ j = 1, 2, \ldots, k\}$$

for every $\alpha = (a_k) \in \Sigma$.

We start with the following fundamental result due to Orihuela [481].

Theorem 4.2.1 (Orihuela) *Let (Z, d) be a compact metric space. Let $\mathcal{K}$ be a set of functions from Y into Z. Suppose that for each $\alpha = (a_m) \in \Sigma$, for each increasing sequence of natural numbers $(m_p)_p$, and for each $y_p \in C_{a_1a_2\ldots a_{m_p}}$, $p \in \mathbb{N}$, one has that each sequence $(f_n)_n$ in $\mathcal{K}$ satisfies the interchangeable double limit property with (y_p). Then for every f in the closure of $\mathcal{K}$ in Z^Y, there is a sequence $(g_n)_n$ in $\mathcal{K}$ converging pointwise to f.*

Proof The proof will be presented in three steps.

Step 1. If $g_i \in Z^Y$, $1 \le i \le m$, and $D \subset Y$, then

$$\{(g_1(x), g_2(x), \ldots, g_m(x)) : x \in D\}$$

is a subset of the compact product space Z^m. Therefore, for each $\varepsilon > 0$, there exists a finite subset L in D such that, for each $x \in D$, there exists $y \in L$ verifying $d(g_i(x), g_i(y)) < \varepsilon$ for $1 \le i \le m$.

Step 2. Since $S := \{(a_1, a_2, \ldots, a_n) : \alpha = (a_m) \in \Sigma, n \in \mathbb{N}\}$ is countable, there exists a bijection $\Psi : \mathbb{N} \to S$. For each $n \in \mathbb{N}$, set $D_n := C_{\Psi(n)}$. By Step 1, applied to f and D_1, there exists a finite set $L_1 \subset D_1$ such that, for each $x \in D_1$, there exists $y \in L_1$ for which $d(f(x), f(y)) < 1$. Since f is in the closure of $\mathcal{K}$, there exists $g_1 \in \mathcal{K}$ such that for each $y \in L_1$, we have $d(g_1(y), f(y)) < 1$. Applying Step 1 to the finite subset $\{f, g_1\}$ of Z^Y and D_2, we obtain a finite subset $L_2 \subset D_2$ such that, for each $x \in D_2$, there exists $y \in L_2$ with $d(f(x), f(y)) < 2^{-1}$, $d(g_1(x), g_1(y)) < 2^{-1}$. Since f belongs to the closure of $\mathcal{K}$, there exists $g_2 \in \mathcal{K}$ such that for each $y \in L_1 \cup L_2$, we have $d(g_2(y), f(y)) < 2^{-1}$. Step 1 applied to the subset $\{f, g_1, g_2\}$ of Z^Y and D_3 provides a finite subset $L_3 \subset D_3$ such that for each $x \in D_3$, there exists $y \in L_3$ that verifies

$$d(f(x), f(y)) < 3^{-1},\ d(g_1(x), g_1(y)) < 3^{-1},\ d(g_2(x), g_2(y)) < 3^{-1}.$$

Again, as f belongs to the closure of $\mathcal{K}$, there exists $g_3 \in \mathcal{K}$ such that for each $y \in L_1 \cup L_2 \cup L_3$, we have $d(g_3(y), f(y)) < 3^{-1}$. We continue this procedure by induction. Consequently, we proved two facts:

Fact 1. If $(n_p)_p$ is an increasing sequence of natural numbers such that $x \in D_{n_p}$, and if z_{n_p} is the element of L_{n_p} such that $d(f(x), f(z_{n_p})) < n_p^{-1}$ and $d(g_i(x), g_i(z_{n_p})) < n_p^{-1}$ for $i < n_p$, then

$$f(x) = \lim_{p\to\infty} f(z_{n_p}), \tag{4.1}$$

and

$$g_i(x) = \lim_{p\to\infty} g_i(z_{n_p}) \tag{4.2}$$

for each $i \in \mathbb{N}$.

Fact 2. Since $d(g_n(y), f(y)) < n^{-1}$ for each $y \in L_1 \cup L_2 \cup \cdots \cup L_n$, we deduce that for each $y \in \bigcup_i L_i$, we have

$$f(y) = \lim_{n\to\infty} g_n(y). \tag{4.3}$$

Step 3. We prove that $f(x) = \lim_{n\to\infty} g_n(x)$ for each $x \in Y$. Since (Z, d) is a compact metric space, we only need to prove that if $(n_k)_k$ is an increasing sequence of natural numbers and

$$w = \lim_{k\to\infty} g_{n_k}(x), \tag{4.4}$$

then $f(x) = w$.

Let $\alpha = (a_n)_n \in \Sigma$ be such that $x \in A_\alpha$. Then we may determine an increasing sequence $(n_p)_p$ of natural numbers such that $x \in D_{n_p} := C_{a_1 a_2 \ldots a_{n_p}}$ and, additionally, we may assume that, if $p < q$, then $n_p < n_q$. If $y_p := z_{n_p}$, from (4.1) and (4.3), we deduce that

$$f(x) = \lim_{p\to\infty} f(z_{n_p}) = \lim_{p\to\infty} f(y_p) = \lim_{p\to\infty} \lim_{n\to\infty} g_n(y_p). \tag{4.5}$$

We rewrite (4.2) to the form

$$g_i(x) = \lim_{p\to\infty} g_i(y_p), \tag{4.6}$$

and from (4.6) and (4.4), we deduce

$$w = \lim_{k\to\infty} g_{n_k}(x) = \lim_{k\to\infty} \lim_{p\to\infty} g_{n_k}(y_p). \tag{4.7}$$

Clearly, (4.5) implies

$$f(x) = \lim_{p\to\infty} \lim_{k\to\infty} g_{n_k}(y_p). \tag{4.8}$$

Applying (4.7) and (4.8), and using the hypothesis concerning the interchangeable double limits, we note $f(x) = w$. This and the compactness of the metric space (Z, d) ensure $f(x) = \lim_{n\to\infty} g_n(x)$ for each $x \in Y$. The proof is completed. □

4.3 Web-Compact Spaces

This section deals with a large class of topological spaces X for which $C_p(X)$ is angelic. Most results presented here are from [481].

Following Orihuela [481], a topological space X is said to be *web-compact* if there exists a non-empty subset Σ of $\mathbb{N}^{\mathbb{N}}$, and a family $\{A_\alpha : \alpha \in \Sigma\}$ of subsets of X (called a web-compact representation for X) such that, if $C_{n_1,n_2,\ldots,n_k} := \bigcup\{A_\beta : \beta = (m_k) \in \Sigma, m_j = n_j, j = 1, 2, \ldots, k\}$, for every $\alpha = (n_k) \in \Sigma$, the following two conditions hold:

(i) $\overline{\bigcup\{A_\alpha : \alpha \in \Sigma\}} = X$.
(ii) If $\alpha = (n_k) \in \Sigma$ and $x_k \in C_{n_1,n_2,\ldots,n_k}$ for all $k \in \mathbb{N}$, then the sequence $(x_k)_k$ has a cluster point in X.

The following proposition characterizes web-compact spaces.

Proposition 4.3.1 *For a topological space X, the following conditions are equivalent:*

(i) X is web-compact.
(ii) There exists a metrizable and separable space P, a map T from P into $\mathcal{P}(X)$ such that

(a) $\bigcup\{T(x) : x \in P\}$ is dense in X.
(b) If $(x_n)_n$ converges in P, then $\bigcup\{T(x_n) : n \in \mathbb{N}\}$ is a relatively countably compact set in X.

Proof (i) ⇒ (ii): Set $P := \Sigma$ endowed with the topology of $\mathbb{N}^{\mathbb{N}}$. Let $T(\alpha) := A_\alpha$.

(ii) ⇒ (i): Since every Polish space is a continuous image of $\mathbb{N}^{\mathbb{N}}$, there exists a continuous map φ from a subset $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ onto P. For $\alpha \in \Sigma$, set $A_\alpha := T\varphi(\alpha)$. Then $\{A_\alpha : \alpha \in \Sigma\}$ is a web-compact representation in X. □

It turns out that the class of web-compact spaces is reasonably rich. We provide some examples of web-compact spaces.

Example 4.3.2 The following spaces are web-compact.

(1) Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a family of relatively countably compact sets in a space X whose union Y is dense in X. Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a resolution in Y.

It is easy to see that X is web-compact. Indeed, if $(\alpha_n)_n$ is a sequence in $\mathbb{N}^{\mathbb{N}}$ which converges to $\alpha \in \mathbb{N}^{\mathbb{N}}$, there exists $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha_n \leq \beta$ for all $n \in \mathbb{N}$. Therefore, $\bigcup_n A_{\alpha_n} \subset A_\beta$. Now Proposition 4.2.1 applies. Consequently, every space X admitting a sequence of relatively countably compact sets whose union is dense in X is web-compact. In particular, all separable spaces are web-compact.

(2) Any Lindelöf Σ-space is web-compact. Hence, K-analytic spaces are web-compact.

(3) Quasi-Suslin spaces are web-compact.

On the other hand, note that there are web-compact spaces that are not quasi-Suslin. Indeed, the space $\mathbb{R}^{\mathbb{R}}$ is web-compact (as separable) but $\mathbb{R}^{\mathbb{R}}$ is not quasi-Suslin by Proposition 3.2.2. In [452, Proposition 80], Monterde Pérez showed that, if E is a lcs such that $(E', \sigma(E', E))$ is web-compact, then on E', there exists a web-compact representation $\{D_\alpha : \alpha \in \Sigma\}$ such that $\bigcup_{\alpha\in\Sigma} D_\alpha$ is a linear subspace of E'.

In [240, Theorem 3.7], Floret proved that, if $X = \overline{\bigcup_n K_n}$, where the sets K_n are relatively countably compact, and Z is a metric space, the space $C_p(X, Z)$ is angelic; see also Proposition 4.3.4. This result summarized some of the results obtained earlier by Grothendieck, Fremlin, Pryce, and De Wilde; see [240] for references. The following result [481, Theorem 3] extends [240, Theorem 3.7 and Corollary 3.1].

Theorem 4.3.3 (Orihuela) *For a web-compact space X, the space $C_p(X)$ is angelic.*

Proof Let (Z, d) be a metric compactification of the space of the real numbers. Clearly, the inclusion map $i : C_p(X) \to C_p(X, Z)$ is continuous and injective. Let $\{A_\alpha : \alpha \in \Sigma\}$ be a family of subsets of X giving on X a web-compact structure. Set $Y := \bigcup\{A_\alpha : \alpha \in \Sigma\}$. Let Z^Y be the topological product of Y-copies of the space Z. Clearly, the restriction map $\theta : C_p(X, Z) \to Z^Y$ is also continuous.

Since the injective map

$$\Phi := \theta \cdot i : C_p(X) \to Z^Y$$

is continuous, it is enough to show (by Corollary 4.1.2) that if A is a relatively countably compact subset of $C_p(X)$, and f belongs to the closure of $\mathcal{K} := \Phi(A)$, then there exists a sequence $(g_n)_n$ in A such that $f(y) = \lim_n g_n(y)$ for each $y \in Y$. Clearly, it is enough to prove that there exists a sequence $(\Phi(g_n))_n$ in $\mathcal{K}$ such that

$$f(y) = \lim_n \Phi(g_n(y)) = \lim_n g_n(y)$$

for each $y \in Y$. This follows from Theorem 4.2.1. Indeed, if $\alpha = (a_m)_m \in \Sigma$, and $(m_p)_p$ is an increasing sequence of natural numbers, and $y_p \in C_{a_1 a_2 \ldots a_{m_p}}$ for $p \in \mathbb{N}$, then $(y_p)_p$ has an adherent point $x \in X$. Therefore, if $(f_n)_n$ is a sequence in A, and the limits involved in $\lim_n \lim_p \Phi(f_n(y_p)) = \lim_n \lim_p f_n(y_p)$,

and $\lim_p \lim_n \Phi(f_n(y_p)) = \lim_p \lim_n f_n(y_p)$ exist, both limits are equal. Indeed, since A is relatively countably compact, it follows that, if f is an adherent point of the sequence $(f_n)_n$, the two preceding double limits are equal to $f(x)$. □

We refer also the reader to [128] for more general results concerning applications of Theorem 4.3.3, which also applies to get the following fact; see [481, Corollary 1.3].

Proposition 4.3.4 *If X is web-compact and Y is a metric space, the space $C_p(X, Y)$ of continuous functions from X to Y with the pointwise topology is angelic.*

Proof By Theorem 4.3.3, the space $C_p(X)$ is angelic. On the other hand, applying Fremlin's Theorem 4.1.5, the space $C_p(X, Y)$ is also angelic. □

To show another consequence of Theorem 4.3.3, we need the following fact (due to Grothendieck); see [240, Theorem 4.2] and Corollary 4.6.23.

Proposition 4.3.5 *Let X be a compact space. Let B be the closed unit ball in the Banach space $C(X)$. Then every compact set in B for the topology of $C_p(X)$ is compact in the weak topology of $C(X)$.*

Proof Since $C_p(X)$ is angelic, it is enough to show that every sequence in B which converges in $C_p(X)$ also converges in the weak topology of $C(X)$. This follows from the Lebesque dominated convergence theorem. □

The following useful result is due to Talagrand [579, Theorem 3.4].

Proposition 4.3.6 (Talagrand) *Let X be a compact space. Then $C_p(X)$ is K-analytic if and only if the weak topology of the Banach space $C(X)$ is K-analytic.*

Proof Let τ_p be the original topology of $C_p(X)$. Since the weak topology of $C_p(X)$ coincides with τ_p, the identity map

$$I : (C(X), \sigma(C(X), C(X)')) \to C_p(X)$$

is continuous. Hence, if $(C(X), \sigma(C(X), C(X)'))$ is K-analytic, then $C_p(X)$ is K-analytic.

To prove the converse, assume $C_p(X)$ is K-analytic. Let $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a compact resolution in $C_p(X)$. We prove that the closed unit ball B in the Banach space $C(X)$ is a K-analytic subspace of the space $(C(X), \sigma(C(X), C(X)'))$. Clearly, $\{K_\alpha \cap B : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution in τ_p. Since $C_p(X)$ is angelic by Theorem 4.3.3, the space B is angelic in τ_p. Hence, by Corollary 4.1.3, the space $(B, \sigma(C(X), C(X)')|B)$ is angelic. Note that $\{K_\alpha \cap B : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution in $(B, \sigma(C(X), C(X)')|B)$ by Proposition 4.3.5. Now Corollary 3.2.9 applies to deduce that B is K-analytic in $\sigma(C(X), C(X)')$. Since $C(X) = \bigcup_n B_n$, where $B_n := nB$ for each $n \in \mathbb{N}$, and each set B_n is K-analytic in the topology $\sigma(C(X), C(X)')$, the space $(C(X), \sigma(C(X), C(X)'))$ is K-analytic as claimed. □

Another consequence of Theorem 4.3.3 is the following interesting fact; see [481].

Corollary 4.3.7 *Let E be a lcs such that $(E', \sigma(E', E))$ is web-compact. Then $(E, \sigma(E', E))$ is angelic.*

Proof This follows from Theorem 4.3.3 and the fact that $(E, \sigma(E, E')) \subset C_p((E', \sigma(E', E))$. □

Corollary 4.3.7 applies to show that the weak topology $\sigma(E, E')$ of a metrizable lcs is angelic. Indeed, $(E', \sigma(E', E))$, as covered by a sequence of compact sets, is web-compact. Note that the Mackey dual $(E', \mu(E', E))$ of a Banach space E need not be an angelic. Indeed, if E is a Banach space and $F \subset E'$ is $\mu(E', E)$-sequentially compact, the $\mu(E', E)$-closure of F is $\mu(E', E)$-precompact and complete; see [313] and [546]. By [453] for a Fréchet space E, any $\mu(E', E)$-relatively countably compact set of E' is $\mu(E', E)$-relatively compact, and for $E := \ell_1[0, 1]$, there exists a $\mu(E', E)$-sequentially compact set in E' which is $\mu(E', E)$-relatively compact and not $\mu(E', E)$-compact. Hence, $(E', \mu(E', E))$ is not angelic. On the other hand, if E is a separable lcs, the space E' admits a metric topology that is weaker than $\sigma(E', E)$. Hence, $\sigma(E', E)$, and then $\mu(E', E)$, is angelic by Corollary 4.1.3. If E is a (WCG) Banach space (which need not be separable), $\sigma(E, E')$ is K-analytic; see Theorem 12.4.6. Consequently, $(E', \sigma(E', E)) \subset C_P(E, \sigma(E, E'))$ and angelic by Theorem 4.3.3. Hence, $(E', \mu(E', E))$ is angelic as well.

By $\mathbb{T}$, we denote the torus $\{z \in \mathbb{C} : |z| = 1\}$. For a topological Abelian group E by $E^\wedge$, we denote the set of all continuous group homomorphisms $\varphi : E \to \mathbb{T}$. The coarsest group topology on E for which all elements of $E^\wedge$ are continuous is called the *Bohr topology* and is denoted by $\sigma\left(E, E^\wedge\right)$.

Proposition 4.3.8 *Let E be a metrizable dual-separating Abelian topological group. Then $(E, \sigma(E, E^\wedge)$ is angelic.*

Proof Let $(U_n)_{n\in\mathbb{N}}$ be a decreasing basis of neighbourhoods of zero in E. Then

$$U_n^\triangleright := \{\varphi \in E^\wedge : \varphi(U_n) \subset \mathbb{T}_+\},$$

where $\mathbb{T}_+ := \{z \in \mathbb{C} : |z| = 1, \operatorname{Re} z \geqslant 0\}$ is closed in the compact space $\mathbb{T}^E$, hence compact in $\sigma(E^\wedge, E)$, and each $U_n^\triangleright$ is equicontinuous. Note that $E^\wedge = \bigcup_n U_n^\triangleright$. Indeed, if $\varphi \in X^\wedge$, then $\varphi^{-1}(\mathbb{T}_+)$ is a neighbourhood of zero, so there exists $m \in \mathbb{N}$ such that $U_m \subset \varphi^{-1}(\mathbb{T}_+)$. Therefore, $\varphi \in U_m^\triangleright$. This proves that $(E^\wedge, \sigma(E^\wedge, E))$ is web-compact. $C_p((E^\wedge, \sigma(E^\wedge, E)), \mathbb{T})$ is angelic by Proposition 4.3.4. Since $(E, \sigma(E, E^\wedge)) \subset C_p((E^\wedge, \sigma(E^\wedge, E)), \mathbb{T})$, the proof is completed. □

4.4 Subspaces of Web-Compact Spaces

Proposition 4.4.1 from [128, Theorem 1] shows some relations between web-compact spaces and Lindelöf Σ-spaces.

Proposition 4.4.1 *Let X be a web-compact regular space. Let $\{A_\alpha : \alpha \in \Sigma\}$ be a web-compact representation of X such that every relatively countably compact set in X is relatively compact. Then X contains a dense Lindelöf Σ-subspace. If $\Sigma = \mathbb{N}^{\mathbb{N}}$, then X contains a dense K-analytic subspace.*

Proof For $\alpha = (a_n) \in \Sigma$, set

$$B_\alpha := \{x \in X : \exists (x_n)_n \subset X, x_n \in C_{a_1,\dots,a_n},\ x \in \overline{\{x_n : n \in \mathbb{N}\}}\}.$$

Clearly, $A_\alpha \subset B_\alpha$ for each $\alpha \in \Sigma$, and $Y := \bigcup_{\alpha\in\Sigma} \overline{B_\alpha}$ is a dense subspace of X. The reader will check that each set B_α is relatively countably compact in X; therefore, B_α is relatively compact (by the assumption). We prove that Y is a Lindelöf Σ-space by showing that the map T from Σ into compact subsets of Y, defined by $T(\alpha) := \overline{B_\alpha}$, is (usco). Let U be a closed (by regularity of X) neighbourhood of $T(\alpha)$. There exists $m \in \mathbb{N}$ such that $C_{a_1,\dots,a_m} \subset U$. Indeed, otherwise there exists a sequence

$$x_n \in C_{a_1,\dots,a_n} \setminus U$$

for each $n \in \mathbb{N}$. Since $\{A_\alpha : \alpha \in \Sigma\}$ is a web-representation of X, the sequence $(x_n)_n$ has an adherent point $x \in B_\alpha$, a contradiction. This proves that for each $\beta = (b_n) \in \Sigma$ such that $a_j = b_j$ for $j = 1, \dots, m$, we have

$$T_\beta = \overline{B_\beta} \subset \overline{C_{b_1,\dots,b_m}} \subset \overline{C_{a_1,\dots,a_m}} \subset U;$$

consequently, T is (usco). □

A compact space X is called *Talagrand compact* if $C(X)$ is K-analytic in its weak topology, [196]. By Proposition 4.3.6, a compact space X is Talagrand compact if and only if $C_p(X)$ is K-analytic. Proposition 4.4.1 applies to prove a couple of consequences observed in [127] and presented below.

Corollary 4.4.2 *Let X be a web-compact space with a web-compact representation for the case $\Sigma = \mathbb{N}^{\mathbb{N}}$. Then every compact set $K \subset C_p(X)$ is a Talagrand compact space.*

Proof The map $\delta : X \to C_p(K)$ defined by $\delta(x) := \delta_x$, where $\delta_x(f) = f(x)$ for all $f \in K$, is continuous. This shows that $\delta(X) \subset C_p(K)$ is a web-compact angelic and regular space. Proposition 4.4.1 provides in $\delta(X)$ a dense K-analytic subspace Y that separates points of K. By Theorem 14.1.1, the space $C_p(K)$ is K-analytic, so K is Talagrand compact. □

Corollary 4.4.3 *Let X be a web-compact space. Assume that $K \subset C_p(X)$ is a compact set. The following assertions are equivalent:*

(i) K is metrizable.
(ii) K is contained in a separable subset of $C_p(X)$.

Proof (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (i): Let D be a countable subset of $C_p(X)$ such that $K \subset \overline{D}$ (the closure in $C_p(X)$). Define the map $\delta : X \to C_p(\overline{D})$ by the formula $\delta(x) := \delta_x$, where $\delta_x(f) := f(x)$ for $x \in X$ and $f \in \overline{D}$. Clearly, δ is continuous, $\delta(X)$ is a web-compact subspace of the angelic (Theorem 4.3.3) regular space $C_p(\overline{D})$, and $C_p(\overline{D})$ admits a weaker metrizable topology ξ (since $\overline{D}$ is separable). Note also that $\delta(X)$ is separable. Indeed, by Proposition 4.4.1, the space $\delta(X)$ contains a dense Lindelöf Σ-subspace Y, so Y is Lindelöf. Therefore, $(Y, \xi|Y)$ is separable and metrizable. Now Lemma 3.1.13 applies to derive that Y in its original topology is separable. Hence, $\delta(X)$ is separable too. Finally, observe that K can be considered as a subset of $C_p(\delta(X))$. Indeed, for every $f \in K$, define the function $f^\wedge$ on $\delta(X)$ by $f^\wedge(\delta_x) := f(x)$, and set $K^\wedge := \{f^\wedge : f \in K\}$. Clearly, $K^\wedge \subset C_p(\delta(X))$, and $K^\wedge$ is homeomorphic to K. Since $C_p(\delta(X))$ admits a weaker metrizable topology, we reach the conclusion. □

Corollary 4.4.4 *Let E be a lcs such that $(E', \sigma(E', E))$ contains a dense web-compact subset X. Then $(E, \sigma(E, E'))$ is angelic, and every weakly compact subset of E is metrizable if and only if it is contained in a separable subspace of E. For the case $\Sigma = \mathbb{N}^{\mathbb{N}}$, every weakly compact subset of E is Talagrand compact.*

Proof By Theorem 4.3.3, the space $C_p(X)$ is angelic. Let A be a relatively countably compact subset of $(E, \sigma(E, E'))$, and let $\Phi : (E, \sigma(E, E')) \to C_p(X)$ be a continuous and injective map defined by the restriction. Then $\Phi(A)$ is a relatively countably compact subset of $C_p(X)$. By the angelicity of $C_p(X)$, we have that $\overline{\Phi(A)}$ is compact, and for each subset B of $\Phi(A)$ and each $b \in \overline{B}$, there exists a sequence $(b_n)_n$ in B such that $b = \lim_n b_n$. By Theorem 4.1.1, the set $\Phi(\overline{A})$ is closed, because $\Phi(\overline{A}) = \overline{\Phi(A)}$ and $\Phi|_{\overline{A}}$ is a homeomorphism. Therefore, $\overline{A}$ is a compact subset of $(E, \sigma(E, E'))$. Moreover, for each $a \in \overline{A}$, there exists a sequence $(a_n)_n$ in A such that $a = \lim_n a_n$. This proves that $(E, \sigma(E, E'))$ is angelic. If $\Sigma = \mathbb{N}^{\mathbb{N}}$, by Corollary 4.4.2, every weakly compact set in E is Talagrand compact, and Corollary 4.4.3 completes the proof. □

Note that the class of lcs E whose *weak** dual $(E', \sigma(E', E))$ includes a dense web-compact subset contains a large class $\mathfrak{G}$ of lcs; see Chap. 11. Therefore, in particular case, the conclusions of Corollary 4.4.4 hold for spaces in the class $\mathfrak{G}$ (which includes all (DF) and (LM)-spaces).

4.5 Angelic Duals of Spaces $C(X)$

This short section provides another application of Corollary 4.4.4 to study the angelicity of the dual of spaces $C(X)$. Let $C_b(X)$ be the space of all continuous bounded real-valued functions on X. By the *strict* topology β_0 on $C_b(X)$, we mean a locally convex topology generated by the family of seminorms $(p_h)_h$ defined by

$$p_h(f) := \sup_{x\in X} |f(x)h(x)|,$$

where $f \in C_b(X)$, and h run over the space of real-valued bounded functions on X vanishing at infinity, i.e. for each $\epsilon > 0$, there exists a compact set $K \subset X$ such that $|h(x)| < \epsilon$ for each $x \in X \setminus K$; see [107]. It is known (see [107]) that the dual $(C_b(X), \beta_0)' = M_t(X)$, is the space of all finite Borel measures on X, which are inner regular with respect to the compact sets.

We need the following simple lemma from [128, Lemma 13].

Lemma 4.5.1 *For a uniform space $(X, \mathfrak{U})$, the space $U_b(X)$ is a dense subspace of $(C_b(X), \beta_0)$.*

Proof Recall that $U_b(X) := \{f \in C_b(X) : f \text{ is } \mathfrak{U} - \text{uniformly continuous}\}$. It is enough to prove that, if $\mu \in M_t(X)$ and $\int_X g d\mu = 0$ for all $g \in U_b(X)$, then $\mu = 0$. Fix $f \in C_b(X)$ and $\epsilon > 0$. Let $|\mu|$ be the variation of μ. Then there exists compact $K \subset X$ such that $|\mu|(X\setminus K) < \epsilon$. Since the restriction $f|K$ is uniformly continuous, we apply [195, Theorem 8.5.6] to get a function $g \in U_b(X)$ such that $f|K = g|K$ and $\|g\|_\infty = \|f\|_\infty$, where $\|f\|_\infty := \sup_{x\in X} |f(x)|$. Then $\int_X g d\mu = 0$ and

$$|\int_X f d\mu| = |\int_X (f-g) d\mu| \le \int_{X\setminus K} |(f-g)d\mu| \le 2\|f\|_\infty \epsilon.$$

□

If a uniform space $(X, \mathfrak{U})$ admits a $\mathfrak{U}$-basis, i.e. a basis $\{N_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that $N_\alpha \subset N_\beta$ for $\beta \le \alpha$, then precompact subsets of $(X, \mathfrak{U})$ are metrizable. This result will be proved in Proposition 6.4.8 by using the concept of the trans-separability; see also [127, Theorem 1]. Proposition 6.4.8 will be used for the proof of Proposition 4.5.2, due to Cascales and Orihuela, [128, Theorem 14].

Proposition 4.5.2 *Let $(X, \mathfrak{U})$ be a uniform space with a $\mathfrak{U}$-basis. Then the dual $M_t(X)$ is $\sigma(M_t(X), C_b(X))$-angelic, and each $\sigma(M_t(X), C_b(X))$-compact set in $M_t(X)$ is metrizable.*

Proof For $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$, set

$$A_\alpha = \{f \in C_b(X) : \|f\|_\infty \le a_1, |f(x) - f(y)| \le n^{-1}, (x, y) \in N_{(a_n.....)}, n \in \mathbb{N}\}.$$

Since the compact-open topology τ_c of $C_b(X)$ coincides with β_0 on uniformly bounded subsets of $C_b(X)$, and each A_α, as uniformly bounded and uniformly equicontinuous, is compact in τ_c (by the Ascoli theorem, [195]), we deduce that A_α is β_0-compact. Note that $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution in $U_b(X)$ (this will show (by applying Lemma 4.5.1) that $(C_b(X), \beta_0)$ is web-compact). Indeed, choose arbitrary $f \in U_b(X)$. Then there exist $L > 0$ and a sequence $\alpha_k = (a_n^k)$, $k \in \mathbb{N}$, in $\mathbb{N}^{\mathbb{N}}$ such that

$$\|f\|_\infty \le L, \ |f(x) - f(y)| \le k^{-1}, \ (x, y) \in N_{\alpha_k}.$$

Define $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$ by the formula

$$a_1 := \max\{L, a_1^1\}, \quad a_n := \max\{a_n^1, a_{n-1}^2, \ldots, a_1^n\}$$

for all $n \geq 2$. Then $f \in A_\alpha$. The ordering condition required for a family to be a resolution is obvious. Consequently, using Corollary 4.4.4, we note that $(M_t(X), \sigma(M_t(X), C_b(X))$ is angelic. Note that every compact set K in $(M_t(X), \sigma(M_t(X), C_b(X))$ is metrizable. Indeed, if K is compact in the topology τ on $M_t(X)$ of the uniform convergence on the sets A_α, by Proposition 6.4.8 (we already mentioned above), K is τ-metrizable. If τ_0 is the topology on $M_t(X)$ of the uniform convergence on uniformly bounded and equicontinuous sets of $C_b(X)$, we have $\tau \leq \tau_0$, and $\sigma(M_t(X), C_b(X)) \leq \tau_0$. Observe that K is also τ_0-compact. Indeed, since τ_0 is angelic (see Theorem 4.1.1), it is enough to show that K is τ_0-sequentially compact: If $(\mu_n)_n$ is a sequence in K, there is $\mu \in K$ and a subsequence $(\mu_{n_k})_k$ converging to μ in $\sigma(M_t(X), C_b(X))$. Then, by [189, Theorem 7], we have $\mu_{n_k} \to \mu$ in τ_0. Hence, $\sigma(M_t(X), C_b(X))|K = \tau|K = \tau_0|K$, and the proof is completed. □

4.6 About Compactness via Distances to Function Spaces $C(K)$

In this section, supplementing the previous one, we survey some classical results about the compactness concepts by using suitable inequalities involving distances to spaces of continuous functions. We present some quantitative versions of Grothendieck's characterization of the weak compactness for spaces $C(K)$ for compact Hausdorff spaces K and quantitative versions of the classical Eberlein–Grothendieck and Krein–Šmulian theorems. The first part of this section will be used to present corresponding results for general Banach spaces E. Results of this section are based on recent works of Angosto and Cascales [11–13]; Angosto [10]; Angosto et al. [14]; and Cascales et al. [121]. In the case of Banach spaces, these quantitative generalizations have been previously studied by Fabian et al. [200] and Granero [283].

Let (Z, d) be a metric space. Let A be a non-empty subset of Z. Set

$$\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}.$$

By the distance between non-empty sets A, B in Z, we mean $d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$. By the Hausdorff non-symmetrized distance from A and B, we mean $\hat{d}(A, B) := \sup\{d(a, B) : a \in A\}$. The product space Z^X for a space X will be considered with the standard supremum metric

$$d(f, g) := \sup_{x \in X} d(f(x), g(x)),$$

that is allowed to take the value $+\infty$. For a subset A of a space X by $\mathrm{clust}_X(\varphi)$, we mean the set of all cluster points in X of a sequence $\varphi \in A^{\mathbb{N}}$. Recall also that $\mathrm{clust}_X(\varphi) = \bigcap_n \overline{\{\varphi(m) : m > n\}}$.

For a compact space K and $\epsilon \geq 0$, we will say that a set $H \subset C(K)$ ϵ-interchanges limits with K; see [200] if

$$|\lim_n \lim_k f_n(x_k) - \lim_k \lim_n f_n(x_k)| \leq \epsilon$$

for any two sequences $(x_n)_n$ in K and $(f_m)_m$ in H, provided the iterated limits exist. This concept for $\epsilon = 0$ is due to Grothendieck; see [240].

Let K be a compact Hausdorff space. Let τ_p be the pointwise topology on the space $\mathbb{R}^K$, and let d be the metric of the uniform convergence on $\mathbb{R}^K$. Let $H \subset \mathbb{R}^K$ be a pointwise bounded set.

By the classical Tichonov theorem, the closure $\overline{H}$ of H in τ_p is τ_p-compact. Hence, to show that H is τ_p-relatively compact in $C(K)$, it is enough to see when $\overline{H} \subset C(K)$. Assume that $\hat{d} := \hat{d}(\overline{H}, C(K))$ is the non-symmetrized distance from $\overline{H}$ to $C(K)$. Then $\hat{d} > 0$ provides us a non-τ_p-compactness measure for H relative to the space $C(K)$. Clearly,

$$\overline{H} \subset C(K) \Leftrightarrow \hat{d} = 0.$$

A pointwise bounded set H is a relatively compact set in $C_p(K)$ if and only if $\hat{d}(\overline{H}, C(K)) = 0$.

It is important to know how to compute $\hat{d}$. The distance of a function $f \in \mathbb{R}^K$ to the space $C(K)$ can be obtained by the following easy formula in Theorem 4.6.2; see [11].

The proof of Theorem 4.6.2 (from [11, Theorem 2.2]) uses the following fact due to Jameson [320, Theorem 12.16].

Lemma 4.6.1 *Let X be a normal space and let $f_1 \leq f_2$ be two real functions on X such that f_1 is upper semicontinuous and f_2 is lower semicontinuous. Then there exists on X a continuous function f such that $f_1 \leq f \leq f_2$.*

Theorem 4.6.2 *Let X be a normal space and let $f \in \mathbb{R}^X$. Then $d(f, C(X)) = 2^{-1} \operatorname{osc}(f)$, where $\operatorname{osc}(f) := \sup_{x \in X} \operatorname{osc}(f, x) = \sup_{x \in X} \inf\{\operatorname{diam} f(U) : U \subset X \text{ open }, x \in U\}$.*

Proof First, we prove

$$2^{-1} \operatorname{osc}(f) \leq d(f, C(X)).$$

It is enough to check this inequality for finite $d := d(f, C(X))$. Fix $x \in X$ and $\epsilon > 0$. Then there exists $g \in C(X)$ such that

$$d(f, g) \leq d + 3^{-1}\epsilon.$$

By the continuity of g, there exists an open neighbourhood U of x such that $\operatorname{diam}(g(U)) < 3^{-1}\epsilon$. Then

$$d(f(y), f(z)) \leq (f(y), g(y)) + d(g(y), g(z)) + d(g(z), f(z)) < 2d + \epsilon$$

for all $y, z \in U$. Hence, $\operatorname{osc}(f, x) \leq 2d$ for each $x \in X$, and the claim holds. Next, we prove

$$d(f, C(X)) \leq 2^{-1} \operatorname{osc}(f).$$

It is enough to show this for finite $t := 2^{-1} \operatorname{osc}(f)$. For $x \in X$, let $\mathfrak{U}_x$ be the family of open neighbourhoods of x. Set

$$\mathfrak{V}_x := \{U \in \mathfrak{U}_x : \operatorname{diam}(f(U)) < \operatorname{osc}(f) + 1\}.$$

Note that the restricted map $f|U$ is upper and lower bounded for each $U \in \mathfrak{V}_x$, and $\mathfrak{V}_x$ forms a basis of neighbourhoods of x in X. Then

$$\inf_{U \in \mathfrak{V}_x} \sup_{y \in U} f(y) - \sup_{U \in \mathfrak{V}_x} \inf_{z \in U} f(z) =$$

$$\inf_{U, V \in \mathfrak{V}_x} \sup_{y \in U, z \in V} (f(y) - f(z)) \leq$$

$$\inf_{U \in \mathfrak{V}_x} \sup_{y, z \in U} (f(y) - f(z)) = \inf_{U \in \mathfrak{V}_x} \operatorname{diam}(f(U)) =$$

$$\inf_{U \in \mathfrak{U}_x} \operatorname{diam}(f(U)) = \operatorname{osc}(f, x) \leq 2t.$$

Now set

$$f_1(x) := \inf_{U \in \mathfrak{V}_x} \sup_{z \in U} f(z) - t,$$

$$f_2(x) := \sup_{U \in \mathfrak{V}_x} \inf_{z \in U} f(z) + t.$$

Then $f_1 \leq f_2$, f_1 is upper semicontinuous, and f_2 is lower semicontinuous. Now Lemma 4.6.1 applies to get a continuous function h such that $f_1 \leq h \leq f_2$ on X. Since

$$f_2(x) - t \leq f(x) \leq f_1(x) + t$$

for each $x \in X$, we have

$$h(x) - t \leq f_2(x) - t \leq f(x) \leq f_1(x) + t \leq h(x) + t.$$

Thus $d(f,h) \le t = 2^{-1}\operatorname{osc}(f)$. □

Corollary 4.6.3 *Let X be a topological space. The following conditions are equivalent:*

(i) X is a normal space.
(ii) For each $f \in \mathbb{R}^X$, there exists $g \in C(X)$ such that $d(f,g) = 2^{-1}\operatorname{osc}(f)$.
(iii) $d(f, C(X)) = 2^{-1}\operatorname{osc}(f)$ for each $f \in \mathbb{R}^X$.

Let X be a topological space and $H \subset \mathbb{R}^X$. Define

$$\operatorname{ck}(H) := \sup_{\varphi \in H^{\mathbb{N}}} d(\operatorname{clust}_{\mathbb{R}^X}(\varphi), C(X)).$$

For $K \subset X$, define $\gamma_K(H)$ as

$$\sup\left\{d(\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)) : (f_m)_m \subset H, (x_n)_n \subset K\right\}$$

provided the iterated limits exist.

If $H \subset C(X)$ is τ_p-relatively countably compact in $C(X)$, then $\operatorname{ck}(H) = 0$. The equality $\gamma_K(H) = 0$ means that in H we interchange limits with K.

The following formula [12, Theorem 2.3] (see also [121] and [11]) describes the distance $\hat{d} := \hat{d}(\overline{H}, C(K))$ (the closure in $\mathbb{R}^K$) by using the above quantities. We present only a sketch of the proof.

Theorem 4.6.4 (Angosto–Cascales) *Let K be a compact space. Let $H \subset C(K)$ be a uniformly bounded set. Then*

$$\operatorname{ck}(H) \le \hat{d}(\overline{H}, C(K)) \le \gamma_K(H) \le 2\operatorname{ck}(H),$$

where the closure is taken in $\mathbb{R}^K$.

Proof The first inequality $\operatorname{ck}(H) \le \hat{d}(\overline{H}, C(K))$ follows from the definitions. To prove the inequality $\gamma_K(H) \le 2\operatorname{ck}(H)$, we use a standard argument to show that τ_p-relatively compactness of H in $C(K)$ implies that H interchanges limits with K. Indeed, let $(f_m)_m$ be a sequence in H, and let $(x_n)_n$ be a sequence in K, and assume that both iterated limits

$$\lim_n \lim_m f_m(x_n), \quad \lim_m \lim_n f_m(x_n)$$

exist in $\mathbb{R}$. Fix any $t \in \mathbb{R}$ such that $\operatorname{ck}(H) < t$. Then the sequence $(f_m)_m$ has a τ_p-cluster point $f \in \mathbb{R}^K$ such that $d(f, C(K)) < t$. Fix $g \in C(K)$ such that $\sup_{x\in K} |f(x) - g(x)| < t$. Let $x \in K$ be a cluster point of $(x_n)_n$. Since $g(x)$ and $f_m(x)$ are cluster points in $\mathbb{R}$ of $(g(x_n))_n$ and $f_m(x_n)_n$, respectively, we get a subsequence $(x_{n_k})_k$ such that

$$\lim_k g(x_{n_k}) = g(x),$$

and $f_m(x)$ is the limit of the convergent sequence $(f_m(x_{n_k})_n$. Hence,

$$|\lim_k f(x_{n_k}) - f(x)| \leq |\lim_k f(x_{n_k}) - \lim_k g(x_{n_k})| + |g(x) - f(x)| \leq 2t$$

and

$$\lim_m \lim_n f_m(x_n) = \lim_m f_m(x) = f(x).$$

Consequently,

$$|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| = |\lim_n \lim_m f_m(x_n) - f(x)| =$$

$$|\lim_k f(x_{n_k}) - f(x)| \leq 2t.$$

Now we prove that $\hat{d}(\overline{H}, C(K)) \leq \gamma_K(H)$. Define

$$\operatorname{osc}^*(f, x) := \inf_U \{\sup_{y \in U} |f(y) - f(x)| : U \subset X \text{ open}, x \in U\}$$

for each $x \in X$ and $f \in \overline{H}$. Then $\operatorname{osc}^*(f, x) \leq \gamma_K(H)$ and $\operatorname{osc}(f) \leq 2\gamma_K(H)$. We apply Theorem 4.6.2 to deduce $d(f, C(K)) \leq \gamma_K(H)$. Finally, $\hat{d}(\overline{H}, c(K)) \leq \gamma_K(H)$. □

We note following Eberlein–Grothendieck's result (see [240]) as a consequence of the above results.

Corollary 4.6.5 *Let K be a compact space and $H \subset C(K)$. The following assertions are equivalent:*

(i) H is τ_p-relatively countably compact in $C(K)$.
(ii) H interchanges limits with K.
(iii) $\overline{H} \subset C(K)$, where the closure is taken in $\mathbb{R}^K$.
(iv) H is τ_p-relatively compact in $C(K)$.

The next result, due to Cascales et al., [121, Theorem 3.3], says that the ϵ-interchanging limit property is preserved when taking convex hulls. We refer also to [200, Theorem 13], where a similar result has been proved for subsets in Banach spaces.

The proof of Theorem 4.6.8, originally presented in [121], uses some ideas from the proof of the Krein–Šmulian theorem; see [364, Chapter 5]. We need the following two lemmas due to Kelley and Namioka [364, Lemma 17.9, Lemma 17.10]. By $\mathcal{P}(X)$, we denote the power set of a set X. The proof of the first Lemma 4.6.6 is omitted; we refer the reader to the book [364].

Lemma 4.6.6 *Let μ be a finitely additive finite measure on an algebra of sets $\mathcal{A}$, and let $(A_n)_n$ be a sequence of sets in $\mathcal{A}$ such that $\mu(A_n) > \delta$ for some $\delta > 0$ and all $n \in \mathbb{N}$. Then there exists a subsequence $(A_{n_k})_k$ such that $\mu(\bigcap_{i=1}^m A_{n_i}) > 0$ for each $m \in \mathbb{N}$.*

Lemma 4.6.7 *Let $(I_n)_n$ be a sequence of pairwise disjoint finite non-empty sets, and let μ_n be a probability measure on $\mathcal{P}(I_n)$ for each $n \in \mathbb{N}$. Let $(A_k)_k$ be a sequence of subsets of $I := \bigcup_n I_n$ such that for some $\delta > 0$, we have $\liminf_n \mu_n(A_k \cap I_n) > \delta$ for each $k \in \mathbb{N}$. Then there exists a subsequence $(A_{k_i})_i$ such that $\bigcap_{i=1}^j A_{k_i} \neq \emptyset$ for each $j \in \mathbb{N}$.*

Proof We denote by $\mathcal{A}$ the (countable) subalgebra of $\mathcal{P}(I)$ generated by the sets A_n. Then there exists an increasing sequence $(n_k)_k$ in $\mathbb{N}$ such that $\lim_k \mu_{n_k}(A \cap I_{n_k})$ exists for each $A \in \mathcal{A}$. Define a finitely additive measure on $\mathcal{A}$ by the following formula:

$$\mu(A) := \lim_k \mu_{n_k}(A \cap I_{n_k})$$

for $A \in \mathcal{A}$. Since $\mu(A_k) > \delta$ for each $k \in \mathbb{N}$, we apply Lemma 4.6.6 to complete the proof. □

We are ready to prove

Theorem 4.6.8 (Cascales–Marciszewski–Raja) *Let Z be a compact convex subset of a normed space E. Let K be a set, and let H be a subspace of Z^K. Then for each $\epsilon \geq 0$, the space H ϵ-interchanges limits with K if and only if* $\mathrm{conv}(H)$ *ϵ-interchanges limits with K.*

Proof Let $(f_n)_n$ and $(x_k)_k$ be sequences in $\mathrm{conv}(H)$ and K, respectively, with the property that both limits

$$\lim_n \lim_k f_n(x_k), \ \lim_k \lim_n f_n(x_k)$$

exist. Set

$$\gamma := \|\lim_n \lim_k f_n(x_k) - \lim_k \lim_n f_n(x_k)\|. \tag{4.9}$$

For each $n \in \mathbb{N}$, there exist $g_s \in H$ and $t_s \in [0, 1]$ such that

$$f_n = \sum_{s \in I_n} t_s g_s, \ \sum_{s \in I_n} t_s = 1.$$

We may assume that the sets I_n are pairwise disjoint. Set $I := \bigcup_n I_n$. By $(x_k)_k$, we denote again a subsequence of $(x_k)_k$ such that, for every $s \in I$, $g_s(x_k) \to r_s (\in Z)$ for some r_s. Then $p_n := \lim_k f_n(x_k) = \sum_{s \in I_n} t_s r_s$. There exists $\xi \in B_{E'}$ such that

$$\gamma = \xi(\lim_n \lim_k f_n(x_k) - \lim_k \lim_n f_n(x_k)) = \tag{4.10}$$

$$\xi(\lim_n p_n - \lim_k \lim_n f_n(x_k)) = \lim_k \xi(\lim_n p_n - \lim_n f_n(x_k)), \tag{4.11}$$

where $B_{E'}$ is the closed unit ball in the dual E' of E. Fix arbitrary $\delta > 0$. Without loss of generality, we may assume that for each $k \in \mathbb{N}$

$$\xi(\lim_n p_n - \lim_n f_n(x_k)) = \lim_n \xi(p_n - f_n(x_k)) > \gamma - \delta. \tag{4.12}$$

Therefore, for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that for all $n \geq n_k$, we have $\xi(p_n - f_n(x_k)) > \gamma - \delta$. For each $k \in \mathbb{N}$, set

$$A_k := \{s \in I : \xi(r_s - g_s(x_k)) > \gamma - 2\delta\}. \tag{4.13}$$

Next, for each $n \in \mathbb{N}$, define a probability measure on I_n by the formula $\mu_n(A) := \sum_{s \in A} t_s$ for each $A \subset I_n$. Let M be the diameter of Z, and we may assume that $M > 0$. Then

$$\gamma - \delta < \xi(p_n - f_n(x_k)) = \xi(\sum_{s \in I_n} t_s r_s - \sum_{s \in I_n} t_s g_s(x_k)) \tag{4.14}$$

$$= \sum_{s \in I_n} t_s \xi(r_s - g_s(x_k)) \tag{4.15}$$

$$= \sum_{s \in I_n \cap A_k} t_s \xi(r_s - g_s(x_k)) + \sum_{s \in I_n \setminus A_k} t_s \xi(r_s - g_s(x_k)) \leq \tag{4.16}$$

$$\sum_{s \in I_n \cap A_k} t_s M + \gamma - 2\delta = \mu_n(I_n \cap A_k) M + \gamma - 2\delta \tag{4.17}$$

for each $k \in \mathbb{N}$ and each $n \geq n_k$. This yields the inequality $\mu_n(I_n \cap A_k) > \delta M^{-1}$. Hence,

$$\liminf_n \mu_n(I_n \cap A_k) \geq \delta M^{-1}$$

for all $k \in \mathbb{N}$. Now Lemma 4.6.7 applies to get a subsequence $(A_{k_i})_i$ with $\bigcap_{i=1}^{j} A_{k_i} \neq \emptyset$ for all $j \in \mathbb{N}$. Then, if $j \in \mathbb{N}$, we can select $s_j \in I$ with

$$\xi(r_{s_j} - g_{s_j}(x_{k_i})) > \gamma - 2\delta$$

for all $i \leq j$. Set

$$h_j := g_{s_j}, \ d_j := r_{s_j}, \ y_i := x_{k_i},$$

for $i, j \in \mathbb{N}$. Then $\xi(d_j - h_j(y_i)) > \gamma - 2\delta$ for all $i, j \in \mathbb{N}$. We proceed as before with the sequence $(h_j)_j$. There exists a subsequence of $(h_j)_j$, which we denote again by $(h_j)_j$, such that $\lim_j h_j(y_i) = w_i$ in Z for each $i \in \mathbb{N}$. Also the corresponding sequence $(d_j)_j$ converges to some $d \in Z$. We select a subsequence of $(y_i)_i$, which we denote again by $(y_i)_i$, such that the corresponding sequence $(w_i)_i$ converges to $w \in Z$. This yields

$$\lim_j \lim_i h_j(y_i) = \lim_j d_j = d, \quad \lim_i \lim_j h_j(y_i) = \lim_i w_i = w,$$

and it follows that

$$\xi(d - w_i) \geq \gamma - 2\delta.$$

We conclude that

$$\lim_i \xi(d - w_i) = \xi(d - w) \geq \gamma - 2\delta.$$

As $\xi \in B_{E'}$ was arbitrary,

$$\|d - w\| = \|\lim_j \lim_i h_j(y_i) - \lim_i \lim_j h_j(y_i)\| \geq \gamma - 2\delta. \tag{4.18}$$

Since, by the assumption, the set H ϵ-interchanges limits with K, we deduce that $\epsilon \geq \gamma - 2\delta$. Hence, $\epsilon \geq \gamma$, since δ was arbitrary. □

Corollary 4.6.9 *For a compact space K and a uniformly bounded set $H \subset C(K)$, we have*

$$\hat{d}(\overline{\text{conv}(H)}, C(K)) \leq 2\hat{d}(\overline{H}, C(K)),$$

where the closure is taken in $\mathbb{R}^K$.

Proof This follows from Theorems 4.6.8 and 4.6.4. We have

$$\hat{d}(\overline{\text{conv}(H)}, C(K)) \leq \gamma_K(\text{conv}(H)) = \gamma_K(H) \leq 2\,\text{ck}(H) \leq 2\hat{d}(\overline{H}, C(K)).$$

□

The following quantitative version of the classical Krein–Šmulian theorem follows also from the above results.

Theorem 4.6.10 (Cascales–Marciszewski–Raja) *If K is a compact space and $H \subset \mathbb{R}^K$ is uniformly bounded,*

$$\hat{d}(\overline{\text{conv}(H)}, C(K)) \leq 5\hat{d}(\overline{H}, C(K)),$$

where the closure is taken in $\mathbb{R}^K$.

Proof Without loss of generality, we may assume that H is uniformly bounded and compact in $\mathbb{R}^K$ since $\overline{\text{conv}(\overline{H})} = \overline{(\text{conv}(H))}$. Hence, we may assume that $H = \overline{H}$. Therefore, it is enough to show that

$$\hat{d}(\overline{\text{conv}(H)}, C(K)) \leq 5\hat{d}(H, C(K)).$$

Take arbitrary $\epsilon > \hat{d}(H, C(K))$. Then for each $f \in H$, one has $d(f, C(K)) < \epsilon$, and then there exists $g_f \in C(K)$ such that $\|f - g_f\|_\infty \leq \epsilon$. The set $H_0 := \{g_f : f \in C(H)\}$ 4ϵ-interchanges with K. Indeed, note that $H_0 \subset H + \overline{B}(0, \epsilon)$, where the closed ball $\overline{B}(0, \epsilon)$ is taken in the Banach space $\ell^\infty(K)$. Then $\overline{H_0} \subset H + \overline{B}(0, \epsilon)$. Since $H \subset H_0 + \overline{B}(0, \epsilon)$, we note (by using the above facts) that

$$\overline{H_0} \subset H_0 + \overline{B}(0, 2\epsilon).$$

Consequently, $\hat{d}(\overline{H_0}, C(K)) \leq 2\epsilon$. By Theorem 4.6.4, we conclude $\text{ck}(H_0) \leq 2\epsilon$. Hence, again Theorem 4.6.4 applies to get

$$\gamma_K(H_0) \leq 4\epsilon.$$

This proves the claim. By Theorem 4.6.5, we know that

$$\gamma_K(\text{conv}(H_0)) \leq 4\epsilon.$$

Again by Theorem 4.6.4,

$$\overline{\text{conv}(H_0)} \subset C(K) + \overline{B}(0, 4\epsilon).$$

Finally,

$$\overline{\text{conv}(H)} \subset \overline{\text{conv}(H_0)} + \overline{B}(0, \epsilon) \subset C(K) + \overline{B}(0, 5\epsilon).$$

The proof is completed. □

Theorem 4.6.12 applies to provide a quantitative approach to show already known fact stating that $C_p(K)$ is angelic for a compact space K. We need the following [121, Lemma 5.1]:

Lemma 4.6.11 *Let (Z, d) be a compact metric space, and let K be a set. Let $f_1, f_2, \ldots, f_n \in Z^K$ and $\delta > 0$. Then there exists a finite subset $L \subset K$ such that for each $x \in K$, there exists $y \in L$ such that $d(f_k(y), f_k(x)) < \delta$ for each $1 \leq k \leq n$.*

Proof Set $d_\infty((x_k), (y_k)) := \sup_{1 \leq k \leq n} d(x_k, y_k)$ for each $(x_k), (y_k) \in Z^n$. This defines a metric on compact Z^n. Set

$$B := \{f_1(x), f_2(x), \ldots, f_n(x) : x \in K\}.$$

Note that (B, d_∞) is totally bounded in (Z_n, d_∞). Therefore, there exists a finite set $L \subset K$ such that $\{f_1(x), f_2(x), \ldots, f_n(x) : x \in L\}$ is δ-dense in (B, d_∞). □

Now we are ready to prove the following interesting approximation theorem from [121, Proposition 5.2].

Theorem 4.6.12 (Cascales–Marciszewski–Raja) *Let (Z, d) be a compact metric space and let K be a set. Let H be a set in Z^K that ϵ-interchanges limits with K. Then for each $f \in \overline{H}$ (the closure in $\mathbb{R}^K$), there exists a sequence $(f_n)_n \subset H$ such that we have*

$$\sup_{x \in K} d(g(x), f(x)) \leq \epsilon$$

for any cluster point g of $(f_n)_n$ in Z^K.

Proof For f the corresponding sequence $(f_n)_n$ will be defined by induction. Let $\epsilon \geq 0$ be fixed, and set $f_1 := f$. By Lemma 4.6.11, there exists finite set $L_1 \subset K$ such that

$$\min_{y \in L_1} d(f_1(x), f_1(y)) < 1 \tag{4.19}$$

for each $x \in K$. Since $f \in \overline{H}$, there exists $f_2 \in H$ such that $d(f_2(y), f_1(y)) < 2^{-1}$ for all $y \in L_1$. Assume we have already defined in H functions $f_1, f_2, \ldots, f_n$, and the corresponding finite sets $L_1, L_2, \ldots, L_n$ (according to Lemma 4.6.11) for $n \geq 2$ such that

$$\min_{y \in L_n} \max_{1 \leq k \leq n} \{d(f_k(x), f_k(y))\} < n^{-1}$$

for each $x \in K$, and

$$d(f_{n+1}(y), f_1(y)) < (n+1)^{-1}$$

for each $y \in \bigcup_{k=1}^{n} L_k$. Now set $D := \bigcup_n L_n$. Note that $\lim_k f_k(y) = f_1(y)$ for $y \in D$. Moreover, for each $x \in K$, $n \in \mathbb{N}$, there exists $y_n \in D$ such that

$$\max_{1 \leq k \leq n} \{d(f_k(x), f_k(y_n))\} < n^{-1}.$$

Then $\lim_n f_k(y_n) = f_k(x)$ for fixed $x \in K$ and all $k \in \mathbb{N}$.

Let g be a cluster point of $(f_k)_n$ in Z^K. Let $(f_{k_j})_j$ be a subsequence of $(f_k)_k$ such that for $x \in K$ we have $\lim_j f_{k_j}(x) = g(x)$. Also

$$\lim_j \lim_n f_{k_j}(y_n) = \lim_j f_{k_j}(x) = g(x),$$

and

$$\lim_n \lim_j f_{k_j}(y_n) = \lim_n f_1(y_n) = f_1(x) = f(x).$$

Since H is a set that ϵ-interchanges limits with K, we get $d(g(x), f(x)) \leq \epsilon$. The proof is completed. □

Theorem 4.6.12 can be used to prove the following corollary; see [364, Theorem 8.20] and [240, p.31].

Corollary 4.6.13 *Let Z be a compact metric space and let K be a set. Let $H \subset Z^K$ be a set that ϵ-interchanges limits with K for $\epsilon = 0$. Then for each $f \in \overline{H}$, there exists a sequence $(f_n)_n$ in H such that $f_n(x) \to f(x)$ for each $x \in K$.*

Using Corollaries 4.6.5 and 4.6.13, we obtain the following classical result; see [240], and compare Theorem 4.3.3.

Corollary 4.6.14 *$C_p(K)$ is angelic for any compact space K.*

We complete this section with some applications to study measures of weak non-compactness in Banach spaces; see [13]. We need the following concepts. For a bounded set $H \subset E$ in a Banach space E, we define

$$\omega(H) := \inf\{\epsilon > 0 : H \subset K_\epsilon + \epsilon B_E, K_\epsilon \subset E \text{ is } \sigma(E, E') - \text{compact}\}.$$

The function $\omega(H)$ was defined by de Blasi [89] as a measure of weak non-compactness; see also [41, 68, 377], and [378] for more information.

By $\gamma(H)$, we mean

$$\sup\{|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| : (f_m)_m \subset B_{E'}, (x_n)_n \subset H\},$$

if the involved limits exist; see also [41, 121, 377], and [200] for several situations where this concept has been used. Define

$$\mathrm{k}(H) := \hat{d}(\overline{H}, E) = \sup_{x^{**} \in \overline{H}} d(x^{**}, E),$$

where the closure is taken in the ω^*-topology of the bidual E'', i.e. $\sigma(E'', E')$, and d is the inf-distance for sets generated by the natural norm in E'', and we refer also the reader to works of [121, 200], and [283].

Finally, set

$$\mathrm{ck}(H) := \sup_{\varphi \in H^{\mathbb{N}}} d(\mathrm{clust}_{E''}(\varphi), E),$$

where $\mathrm{clust}_{E''}(\varphi)$ denotes the set of all cluster points of $\varphi \in H^{\mathbb{N}}$ in (E'', ω^*). Observe that

$$\mathrm{k}(H) = \inf\{\epsilon > 0 : \overline{H} \subset E + \epsilon B_{E''}\}. \tag{4.20}$$

Note that $\mathrm{k}(H) = 0$ if and only if H is relatively weakly compact in E (since $\overline{H}$ in E'' is ω^*-compact, and then $\mathrm{k}(H) = \hat{d}(\overline{H}, E) = 0$ if and only if $\overline{H} \subset E$ if and only if H is weakly relatively compact.)

We need the following fact from [13, Proposition 2.1].

Lemma 4.6.15 *Let H be a bounded set in a Banach space E. Then H $2\,\mathrm{ck}(H)$-interchanges limits with the dual ball $B_{E'}$ in E'.*

Proof Fix a sequence $(f_m)_m$ in $B_{E'}$. Let $(x_n)_n$ be a sequence in H, and assume that both iterated limits exist in $\mathbb{R}$. Fix $t > \mathrm{ck}(H)$. Then $(x_n)_n$ has a ω^*-cluster point $z \in E''$ such that $d(z, E) < t$. Let $z' \in E$ be such that $\|z - z'\| < t$. Next, let $f \in B_{E'}$ be a ω^*-cluster point of $(f_m)_m$. Then $f(z')$ and $f(x_n)$ are cluster points of $(f_m(z'))_m$ and $(f_m(x_n))_m$, respectively. There exists a subsequence $(f_{m_k})_k$ such that

$$f_{m_k}(z') \to f(z'), \ k \to \infty.$$

This implies

$$|\lim_k f_{m_k}(z) - f(z)| \leq \tag{4.21}$$

$$|\lim_k f_{m_k}(z) - \lim_k f_{m_k}(z')| + |f(z') - f(z)| \leq 2t. \tag{4.22}$$

Hence,

$$|\lim_m \lim_n f_m(x_n) - \lim_n \lim_m f_m(x_n)| =$$

$$|\lim_m \lim_n f_m(x_n) - f(z)| = |\lim_k f_{m_k}(z) - f(z)| \leq 2t. \qquad \square$$

Theorem 4.6.18, originally from [121, Corollary 4.2], shows a way to transfer previous results to the context of Banach spaces. Its proof uses an argument from the proof of Theorem 4.6.2, and, instead of Lemma 4.6.1, the fact from [139, Theorem 21.20]; see [121, Corollary 4.2].

Proposition 4.6.16 *If $f_1 < f_2$ are two real functions on $B_{E'}$ such that f_1 is concave and $\sigma(E', E)$-upper semicontinuous, and f_2 is convex and $\sigma(E', E)$-lower semicontinuous, there exists a $\sigma(E', E)$-continuous affine function h on $B_{E'}$ such that $f_1 < h < f_2$.*

In order to prove Theorem 4.6.18, we need the following lemma; see [121].

Lemma 4.6.17 *Let K be a convex compact set in a lcs E. Let $\mathcal{A}(K)$ be the set of all affine real functions on K. Then $d(f, C(K)) = d(f, \mathcal{A}^c(K))$ for every bounded $f \in \mathcal{A}(K)$, where $\mathcal{A}^c(K) := C(K) \cap \mathcal{A}(K)$.*

Proof Note that $d(f, C(K)) = 2^{-1}\,\mathrm{osc}(f)$ by Theorem 4.6.2. Thus, it is enough to show that $d(f, \mathcal{A}^c(K)) \leq 2^{-1}\,\mathrm{osc}(f)$. Fix arbitrary $\epsilon > 2^{-1}\,\mathrm{osc}\, f$ and set

$$f_1(x) := \inf_U \sup\{f(z) : z \in U\} - \epsilon, \quad f_2(x) := \sup_U \inf\{f(z) : z \in U\} + \epsilon,$$

where the infimum, in the first equality, and the supremum in the other one, are taken over neighbourhoods U of the point x. We prove that f_1 is concave upper semicontinuous and f_2 is concave lower semicontinuous. Indeed, we prove only the claim for the concavity of the function f_1. The other case is similar. Let $t > 0$, $x, y \in K$, and $s \in (0, 1)$. Fix a neighbourhood U of $sx + (1 - s)y$ such that

$$\sup\{f(z) : z \in U\} - \epsilon \leq f_1(sx + (1 - s)y) + t.$$

Next, choose neighbourhoods V and W of x and y, respectively, such that $sV + (1 - s)W \subset U$. Then

$$sf_1(x) + (1 - s)f_1(y) \leq$$

$$s \sup\{f(z) : z \in V\} + (1 - s)\sup\{f(z) : z \in W\} - \epsilon$$

$$\leq \sup\{f(z) : z \in U\} - \epsilon \leq f_1(sx + (1 - s)y) + t.$$

This proves the claim.

Note that $f_1 < f_2$, and we apply Proposition 4.6.16 to get $h \in \mathcal{A}^c(K)$ such that $f_1 < h < f_2$ on K. Since $h(x) - \epsilon < f(x) < h(x) + \epsilon$ for all $x \in K$,

$$\sup_{x \in K} |f(x) - h(x)| \leq \epsilon.$$

Hence, $d(f, \mathcal{A}^c(K)) \leq \epsilon$. □

We are ready to prove Theorem 4.6.18.

Theorem 4.6.18 (Cascales–Marciszewski–Raja) *Let E be a Banach space, and let $B_{E'}$ be the closed unit ball in E' endowed with the topology $\sigma(E', E)$. Let $i : E \to E''$ and $j : E'' \to \ell_\infty(B_{E'})$ be the natural embeddings. Then $d(x^{**}, i(E)) = d(j(x^{**}), C(B_{E'}))$ for each $x^{**} \in E''$.*

Proof Since $x^{**} \in E''$ is affine and bounded on $B_{E'}$, we apply Lemma 4.6.17 to get a function $h_1 \in \mathcal{A}^c(B_{E'})$ such that $\|x^{**} - h_1\| \leq \epsilon$ for $\epsilon > d(x^{**}, C(B_{E'}))$. Then $\|x^{**} - h_2\| \leq \epsilon$, where $h_2(x^*) = -h_1(-x^*)$ on $B_{E'}$. The function $g : B_{E'} \to \mathbb{R}$ defined by $g := 2^{-1}(h_1 + h_2)$ belongs to $\mathcal{A}^c(B_{E'})$ and $g(0) = 0$. Moreover,

$$\|x^{**} - g\| \leq 2^{-1}\left\{\|x^{**} - h_1\| + \|x^{**} - h_2\|\right\} \leq \epsilon.$$

There exists a linear functional y^{**} on E' such that $y^{**}|B_{E'} = g$. Then, by Grothendieck's completeness theorem, (see [322, Theorem 9.2.2]), there exists $x \in E$ such that $y^{**} = i(x)$. As $\|x^{**} - i(x)\| \leq \epsilon$, the proof is completed. □

Since $\ell^\infty(B_{E'})$ can be considered as a subspace of $(\mathbb{R}^{B_{E'}}, \tau_p)$, the map $j : (E'', \omega^*) \to (\ell^\infty(B_{E'}), \tau_p)$ is continuous. Let $H \subset E''$ be a bounded set. Then the closure $\overline{H}^{\omega^*}$ is ω^*-compact, and then

$$\overline{j(H)}^{\tau_p} = j(\overline{H}^{\omega^*}).$$

This combined with Theorem 4.6.18 yields

$$\hat{d}(\overline{j(H)}^{\tau_p}, C(B_{E'})) = \hat{d}(j(\overline{H}^{\omega^*}), C(B_{E'})) = \tag{4.23}$$

$$\sup_{z \in \overline{H}^{\omega^*}} d(j(z), C(B_{E'})) = \sup_{z \in \overline{H}^{\omega^*}} d(z, i(E)) = \hat{d}(\overline{H}^{\omega^*}, i(E)). \tag{4.24}$$

Therefore,

$$\hat{d}(\overline{j(H)}^{\tau_p}, C(B_{E'})) = \hat{d}(\overline{H}^{\omega^*}, i(E)). \tag{4.25}$$

For the next result, we refer to [13, Theorem 2.3].

Theorem 4.6.19 *(a) For a bounded set H in a Banach space E, we have* $\mathrm{ck}(H) \leq k(H) \leq \gamma(H) \leq 2\,\mathrm{ck}(H) \leq 2\,\mathrm{k}(H) \leq 2\omega(H)$, $\gamma(H) = \gamma(\mathrm{conv}(H))$, $\omega(H) = \omega(\mathrm{conv}(H))$.

(b) For every $x^{**} \in \overline{H}^{\omega^*}$, there exists a sequence $(x_n)_n$ in H such that $\|x^{**} - y^{**}\| \leq \gamma(H)$ for any cluster point y^{**} of $(x_n)_n$ in E''. Moreover, H is weakly relatively compact in E if and only if one (equivalently all) of the numbers $\mathrm{ck}(H)$, $\mathrm{k}(H)$, $\gamma(H)$, *or* $\omega(H)$ *equals zero.*

Proof First, observe that $k(H) \leq \epsilon$ if H ϵ-interchanges limits with $B_{E'}$; see [13, Proposition 2.1(i)]. Hence, $k(H) \leq \gamma(H)$. By Lemma 4.6.15, we deduce that $\gamma(H) \leq 2\,\mathrm{ck}(H)$. The equality

$$\gamma(H) = \gamma(\mathrm{conv}(H))$$

follows from Theorem 4.6.8. The equality $\omega(H) = \omega(\mathrm{conv}(H))$ is a consequence of the definition of $\omega(H)$, and the well-known fact stating that the closed convex hull of a weakly compact set in E is weakly compact; see [322, Theorem 9.8.5]. The last inequality is also easy. Indeed, take $\epsilon > 0$ and a weakly compact set $K_\epsilon \subset E$ such that $H \subset K_\epsilon + \epsilon B_E$. Then

$$\overline{H}^{\omega^*} \subset K_\epsilon + \epsilon B_{E''} \subset E + \epsilon B_{E''}.$$

Applying (4.20), we note $k(H) \leq \omega(H)$. To prove the first part of (b), it is enough to use Theorem 4.6.12 (note that $(\overline{H}^{\omega^*}, \omega^*)$ can be looked as a subspace

of $([-M, M]^{B_{E'}}, \tau_p)$, where M is a bound of H). Note also that $\omega(H) = 0$ if and only if H is weakly relatively compact; see, for example, [170, Lemma 2, p.227]. This applies to complete the proof. □

Lastly, Theorem 4.6.19 applies to fix the following:

Corollary 4.6.20 *Let E be a normed space. Then $(E, \sigma(E, E'))$ is angelic.*

Proof We may assume that E is a Banach space. Let H be a weakly relatively countably compact set in E. Hence, every sequence in H has a cluster point in $\sigma(E, E')$, and this shows that $\mathrm{ck}(H) = 0$. By Theorem 4.6.19, H is weakly relatively compact. Fix $x \in \overline{H}$, where the closure is taken in $\sigma(E, E')$. Then $\gamma(H) = 0$ by Theorem 4.6.19, and we can apply (b) to get a sequence $(x_n)_n$ in H such that

$$0 \leq \|y - x\| \leq \gamma(H) = 0$$

for every $\sigma(E, E')$-cluster point $y \in E$ of $(x_n)_n$. This implies $x_n \to x$ in $\sigma(E, E')$ (since H is $\sigma(E, E')$ relatively compact and x is the unique $\sigma(E, E')$-cluster point of $(x_n)_n$). □

We apply the previous results to show Theorem 4.6.22 is proved in [13, Theorem 3.5]. We need the following lemma; see [13].

Lemma 4.6.21 *Let $D \subset K$ be a dense subset of a compact space K, and let $H \subset C(K)$ be a uniformly bounded set. If H ϵ-interchanges limits with D, then H 2ϵ-interchanges limits with K.*

Proof Let $\delta > \epsilon$ be arbitrary. We claim that, if $f \in \overline{H}$ (the closure in $\mathbb{R}^K$), for each $y \in K$, there exists a neighbourhood V of y such that

$$\sup_{d \in V \cap D} |f(d) - f(y)| \leq \delta.$$

Indeed, assume that there exists $y \in K$ such that

$$\sup_{d \in U \cap D} |f(d) - f(y)| > \delta$$

for each neighbourhood U of y. By a simple induction, we construct two sequences $(g_n)_n$ in H and $(d_n)_n$ in D such that

$$|g_n(d_i) - f(d_i)| \leq n^{-1}$$

for $0 \leq i \leq n - 1$,

$$|g_j(d_n) - g_j(d_0)| \leq n^{-1}$$

for $1 \leq j \leq n$, and $|f(d_n) - f(d_0)| > \delta$, where $d_0 := y$. Taking a subsequence if necessary, we may assume that $f(d_n) \to r \in \mathbb{R}$. Then

$$\lim_n \lim_m g_m(d_n) = \lim_n f(d_n),$$

$$\lim_m \lim_n g_m(d_n) = \lim_m g_m(d_0) = f(d_0) = f(y).$$

Hence,

$$|\lim_m \lim_n g_m(d_n) - \lim_n \lim_m g_m(d_n)| = |\lim_n f(d_n) - f(y)| \geq \delta > \epsilon.$$

This provides a contradiction with the assumption that H ϵ-interchanges limits with D. The claim is proved. Next, choose arbitrary sequences $(x_n)_n$ in K (with a cluster point $x \in K$) and $(f_m)_m$ in H (with a cluster point $f \in \overline{H}$) such that the limits $\lim_n \lim_m f_m(x_n)$ and $\lim_m \lim_n f_m(x_n)$ exist. Then

$$\lim_m \lim_n f_m(x_n) = \lim_m f_m(x) = f(x),$$

$$\lim_n \lim_m f_m(x_n) = \lim_n f(x_n).$$

Consequently,

$$|\lim_m \lim_n f_m(x_n) - \lim_n \lim_m f_m(x_n)| = |\lim_n f(x_n) - f(x)| = A.$$

We show that $A \leq 2\delta$. Since $\delta > \epsilon$, the proof will be finished. By the first part of the proof, we find a neighbourhood U of x such that $\sup_{d \in U \cap D} |f(x) - f(d)| \leq \delta$. Since for each $n \in \mathbb{N}$ there exists $k > n$ such that $x_k \in U$, the same argument applies to get a neighbourhood $V_k \subset U$ of x_k such that

$$\sup_{d \in V_k \cap D} |f(x_k) - f(d)| \leq \delta.$$

Take $d_k \in V_k \cap D$. Then

$$|f(x_k) - f(x)| \leq |f(x_k) - f(d_k)| + |f(d_k) - f(x)| \leq 2\delta.$$

□

Finally, we prove the following [13, Theorem 3.5]:

Theorem 4.6.22 *Let K be a compact space. Let H be a uniformly bounded set in $C(K)$. Then $\gamma_K(H) \leq \gamma(H) \leq 2\gamma_K(H)$.*

Proof Let $M > 0$ be a uniform bound of H. For each $x \in K$, let $\delta_x : C(K) \to \mathbb{R}$ be the Dirac measure at x. Set $D := \{\pm\delta_x : x \in K\}$. Note that $\text{conv}(D)$ is ω^*-dense in $B_{C(K)'}$. If we show that H $\gamma_K(H)$-interchanges limits with $\text{conv}(D)$, we can apply

Lemma 4.6.21 to deduce that H $2\gamma_K(H)$-interchanges limits with $B_{C(K)'}$, which means $\gamma(H) \leq 2\gamma_K(H)$. To get this, note that $D|H \subset [-M, M]^H$. Hence, $D|H$ $\gamma_K(H)$-interchanges limits with H. By Theorem 4.6.8, we deduce that $\mathrm{conv}(D)|H$ $\gamma_K(H)$-interchanges limits with H. Therefore, H $\gamma_K(H)$-interchanges limits with $\mathrm{conv}(D)$. □

Theorem 4.6.22 applies to extend Grothendieck's characterization of weakly compact sets in Banach spaces; see [13, Corollary 3.6].

Corollary 4.6.23 *Let K be a compact space. Then a uniformly bounded set $H \subset C(K)$ is τ_p-relatively compact if and only if H is weakly relatively compact.*

Proof By Corollary 4.6.5, the set H is τ_p-relatively compact if and only if $\gamma_K(H) = 0$. We apply Theorem 4.6.22. □

Chapter 5
Strongly Web-Compact Spaces and a Closed Graph Theorem

Abstract In this chapter, we continue the study of web-compact spaces. A subclass of web-compact spaces, called strongly web-compact, is introduced, and a closed graph theorem for such spaces is provided. We prove that an own product of a strongly web-compact space need not be web-compact.

5.1 Strongly Web-Compact Spaces

In this section we introduce the class of strongly web-compact spaces [222].

A space X will be called *strongly web-compact*, if X admits a family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of X (called a representation of X) covering X, and such that for every $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$, if $x_k \in C_{n_1,n_2,\ldots,n_k}$ for all $k \in \mathbb{N}$, the sequence $(x_k)_k$ has a cluster point in X. Clearly every strongly web-compact space is web-compact. We start with the following simple characterization.

Proposition 5.1.1 *A space X is strongly web-compact if and only if X admits a resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of relatively countably compact sets.*

Proof If X is strongly web-compact, there exists a representation $\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of X that is a resolution of relatively countably compact sets. Conversely, if X admits a resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of relatively countably compact sets covering X, then X is strongly web-compact. Indeed, let $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$, and assume that

$$x_k \in C_{n_1,n_2,\ldots,n_k} := \bigcup \{A_\alpha : \alpha|_k = (n_1, n_2, \ldots, n_k), \alpha \in \mathbb{N}^\mathbb{N}\}$$

for all $k \in \mathbb{N}$. Then there exists $\beta_k = (m_n^k)_n \in \mathbb{N}^\mathbb{N}$ such that $x_k \in A_{\beta_k}$, $n_j = m_j^k$ for $j = 1, 2, \ldots, k$. Let

$$a_n = \max\{m_n^k : k \in \mathbb{N}\}$$

for $n \in \mathbb{N}$ and $\gamma = (a_n)$. Since $\gamma \geq \beta_k$ for every $k \in \mathbb{N}$, $A_{\beta_k} \subset A_\gamma$. Hence $x_k \in A_\gamma$ for all $k \in \mathbb{N}$. By the assumption the sequence $(x_k)_k$ has a cluster point in X. □

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5.2 Products of Strongly Web-Compact Spaces

We provide an example from [222] showing that the square of a strongly web-compact space need not be strongly web-compact. Our approach uses some argument of [275, 9.15 Example] and some idea presented in Novák's example [471, Theorem 4].

Example 5.2.1 There exists a countably compact topological space G such that the product $G \times G$ cannot be covered by a resolution $\{A_\alpha : \alpha \in N^N\}$ of relatively countably compact sets.

Proof Let X be a discrete space of cardinality $\mathfrak{c}$, and let X_1 and X_2 be two subspaces of X such that:

(i) $X_1 \cap X_2 = \emptyset$.
(ii) $X_1 \cup X_2 = X$.
(iii) $|X_1| = |X_2| = \mathfrak{c}$.

By (iii) there exists a bijection σ from X_1 onto X_2. Then its Čech–Stone extension σ^β is a homeomorphism from βX_1 onto βX_2. Since X is a discrete space, we have

$$\overline{X_1}^{\beta X} \cap \overline{X_2}^{\beta X} = \emptyset$$

and

$$\overline{X_1}^{\beta X} \cup \overline{X_2}^{\beta X} = \overline{X}^{\beta X};$$

this follows from [195, 3.6.2]. If Y is a subspace of X, we can identify βY with $\overline{Y}^{\beta X}$; see again [195, 3.6.8]. So, it follows that $\beta X_1 \cap \beta X_2 = \emptyset$ and $\beta X_1 \cup \beta X_2 = \beta X$. Moreover, if N is a countable infinite subspace of X,

$$\left|\overline{N}^{\beta X}\right| = |\beta N| = |\beta \mathbb{N}| = 2^{\mathfrak{c}}.$$

Now define a homeomorphism $\varphi : \beta X \to \beta X$ by $\varphi(x) = \sigma^\beta(x)$, if $x \in \beta X_1$ and $\varphi(x) = \left(\sigma^\beta\right)^{-1}(x)$ if $x \in \beta X_2$. Clearly $\varphi(\varphi(p)) = p$ for every $p \in \beta X$, and $p \in X$ if and only if $\varphi(p) \in X$.

Since $\varphi(\beta X_1) = \beta X_2$ and $\varphi(\beta X_2) = \beta X_1$, the map φ does not have fixed points.

Set

$$Z := \bigcup \left\{\overline{N}^{\beta X} : N \in \mathcal{N}\right\},$$

where $\mathcal{N}$ denotes the family of all countable infinite subsets of X. By $\mathcal{M}$ we denote the family of all countable infinite subsets of Z.

Since

$$|\mathcal{N}| = \mathfrak{c}^{\aleph_0},$$

we have

$$|Z| = \mathfrak{c}^{\aleph_0} \times 2^{\mathfrak{c}} = 2^{\mathfrak{c}},$$

and hence $|\mathcal{M}| = 2^{\mathfrak{c}}$. So, if $\mathfrak{m}$ is the first ordinal of cardinality $2^{\mathfrak{c}}$, we have that $\mathcal{M} = \{M_\alpha : 0 \leq \alpha < \mathfrak{m}\}$. Note that $\alpha < \mathfrak{m}$ implies that $|\alpha| = |[0, \alpha)| < 2^{\mathfrak{c}}$, and that X is contained in Z.

We claim that, if $M \in \mathcal{M}$, then $\left|\overline{M}^{\beta X}\right| = 2^{\mathfrak{c}}$. Indeed, as M is a countable infinite subset of Z, there is a countable family

$$\{N_i : i \in \mathbb{N}\} \subseteq \mathcal{N}$$

with

$$\overline{M}^{\beta X} \subseteq \overline{\bigcup_{i=1}^{\infty} N_i}^{\beta X}.$$

Since $\bigcup_{i=1}^{\infty} N_i \in \mathcal{N}$, and every infinite closed subset of $\beta\mathbb{N}$ has cardinality $2^{\mathfrak{c}}$, see [195, 3.6.14], $\left|\overline{M}^{\beta X}\right| = 2^{\mathfrak{c}}$.

Let

$$y_0 \in \overline{M_0}^{\beta X} \setminus M_0.$$

Let $1 \leq \alpha < \mathfrak{m}$. Assume that for each $\beta < \alpha$ we have already chosen $y_\beta \in \overline{M_\beta}^{\beta X} \setminus (M_\beta \cup \{\varphi(y_\gamma) : 0 \leq \gamma < \beta\})$. Next, choose

$$y_\alpha \in \overline{M_\alpha}^{\beta X} \setminus (M_\alpha \cup \{\varphi(y_\gamma) : 0 \leq \gamma < \alpha\}).$$

This is possible since $\left|\overline{M_\alpha}^{\beta\mathbb{N}}\right| = 2^{\mathfrak{c}}$, $|M_\alpha| = \aleph_0$ and $|\alpha| < 2^{\mathfrak{c}}$. Thus we have a set $\Gamma = \{y_\gamma : 0 \leq \gamma < \mathfrak{m}\}$ such that

$$y_\alpha \in \overline{M_\alpha}^{\beta X} \setminus (M_\alpha \cup \{\varphi(y_\gamma) : 0 \leq \gamma < \alpha\})$$

for every $0 \leq \alpha < \mathfrak{m}$.

Let $\alpha \in [0, \mathfrak{m})$. The selection of y_α ensures that y_α is a limit point of the countable set M_α. Moreover, $\varphi(y_\alpha) \notin \Gamma$. Indeed, if $\varphi(y_\alpha) = y_\gamma$, then $y_\alpha = \varphi(y_\gamma)$, and hence $\alpha \leq \gamma$. As $y_\alpha \neq \varphi(y_\alpha)$, so $\alpha < \gamma$, which contradicts the fact that $y_\gamma = \varphi(y_\alpha)$. Therefore, if $p \in \Gamma$ then $\varphi(p) \notin \Gamma$. Set $G := X \cup \Gamma$. Then $G \subseteq Z$.

Since every countable infinite subset A of G is equal to M_α for some $0 \leq \alpha < \mathfrak{m}$, the space G contains a limit point of A. Therefore G is countably compact.

If $p \in G \setminus X$, then $p \in \Gamma$, what according to we have already seen above, implies $\varphi(p) \notin \Gamma$. Since for $p \notin X$ we have $\varphi(p) \notin X$, we conclude that if $p \in G \setminus X$, then $\varphi(p) \notin G$. Likewise $\varphi(p) \in G \setminus X$ leads to $p = \varphi(\varphi(p)) \notin G$. So, if $p \in \beta X \setminus X$, then G does not contain both p and $\varphi(p)$, i.e. $(p, \varphi(p)) \notin G \times G$ for every $p \in \beta X \setminus X$. Hence

$$(G \times G) \cap \{(p, \varphi(p)) : p \in \beta X\} = \{(x, \varphi(x)) : x \in X\}.$$

Since the graph $\{(p, \varphi(p)) : p \in \beta X\}$ of the continuous map $\varphi : \beta X \to \beta X$ is closed in $\beta X \times \beta X$, we conclude that

$$S := \{(x, \varphi(x)) : x \in X\}$$

is a *closed subspace of* $G \times G$. Moreover, X is homeomorphic to S, so S is uncountable and discrete. Finally, assume that $G \times G$ is covered by resolution of relatively countably compact subsets. Then S is covered also by such a resolution; hence S is covered by a resolution consisting of finite sets. This means that S is countably, by Proposition 3.2.1, a contradiction. □

Recall that countable products of K-analytic spaces are also K-analytic. Example 5.2.1 yields the following:

Corollary 5.2.2 *There exists a quasi-Suslin space X such that $X \times X$ is not quasi-Suslin.*

5.3 A Closed Graph Theorem for Strongly Web-Compact Spaces

It turns out that the concept of *strongly web-compact spaces* can be used to extend some classical closed graph theorems; see [218].

In [611, I.4.2 (11)] Valdivia proved that a linear map with closed graph from a metrizable Baire lcs E into a quasi-Suslin lcs F is continuous. Drewnowski [180, Corollary 4.10] proved that every continuous linear map from a tvs having a compact resolution onto an F-space, i.e. a metrizable and complete tvs, is open. In this section we use some techniques of [611] to get a closed graph theorem, which extends Valdivia's [611, I.4.2 (11)] and Drewnowski's [180, Corollary 4.10, Corollary 4.11]. In Theorem 5.3.1 we need only to assume that F is a tvs with a relatively countably compact resolution and E is a Baire tvs.

We start with some additional facts which will be used in this section.

Fact 1. Let B be a subset of a topological space E. Assume B is covered by a web $\left\{B_{n_1,n_2,\ldots,n_p} : p, n_1, n_2, \ldots, n_p \in \mathbb{N}\right\}$ and that

$$\bigcap_p O(B_{n_1,n_2,\ldots,n_p}) \subset B$$

for each sequence $(n_p)_p$ in $\mathbb{N}$. By Proposition 2.1.5 the set B has the Baire property. Indeed, $H := O(B) \setminus \bigcup_{n_1 \in \mathbb{N}} O(B_{n_1})$ and

$$H_{n_1,n_2,\ldots,n_p} := O(B_{n_1,n_2,\ldots,n_p}) \setminus \bigcup_{m \in \mathbb{N}} O(B_{n_1,n_2,\ldots,n_p,m}),$$

where $p, n_1, n_2, \ldots, n_p \in \mathbb{N}$, are nowhere dense sets. By the hypothesis

$$H \cup \left\{ \bigcup \left\{ H_{n_1,n_2,\ldots,n_p} : p, n_1, n_2, \ldots, n_p \in \mathbb{N} \right\} \right\}$$

contains $O(B) \setminus B$. Therefore the set B has the Baire property.

Fact 2. Let E be a topological space admitting a weaker first countable topology τ. If B is a subset of E having a τ-closed resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, the set B has the Baire property. Indeed, choose $(n_p)_p$ in $\mathbb{N}$ and $x \in O(C_{n_1,n_2,\ldots,n_p})$, $p \in \mathbb{N}$, where, as usual, $C_{n_1,n_2,\ldots,n_p} := \bigcup \{A_\alpha : \alpha | p = (n_1, n_2, \ldots, n_p)\}$. Let $(U_p)_p$ be a τ-neighbourhood basis of x. Select $x_p \in U_p \cap C_{n_1,n_2,\ldots,n_p}$. There exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $x_p \in A_\alpha$ for $p \in \mathbb{N}$. By the τ-closedness condition we get that $x \in A_\alpha \subset B$ and Fact 1 applies.

By $\mathfrak{F}(E)$ we denote a basis of balanced neighbourhoods of zero in a tvs E. Now we prove the following closed graph theorem from [218]:

Theorem 5.3.1 *Let E and F be tvs such that E is Baire and F admits a relatively countably compact resolution $\left\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$. If $f : E \to F$ is a linear map with closed graph, there is a sequence $(U_n)_n$ in $\mathfrak{F}(E)$ such that for every $V \in \mathfrak{F}(F)$ there exists $m \in \mathbb{N}$ with $m^{-1}U_m \subset f^{-1}(V)$. Hence f is continuous. If $E = F$, then E is a separable F-space.*

Proof Since E is a Baire space, there exists a sequence $(r_p)_p$ in $\mathbb{N}$ such that $\overline{f^{-1}\left(H_p\right)} - \overline{f^{-1}\left(H_p\right)}$ is a neighbourhood of zero for each $p \in \mathbb{N}$, where $H_p := C_{r_1,r_2,\ldots,r_p}$. Let $(U_p)_p$ be a sequence of balanced neighbourhoods of zero in E such that

$$U_{p+1} + U_{p+1} \subset U_p$$

and

$$U_p \subseteq \overline{f^{-1}\left(H_p\right)} - \overline{f^{-1}\left(H_p\right)}$$

for each $p \in \mathbb{N}$. Let τ be the semi-metrizable translation invariant vector topology on E defined by the basis $(p^{-1}U_p)_p$ of neighbourhoods of zero. Since the graph of

f is closed, there is a coarser linear topology ϱ on F such that the map $f : E \to (F, \varrho)$ is continuous. We claim that

$$f : (E, \tau) \to (F, \varrho)$$

is continuous. Indeed, if V is a closed neighbourhood of zero in (F, ϱ), there exists $q \in \mathbb{N}$ such that

$$q^{-1}(H_q - H_q) \subset V.$$

As $f^{-1}(V)$ is closed in E,

$$q^{-1}U_q \subset q^{-1}(\overline{f^{-1}(H_q)} - \overline{f^{-1}(H_q)}) \subset f^{-1}(V).$$

This proves the claim. If $W \in \mathfrak{F}(F)$ is closed and balanced, and

$$B_\alpha := f^{-1}\left(A_\alpha \cap W\right),$$

then (using the fact that $\left\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ is relatively countably compact) we deduce $\overline{B_\alpha}^{\tau} \subset f^{-1}(W)$. Thus $\left\{\overline{B_\alpha}^{\tau} : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ is a τ-closed resolution for $f^{-1}(W)$, and $f^{-1}(W)$ has the Baire property by Fact 2. Since $f^{-1}(W)$ is of second category, Proposition 2.1.8 ensures that $f^{-1}(W) - f^{-1}(W)$ is a neighbourhood of zero in E. Hence $f : E \to F$ is continuous. If $V \in \mathfrak{F}(F)$ is closed, there is $m \in \mathbb{N}$ such that

$$m^{-1}\overline{H_m} - m^{-1}\overline{H_m} \subset V,$$

so $m^{-1}U_m \subset f^{-1}(V)$. If $E = F$, then E is a metrizable tvs having a compact resolution, and Corollary 6.2.5 (below) applies to deduce that (E, τ) is analytic, hence separable. Since every analytic Baire tvs is metrizable and complete (see Theorem 7.1.2), the proof is completed. □

Theorem 5.3.1 fails for topological groups in general. If a compact group of Ulam measurable cardinality is either Abelian or connected, then it admits a strictly finer countably compact group topology [144].

Corollary 5.3.2 *Every linear map f from a Baire tvs E into a tvs F whose graph G admits a relatively countably compact resolution is continuous. Hence, every linear map from a metrizable and complete tvs into a separable metrizable tvs whose graph admits a complete resolution is continuous.*

Proof The projection $P(x, f(x)) = x$ of G onto E is continuous, so P^{-1} is continuous by Theorem 5.3.1. Note that $f = Q \circ P^{-1}$, where $Q : G \to F$ is the projection. To complete the proof we show that f is continuous on every closed separable vector subspace E_0 of E. Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a complete resolution

in G. The sets $A_\alpha \cap (E_0 \times F)$ form a complete resolution on $G \cap (E_0 \times F)$. By Corollary 6.2.5 every metrizable and separable tvs having a complete resolution is analytic. □

Corollary 5.3.3 is a special case of Theorem 7.2.2 below.

Corollary 5.3.3 *Let f be a linear functional on a metrizable and complete tvs E. The following conditions are equivalent:*

(i) *E admits a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that f is continuous on each A_α.*
(ii) *f is continuous on E.*
(iii) *The kernel $N := \{x \in E : f(x) = 0\}$ has a complete resolution.*

Proof (i) $\Rightarrow$ (ii): Observe that f is continuous on the closure each A_α. Indeed, (we follow the argument from [180, Proposition 4.1]) fix A_α and $x \in \overline{A_\alpha}$. There exists $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $A_\alpha \cup \{x\} \subset A_\beta$. Since, by the assumption, $f|A_\beta$ is continuous at x, $f|(A_\alpha \cup \{x\})$ is also continuous at x. This means that the limit of $f|A_\alpha$ at point x exists and equals $f(x)$. Hence $f|\overline{A_\alpha}$ is continuous. We may assume that each set A_α is closed. Also we may assume that E is separable. As every metrizable and complete separable tvs admits a compact resolution, fix a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ on E. Then $D_\alpha := A_\alpha \cap K_\alpha$ compose a compact resolution in E, and f is continuous on each D_α. Assume that f is discontinuous, and let H be its (dense) kernel. Clearly $\{H \cap D_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution on H. Also H admits a strictly weaker metrizable and complete vector topology. Indeed, if D is an algebraic complement to H in E, the restriction of the quotient map $q|_H : H \to E/D$ generates on H such a topology. Theorem 5.3.1 applies to reach a contradiction. A similar argument as in (i) $\Rightarrow$ (ii) applies to get (iii) $\Rightarrow$ (i). □

Chapter 6
Weakly Analytic Spaces

Abstract This chapter studies analytic spaces. We show that a regular space X is analytic if and only if it has a compact resolution and admits a weaker metric topology. This fact, essentially due to Talagrand, extends Choquet's theorem (every metric K-analytic space is analytic). Several applications will be provided. We show Christensen's theorem stating that a separable metric topological space X is a Polish space if and only if it admits a compact resolution swallowing compact sets. We also study the following general problem: When can analyticity or K-analyticity of the weak topology $\sigma(E, E')$ of a dual pair (E, E') be lifted to stronger topologies on E compatible with the dual pair? We prove that, if X is an uncountable analytic space, the Mackey duals $L_\mu(X)$ of $C_p(X)$ is weakly analytic and not analytic. The density condition, due to Heinrich, motivates us to study the analyticity of the Mackey and strong duals of (LF)-spaces. We study trans-separable spaces and show that a tvs with a resolution of precompact sets is trans-separable. This is applied to prove that precompact sets are metrizable in any uniform space whose uniformity admits a U-basis.

6.1 A Few Facts about Analytic Spaces

In this section we collect some general facts about analytic tvs. Recall again that a topological space E is *analytic* if E is a continuous image of a Polish space (or, equivalently, of the space $\mathbb{N}^{\mathbb{N}}$).

Proposition 6.1.1 *(i) Every open (closed) subspace of an analytic space E is analytic.*
(ii) Any countable product $\prod_n E_n$ of analytic spaces is an analytic space.
(iii) Every countable union (intersection) of analytic subspaces of a topological space is an analytic space.
(iv) If E is an analytic space and F is its closed subspace, then the quotient E/F is an analytic space.

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Proof (i) Let U be an open (closed) subspace of E, and let $T : P \to E$ be a continuous map from a Polish P space onto E. Then $T^{-1}(U)$ is a Polish subspace of E by Proposition 2.1.10. The map $T|T^{-1}(U) : T^{-1}(U) \to U$ is a continuous surjection from a Polish space $T^{-1}(U)$. Hence U is analytic.

(ii) For each $n \in \mathbb{N}$ let P_n be a Polish space, and let $T_n : P_n \to E_n$ be a continuous map onto E_n. Clearly $P = \prod_n P_n$ is a Polish space and the map $T : P \to \prod_n E_n$ defined by $T((x_n)) := (T_n(x_n))$ is continuous surjection onto $\prod_n E_n$. Hence $\prod_n E_n$ is analytic. The remaining claims are left to the reader. □

Proposition 6.1.2 *Every Borel subset of an analytic space E is analytic.*

Proof Set $\mathcal{U} := \{A \subset E : A, E \setminus A \text{ are analytic}\}$. Note that $\mathcal{U}$ contains all open subsets of E. It is enough to prove that $\mathcal{U}$ is an σ-algebra in E. The proof will be completed if we realize that, if $(A_n)_n$ is a sequence of subsets of $\mathcal{U}$, then $\bigcup_n A_n \in \mathcal{U}$. □

To prove Proposition 6.1.4 we need the following simple lemma; see [195].

Lemma 6.1.3 *A Lindelöf regular topological space X is normal.*

Proof Let A and B be two disjoint closed subsets of X. For points $a \in A$ and $b \in B$ let U_a and V_b be open neighbourhoods of a and b, respectively, such that $\overline{U_a} \cap B = \emptyset$ and $\overline{V_b} \cap A = \emptyset$. Clearly

$$\{U_a : a \in A\} \cup \{V_b : b \in B\} \cup \{X \setminus (A \cup B)\}$$

is an open cover of X. By the assumption on X there exist a sequence $(U_n)_n$ in $\{U_a : a \in A\}$ and a sequence $(V_n)_n$ in $\{V_b : b \in B\}$, such that $A \subset \bigcup_n U_n$ and $B \subset \bigcup_n V_n$. Set

$$U_n^* = U_n \setminus \bigcup \left\{\overline{V_m} : m \leq n\right\},$$

and

$$V_n^* = V_n \setminus \bigcup \left\{\overline{U_m} : m \leq n\right\}.$$

Then open sets $U = \bigcup_n U_n^*$ and $V = \bigcup_n V_n^*$ are disjoint, $A \subset U$ and $B \subset V$. Hence X is normal. □

The following applicable result was obtained by Talagrand [577]: Proposition 6.1.4 extends Choquet's theorem from [138] (every metric K-analytic space is analytic). The proof presented below is a modification of the proof due to Cascales and Oncina; see [125, Corollary 4.3] and also [522, Theorem 5.5.1] and [550, Corollary 1, p.105]

Proposition 6.1.4 (Talagrand) *Let (X, τ) be a K-analytic space. Let d be a metric on X whose topology is coarser than τ. Then (X, τ) is analytic. Every regular analytic space X admits a weaker metric topology.*

Proof Let $\left\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ be a compact resolution on (X, τ), and let $\{z_n : n \in \mathbb{N}\}$ be a dense subset of (X, d). By $B_d(z, r)$ denote the d-closed ball in (X, d) of centre z and radius $r > 0$. For $\beta = (b_n) \in \mathbb{N}^{\mathbb{N}}$ let

$$D_\beta := \bigcap_{n\in\mathbb{N}} B_d(z_{b_n}, n^{-1}).$$

Each set D_β is unitary or void. For $y \in X$ there exists $(\alpha, \beta) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $K_\alpha \cap D_\beta = \{y\}$.

For $K_\alpha \cap D_\beta \neq \emptyset$, we denote by $y_{\alpha\beta}$ the element of X such that $K_\alpha \cap D_\beta = \left\{y_{\alpha\beta}\right\}$. If

$$T := \left\{(\alpha, \beta) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} : \emptyset \neq K_\alpha \cap D_\beta = \left\{y_{\alpha\beta}\right\}\right\},$$

the map $f : T \to X$ defined by $f((\alpha, \beta)) = y_{\alpha\beta}$ is surjection.

Let $(\alpha(p), \beta(p))_p$ be a sequence in T that converges to (α, β) in $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, and let $(\alpha(p), \beta(p))_{p(m)}$ be a subsequence.

By the K-analyticity $\left(y_{\alpha(p),\beta(p)}\right)_{p(m)}$ has an adherent point $y \in K_\alpha$. Since $\beta(p)$ converges to $\beta = (b_n)_n \in \mathbb{N}^{\mathbb{N}}$, the sequence $\left(y_{\alpha(p),\beta(p)}\right)_{p(m)}$ is eventually in each $B_d(z_{b_n}, n^{-1})$; hence its adherent point y belongs to $B_d(z_{b_n}, n^{-1})$.

This shows that

$$y \in K_\alpha \cap D_\beta = \left\{y_{\alpha\beta}\right\}.$$

We proved that $(\alpha, \beta) \in T$, i.e. T is a closed subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ and, therefore, T is a Polish space. Moreover, we proved that $y_{\alpha\beta}$ is an adherent point of each subsequence of $\left(y_{\alpha(p),\beta(p)}\right)_p$. This implies that $y_{\alpha(p),\beta(p)}$ converges to $y_{\alpha\beta}$, i.e. $f(\alpha(p), \beta(p))$ converges to $f(\alpha, \beta)$. Hence f is a continuous mapping from the Polish space T onto (Y, τ), and this proves that (Y, τ) is analytic.

In order to prove the second part of the proposition, assume that $\Delta = \{(x, x) : x \in X\}$ is the diagonal of the analytic space $X \times X$. Clearly Δ and $(X \times X)\backslash\Delta$ are analytic and, therefore, they are Lindelöf.

If $x \neq y$, there exist two closed neighbourhoods F_x and F_y of x and y, respectively, such that

$$F_x \times F_y \subset (X \times X)\backslash\Delta.$$

The Lindelöf property enables to determine a sequence $(x_n, y_n)_n$ such that

$$X \times X \backslash \Delta = \bigcup_n F_{x_n} \times F_{y_n}.$$

Therefore Δ is a G_δ-subset of $X \times X$ since $\Delta = \bigcap_n G_n$, where

$$G_n = (X \times X) \backslash (F_{x_n} \times F_{y_n}).$$

For each $(x, x) \in \Delta$ and $n \in \mathbb{N}$ there exists an open set $U_{x,n}$ in X such that

$$(x, x) \in U_{x,n} \times U_{x,n} \subset G_n.$$

As the space X is completely regular by Lemma 6.1.3, we may assume that there exists a continuous function $f_{x,n} : X \to [0, 1]$ such that

$$f_{x,n}(U_{x,n}) \subset]\frac{1}{2}, 1], \quad f_{x,n}(X \backslash U_{x,n}) \subset [0, \frac{1}{2}].$$

By the Lindelöf property of Δ the family $\left\{U_{x,n} : x \in X\right\}$ contains a sequence $(U_{x_{(i,n)},n})_i$ such that

$$\Delta \subset \bigcup_i U_{x_{(i,n)},n} \times U_{x_{(i,n)},n} := G_n^*.$$

Since $\Delta = \bigcap_n G_n^*$, we deduce that, if $x \neq y$ are two different points of X, there exists $n \in \mathbb{N}$ such that $(x, y) \notin G_n^*$. Then, since $(x, x) \in G_n^*$, there exists $j \in \mathbb{N}$ such that $x \in U_{x_{(j,n)},n}$. This implies that $y \notin U_{x_{(j,n)},n}$, since $(x, y) \notin G_n^*$. By the construction

$$f_{x_{(j,n)},n}(x) \neq f_{x_{(j,n)},n}(y).$$

Then, X, endowed with the topology that makes continuous the countable family of functions $\left\{f_{x_{(i,n)},n} : (i, n) \in \mathbb{N}^2\right\}$, is metrizable with the metric defined by the formula

$$d(x, y) = \sum \left\{2^{-i-n} \left| f_{x_{(i,n)},n}(x) - f_{x_{(i,n)},n}(y)\right| : (i, n) \in \mathbb{N}^2\right\}.$$

Clearly $d(x, y)$ defines a metric topology weaker than τ. □

Corollary 6.1.5 *A compact space is analytic if and only if it is metrizable.*

Note that every separable and complete metric space is a continuous and open image of $\mathbb{N}^{\mathbb{N}}$; see [522, Part 3, Theorem 1.2.14] and Theorem 3.5.5. In order to fix another sufficient and necessary conditions for a K-analytic space to be analytic (see [522, Theorem 5.5.1]), we need the following simple lemma:

Lemma 6.1.6 *Suppose that U is an open subset of a topological space X. If there exist compact subsets K_i, $1 \leq i \leq n$, such that $\bigcap_{i=1}^{n} K_i \subset U$, then there exist neighbourhoods U_i of K_i, $1 \leq i \leq n$, such that $\bigcap_{i=1}^{n} U_i \subset U$.*

Proof For $n = 2$ the lemma follows from the well-known fact stating that two disjoint compact subsets have disjoint open neighbourhoods. Therefore, there exist open neighbourhoods V_i of $K_i \backslash U$, $i = 1, 2$, such that $V_1 \cap V_2 = \emptyset$. Then the sets $U_i := U \cup V_i$, $i = 1, 2$, are as desired. Therefore, if $\bigcap_{i=1}^{n} K_i \subset U$, then there exist two open subsets V_{n-1} and U_n such that $\bigcap_{i=1}^{n-1} K_i \subset V_{n-1}$, $K_n \subset U_n$, $V_{n-1} \cap U_n \subset U$. Then if the claim is true for $n - 1$, it is also true for n. Now lemma follows by a simple induction. □

We prove a part of [522, Theorem 5.5.1].

Proposition 6.1.7 *Let X be a K-analytic space. The following statements are equivalent:*

(a) X is analytic.
(b) X is a continuous image of a separable metric space.

Proof Clearly (a) implies (b). Therefore, we only need to prove that (b) implies (a). If X is a continuous image of a separable metric space, $X \times X$ is also a continuous image of a separable metric space. The space $X \times X$ is hereditarily Lindelöf. Therefore, if $\Delta := \{(x, x) : x \in X\}$, then $(X \times X) \backslash \Delta$ is an open Lindelöf subset of $X \times X$, and then there exists a sequence $(G_n^1, G_n^2)_n$ of pairs of open sets such that

$$(X \times X) \backslash \Delta = \bigcup_n G_n^1 \times G_n^2. \tag{6.1}$$

For each $n \in \mathbb{N}$ let

$$A_n^i := X \backslash G_n^i, \quad i = 1, 2. \tag{6.2}$$

By the K-analyticity of X we deduce that each closed set A_n^i is K-analytic, and then there exists a compact-valued (usco) map $K_n^i : \mathbb{N}^{\mathbb{N}} \to 2^{A_n^i}$ yielding the K-analyticity of A_n^i. Clearly, for each $n \in \mathbb{N}$ the compact valued map

$$L_n : \{1, 2\} \times \mathbb{N}^{\mathbb{N}} \to 2^X$$

defined by

$$L_n((i, \omega)) := K_n^i(\omega)$$

is also (usco).

Let L be the map from $(\{1,2\}\times\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ into 2^X defined by

$$L((i(n),\omega(n))_n) := \bigcap_n L_n((i(n),\omega(n))),$$

where each $i(n)\in\{1,2\}$, $\omega(n)\in\mathbb{N}^{\mathbb{N}}$. Then

$$L((i(n),\omega(n))_n) = \bigcap_n K_n^{i(n)}(\omega(n)). \tag{6.3}$$

Let U be an open subset of X such that $\bigcap_n K_n^{i(n)}(\omega(n))\subset U$. By the compactness there exists n_0 such that

$$\bigcap_{m=1}^{n_0} K_m^{i(m)}(\omega(m))\subset U.$$

Lemma 6.1.6 implies that for each $1\le m\le n_0$ there exists a neighbourhood U_m of $K_m^{i(m)}(\omega(m))$ such that $\bigcap_{m=1}^{n_0} U_m\subset U$. Then, by the upper semicontinuity of each $K_m^{i(m)}$ there exists a neighbourhood V_m of $\omega(m)$ such that $K_m^{i(m)}(V_m)\subset U_m$ for each $1\le m\le n_0$. Therefore

$$\bigcap_{m=1}^{n_0} K_m^{i(m)}(V_m)\subset U.$$

Hence for the open set

$$V := \{(i(n),\omega(n))_n\in(\{1,2\}\times\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} : i(m)=i_m,\omega(m)\in V_m : 1\le m\le n_0\}$$

we note $L(V)\subset U$. For $U=\emptyset$ the set

$$\{(i(n),\omega(n))_n\in(\{1,2\}\times\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} : L((i(n),\omega(n))_n)=\emptyset\}$$

is open, and then

$$F := \{(i(n),\omega(n))_n\in(\{1,2\}\times\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} : L((i(n),\omega(n))_n)\neq\emptyset\}$$

is a closed subset of the separable, complete, and metrizable space $(\{1,2\}\times\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. The restriction of L to the Polish space F is a compact-valued (usco) map. In order to complete the proof we only need to show that $L(F)=X$ and the compact values of $L_{|F}$ are unitary sets.

From (6.1) and (6.2) it follows that for each $n\in\mathbb{N}$ we have $A_n^1\cup A_n^2=X$, and this clearly implies that $L(F)=X$. Finally, we prove that if $(i(n),\omega(n))_n\in F$, then $L((i(n),\omega(n))_n)$ is a unitary set. Indeed, from (6.1) we deduce that for $x\neq y$

there exists q such that $(x, y) \in G_q^1 \times G_q^2$. This and (6.2) yield $x \notin A_q^1$ and $y \notin A_q^2$. This implies that $\{x, y\} \not\subseteq A_q^i$ for $i = 1, 2$. From $K_q^i(\omega) \subset A_q^i$ it follows

$$\{x, y\} \not\subseteq K_q^i(\omega)$$

and, in particular,

$$\{x, y\} \not\subseteq K_q^{i(q)}(\omega(q)). \tag{6.4}$$

Then from (6.3) and (6.4) we deduce

$$\{x, y\} \not\subseteq L((i(n), \omega(n))_n).$$

This proves that for $(i(n), \omega(n))_n \subset F$ and different points x and y in X, the set $\{x, y\}$ is not contained in $L((i(n), \omega(n))_n)$.

We proved that the values of $L_{|F}$ are compact unitary sets and the map $f : F \to X$ defined by

$$\{f((i(n), \omega(n))_n)\} = L((i(n), \omega(n))_n)$$

is a continuous surjection. We showed that $(b) \Rightarrow (a)$, since F is a Polish space. □

Proposition 6.1.8 *Let E be a separable lcs. Then $(E', \sigma(E', E))$ is strongly web-compact if and only if $(E', \sigma(E', E))$ is analytic.*

Proof Clearly every analytic space is strongly web-compact. For the converse, assume $(E', \sigma(E', E))$ is a strongly web-compact space. Since E is separable, there exists on E' a metric topology ξ weaker than the topology $\sigma(E', E)$. Hence $(E', \sigma(E', E))$ is angelic. Consequently, every relatively countably compact set in $\sigma(E', E)$ is relatively compact. Therefore $(E', \sigma(E', E))$ admits a compact resolution. Then, by Corollary 3.2.9 the space $(E', \sigma(E', E))$ is K-analytic. Now Proposition 6.1.4 applies. □

6.2 Christensen's Theorem

We know already that every Polish space E admits a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ *swallowing compact sets*, i.e. every compact set in E is contained in some K_α. Clearly, every hemicompact space E with a fundamental (increasing) sequence $(K_n)_n$ of compact sets generates a compact resolution swallowing compact sets. Indeed, it is enough to set $K_\alpha := K_{n_1}$ for any $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$. The space $\mathbb{Q}$ of the rational numbers is analytic and does not admit a compact resolution swallowing compact sets; this follows from Theorem 6.2.4, since $\mathbb{Q}$ is not complete.

Clearly every σ-compact space is Lindelöf. Nevertheless, there exist locally compact spaces having a compact resolution swallowing compact sets that are not Lindelöf. The following interesting example is due to Tkachuk [592]:

Example 6.2.1 (Tkachuk) There exists a locally compact space E that is a countably compact and non-compact space. Moreover, E has a compact resolution swallowing compact sets.

Proof Let $\mathbb{N}^{\mathbb{N}}$ be endowed with the discrete topology. For any $\alpha \in \mathbb{N}^{\mathbb{N}}$ set

$$A_\alpha := \{\beta \in \mathbb{N}^{\mathbb{N}} : \beta \leq^* \alpha\},$$

where the relation $\beta = (b_n) \leq^* \alpha = (a_n)$ means that there exists $m \in \mathbb{N}$ such that $b_n \leq a_n$ for all $n \geq m$. Clearly $A_\alpha \subset A_\beta$ for $\alpha \leq \beta$. Set $E := \bigcup_\alpha K_\alpha$, where $K_\alpha := \overline{A_\alpha}$ and closure is taken in $\beta\mathbb{N}^{\mathbb{N}}$. Hence $\mathbb{N}^{\mathbb{N}} \subset E \subset \beta\mathbb{N}^{\mathbb{N}}$, and E is locally compact having a compact resolution of open subsets of $\beta\mathbb{N}^{\mathbb{N}}$. Note that E is countably compact. Indeed, choose an arbitrary countable set $A \subset E$. Then there exists a countable set $D \subset \mathbb{N}^{\mathbb{N}}$ such that $A \subset \bigcup_{\alpha\in D} K_\alpha$. Choose $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha \leq^* \beta$ for any $\alpha \in D$. Then

$$\overline{A} \subset \overline{\bigcup_{\alpha\in D} K_\alpha} \subset K_\beta \subset E;$$

hence, the closure of every countable subset of E is compact and, therefore, E is countably compact. Note also that the resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ swallows compact sets. Indeed, the sets K_α are also open sets, and if $K \subset E$ is a compact set in E, there exists a finite set $C = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subset \mathbb{N}^{\mathbb{N}}$ such that $K \subset \bigcup_{\alpha\in C} K_\alpha$. Set $b_n := \sum_{j=1}^k a_n^j$, where $\alpha_j := (a_n^j)$. Let $\beta = (b_n)$. Then

$$K \subset \bigcup_{\alpha\in C} K_\alpha \subset K_\beta.$$

Next, observe that E is non-compact; hence as countably compact E cannot be Lindelöf. Indeed, set $\mathcal{F} = \{W_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, where $W_\alpha := \mathbb{N}^{\mathbb{N}} \setminus A_\alpha$. The family $\mathcal{F}$ has the finite intersection property. To see this, for a finite set $B = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ in $\mathbb{N}^{\mathbb{N}}$ set $b_n := \sum_{j=1}^k a_n^j + 1$, where $\alpha_j := (a_n^j)_n$ for $1 \leq j \leq k$. If $\beta := (b_n)$, we have $\beta \in \bigcap_{\alpha\in B} W_\alpha$. Consequently,

$$U := \bigcap_\alpha \overline{W_\alpha} \neq \emptyset.$$

Since $K_\alpha \cap \overline{W_\alpha} = \emptyset$ for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, we have $U \subset \beta\mathbb{N}^{\mathbb{N}} \setminus E$. As $\overline{E} = \beta\mathbb{N}^{\mathbb{N}}$, we conclude that E is non-compact. □

Theorem 6.2.4, due to Christensen [140, Theorem 3.3], provides also a good motivation to study spaces having a compact resolution swallowing compact sets.

First we recall some typical notations. Following Suslin scheme notation (see Sect. 3.3), for

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots) \in \mathbb{N}^{\mathbb{N}}$$

set $\sigma|0 := \emptyset$, and for $n \in \mathbb{N}$ set $\sigma|n := (\sigma_1, \sigma_2, \ldots, \sigma_n)$. Let $\mathbb{N}^0 := \emptyset$, and let

$$\mathbb{N}^{(\mathbb{N})} := \bigcup \{\mathbb{N}^n : n \in \mathbb{N}\} = \{\sigma|n : \sigma \in \mathbb{N}^{\mathbb{N}}, n \in \mathbb{N}\}.$$

By $B(x, r)$ we denote (as usual) the open ball with the centre x and radius r for a metrizable space X. We need two claims.

Lemma 6.2.2 *If a metrizable space X admits a compact resolution $\{K_\sigma : \sigma \in \mathbb{N}^{\mathbb{N}}\}$ swallowing compact subsets of X, there exist an open covering*

$$\{A((\sigma_1)), \sigma_1 \in \mathbb{N}\}$$

of X and a map $q : \mathbb{N} \to \mathbb{N}$ such that for each compact subset K of $A((\sigma_1))$ there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha|1 = q((\sigma_1))$ and $K \subset K_\alpha$.

Proof First we prove that for each $x \in X$ there exists $n_x \in \mathbb{N}$ such that for each compact subset K of $B(x, n_x^{-1})$ there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha|1 = (n_x)$ and $K \subset K_\alpha$. Assume the contrary. Then there exists $x \in X$ such that for each $n \in \mathbb{N}$ there exists a compact subset K_n of X such that $K_n \subset B(x, n^{-1})$, and for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ with $\alpha|1 = (n)$

$$K_n \not\subseteq K_\alpha. \tag{6.5}$$

Clearly $K := \{x\} \cup \bigcup_n K_n$ is a compact subset of X. By the assumption there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that

$$K \subset K_\alpha. \tag{6.6}$$

If $\alpha|1 = (m)$, then (6.6) implies $K_m \subset K_\alpha$ that provides a contradiction with $n = m$. This proves the claim. Since X is Lindelöf, the open covering $\{B(x, n_x^{-1}) : x \in X\}$ admits a countable subfamily

$$\{B(x_{\sigma_1}, n_{x_{\sigma_1}}^{-1}) : \sigma_1 \in \mathbb{N}\}$$

covering X. Lemma is proved with

$$A((\sigma_1)) = B(x_{\sigma_1}, n_{x_{\sigma_1}}^{-1})$$

and $q : \mathbb{N} \to \mathbb{N}$ which is defined by $q((\sigma_1)) = (n_{x_{\sigma_1}})$. □

Clearly we may apply Lemma 6.2.2 to each of the sets $A((\sigma_1))$. Then we can obtain a covering $\{A((\sigma_1, \sigma_2)), \sigma_2 \in \mathbb{N}\}$ of each of the sets $A((\sigma_1))$ constructed in Lemma 6.2.2. This process can be continued by induction. Next Lemma 6.2.3 describes this construction.

Lemma 6.2.3 *Let X be a metrizable topological space that admits a compact resolution $\{K_\sigma : \sigma \in \mathbb{N}^{\mathbb{N}}\}$ swallowing compact sets of X. Let $A((\sigma_1, \sigma_2, \ldots, \sigma_h))$ be an open subset of X, and let $q((\sigma_1, \sigma_2, \ldots, \sigma_h))$ be a multiindex of $\mathbb{N}^h$ such that for each compact subset K of $A((\sigma_1, \sigma_2, \ldots, \sigma_h))$ there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that*

$$\alpha|h = q((\sigma_1, \sigma_2, \ldots, \sigma_h))$$

and $K \subset K_\alpha$. Then there exist an open covering

$$\{A((\sigma_1, \sigma_2, \ldots, \sigma_h, \sigma_{h+1})) : \sigma_{h+1} \in \mathbb{N}\}$$

of $A(\sigma_1, \sigma_2, \ldots, \sigma_h)$ and a map $q' : \{(\sigma_1, \sigma_2, \ldots, \sigma_h)\} \times \mathbb{N}^1 \to \mathbb{N}^1$ such that for each compact subset K of $A((\sigma_1, \sigma_2, \ldots, \sigma_h, \sigma_{h+1}))$ there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that

$$\alpha|(h+1) = (q((\sigma_1, \sigma_2, \ldots, \sigma_h)), q'((\sigma_1, \sigma_2, \ldots, \sigma_h, \sigma_{h+1})))$$

and $K \subset K_\alpha$.

Proof As X is a Lindelöf space we need only to prove that for each $x \in A(\sigma_1, \sigma_2, \ldots, \sigma_h)$ there exists $n \in \mathbb{N}$ such that $B(x, n^{-1}) \subset A(\sigma_1, \sigma_2, \ldots, \sigma_h)$, and for each compact set $K \subset B(x, n)$ there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha|k+1 = (q((\sigma_1, \sigma_2, \ldots, \sigma_h)), n)$ and $K \subset K_\alpha$. Assume the contrary. Then there exists $x \in A(\sigma_1, \sigma_2, \ldots, \sigma_h)$, and let n_0 be the minimum natural number such that $B(x, n_0^{-1}) \subset A(\sigma_1, \sigma_2, \ldots, \sigma_h)$. Then for each $n = n_0, n_0+1, \ldots$, there exists a compact subset K_n of $B(x, n^{-1})$ such that for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ with

$$\alpha|h+1 = (q((\sigma_1, \sigma_2, \ldots, \sigma_h)), n)$$

we have

$$K_n \not\subset K_\alpha. \tag{6.7}$$

Clearly $K := \{x\} \cup \left[\bigcup_n K_n\right]$ is a compact subset of $A(\sigma_1, \sigma_2, \ldots, \sigma_h)$. By the assumption there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$, $\alpha|h = q((\sigma_1, \sigma_2, \ldots, \sigma_h))$, such that

$$K \subset K_\alpha. \tag{6.8}$$

Then, $K_p \subset K_\alpha$ for $\alpha|h+1 = (q((\sigma_1, \sigma_2, \ldots, \sigma_h)), p)$. This contradicts (6.7) for $n = p$. The map q' is defined similarly as in Lemma 6.2.2. □

Now we prove the following deep result due to Christensen [140]:

Theorem 6.2.4 (Christensen) *If a metrizable topological space X admits a compact resolution $\{K_\sigma : \sigma \in \mathbb{N}^{\mathbb{N}}\}$ swallowing compact subsets, then X is a Polish space.*

Proof Using Lemmas 6.2.2 and 6.2.3 we construct a Suslin scheme $A(.)$ of non-empty open subsets of X and a map $q : \mathbb{N}^{(\mathbb{N})} \to \mathbb{N}^{(\mathbb{N})}$ such that

$$A(\emptyset) = X = \bigcup\{A(\sigma_1) : \sigma_1 \in \mathbb{N}\}. \tag{6.9}$$

For each $k \in \mathbb{N}$ and each $(\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathbb{N}^k$ we have

$$A((\sigma_1, \sigma_2, \ldots, \sigma_k)) = \bigcup\{A((\sigma_1, \sigma_2, \ldots, \sigma_k, \sigma_{k+1})) : \sigma_{k+1} \in \mathbb{N}\}, \tag{6.10}$$

$q(\emptyset) = \emptyset$, and for each $k \in \mathbb{N}$ and each $(\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathbb{N}^k$

$$q((\sigma_1, \sigma_2, \ldots, \sigma_k))|i = q((\sigma_1, \sigma_2, \ldots, \sigma_i)) \tag{6.11}$$

for $1 \leq i \leq k-1$. For each $(\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathbb{N}^k$ and any compact set $K \subset A((\sigma_1, \sigma_2, \ldots, \sigma_k))$ there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that

$$\alpha|k = q((\sigma_1, \sigma_2, \ldots, \sigma_k)) \tag{6.12}$$

and

$$K \subset K_\alpha. \tag{6.13}$$

Let d be a metric on X, and let $\widetilde{X}$ be the d-completion of X. For each $A((\sigma_1, \sigma_2, \ldots, \sigma_k))$ let

$$B((\sigma_1, \sigma_2, \ldots, \sigma_k)) := \widetilde{X} \backslash \{\overline{X \backslash A((\sigma_1, \sigma_2, \ldots, \sigma_k))}^{\widetilde{X}}\}.$$

Since $A((\sigma_1, \sigma_2, \ldots, \sigma_k))$ is an open subset of X,

$$\{\overline{X \backslash A((\sigma_1, \sigma_2, \ldots, \sigma_k))}^{\widetilde{X}}\} \cap X = X \backslash A((\sigma_1, \sigma_2, \ldots, \sigma_k)).$$

Hence

$$B((\sigma_1, \sigma_2, \ldots, \sigma_k)) \cap X = A((\sigma_1, \sigma_2, \ldots, \sigma_k)).$$

Let

$$M := \bigcup \left\{ \bigcap \{B((\sigma_1, \sigma_2, \ldots, \sigma_k)), k \in \mathbb{N}\}, \sigma = (\sigma_1, \ldots) \in \mathbb{N}^{\mathbb{N}} \right\}. \tag{6.14}$$

Note that $X \subset M$ by (6.12) and (6.10). For each $y \in M$ we may choose $\sigma = (\sigma_1, \sigma_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ such that $y \in B((\sigma_1, \sigma_2, \dots, \sigma_k))$ for each $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$ there exists

$$x_k \in A((\sigma_1, \sigma_2, \dots, \sigma_k)) \tag{6.15}$$

such that

$$d(y, x_k) < k^{-1}. \tag{6.16}$$

By (6.12), (6.13), and (6.15) there exists $\alpha(k) \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha(k)|k = q((\sigma_1, \sigma_2, \dots, \sigma_k))$ and $\{x_k\} \subset K_{\alpha(k)}$. Then $\beta := \sup\{\alpha(k) : k \in \mathbb{N}\} \in \mathbb{N}^{\mathbb{N}}$ by (6.11). Since $\{x_k : k \in \mathbb{N}\} \subset K_\beta$, the sequence $(x_k)_k$ has an adherent point in X. Then (6.16) implies that $y \in X$. From (6.14) it follows that $\widetilde{X} \setminus M$ is the union of the sets

$$\widetilde{X} \setminus \left\{ \bigcup [B((\sigma_1)) : \sigma_1 \in \mathbb{N}] \right\}, \tag{6.17}$$

$$B((\sigma_1, \sigma_2, \dots, \sigma_k)) \setminus \left\{ \bigcup [B((\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_{k+1})) : \sigma_{k+1} \in \mathbb{N}] \right\} \tag{6.18}$$

for each $k \in \mathbb{N}$, $(\sigma_1, \sigma_2, \dots, \sigma_k) \in \mathbb{N}^k$. By the separability of X we note that each set in (6.18) is a countable union of closed sets. Since the set in (6.17) is closed, $\widetilde{X} \setminus M$ is a F_σ- subset of $\widetilde{X}$, and then $M = X$ is a G_δ subset of $\widetilde{X}$. Hence X is Polish as a G_δ-subset of the Polish space $\widetilde{X}$. □

Proposition 6.1.4 applies to provide the following characterization of the analyticity for metric spaces:

Corollary 6.2.5 *For a metric space X the following assertions are equivalent:*

(i) X is separable and admits a complete resolution.
(ii) X admits a compact resolution.
(iii) X is analytic.

Proof (i) $\Rightarrow$ (ii): Let $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a complete resolution, and let $\{T_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a compact resolution on the completion Y of X. Then the sets $K_\alpha := T_\alpha \cap B_\alpha$ form a compact resolution on X for $\alpha \in \mathbb{N}^{\mathbb{N}}$.
(ii) $\Rightarrow$ (iii): Follows from Proposition 6.1.4.
(iii) $\Rightarrow$ (i): Since every analytic space is separable and admits a compact resolution.

□

Corollary 6.2.5 may suggest the following problem. Let X be a Banach space which is weakly K-analytic. Assume that every weakly compact set is separable. Is E separable? The answer is positive under Martin's axiom plus the negation of the Continuum Hypothesis; see [246]. It fails under CH; see, for example, [525].

In [443] Mercourakis and Stamati introduced a class of Banach spaces E (under the name *strongly weakly K-analytic* (SWKA)) whose weak topology $\sigma(E, E')$ admits a (usco) map $T : \mathbb{N}^{\mathbb{N}} \to \mathcal{K}(E)$, where $\mathcal{K}(E)$ is the family of all $\sigma(E, E')$-compact sets in E, and such that for every compact set K in $(E, \sigma(E, E'))$ there exists $T_\alpha := T(\alpha)$ such that $K \subset T_\alpha$.

On the other hand, if E is a metrizable lcs whose weak topology $\sigma(E, E')$ admits a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ swallowing compact sets, then a map $T : T : \mathbb{N}^{\mathbb{N}} \to \mathcal{K}(E)$, defined by $T(\alpha) := \bigcap_k C_{n_1,n_2,\ldots,n_k}$, is (usco) for the weak topology $\sigma(E, E')$. Indeed, since $(E, \sigma(E, E'))$ is angelic (Corollary 4.3.7), countable compact sets $\bigcap_k C_{n_1,n_2,\ldots,n_k}$ are compact; see Corollary 3.2.9. Clearly $K_\alpha \subset C_{n_1,n_2,\ldots,n_k}$ for each $k \in \mathbb{N}$. We provided a (usco) map $T : \mathbb{N}^{\mathbb{N}} \to \mathcal{K}(E)$ for the weak topology $\sigma(E, E')$ such that for every K in $(E, \sigma(E, E'))$ there exists T_α such that $K \subset T_\alpha$.

For a Banach space E, let $B(E)$ be the unit closed ball in E endowed with the weak topology $\sigma(E, E')$. If $B(E)$ is a Polish space, then $B(E)$ is metrizable, and it is well known that the *weak** dual E' of E is separable. It is also a classical fact (see [277]) that for a Banach space E such that E'' is separable, the ball $B(E)$ with the weak topology of E is a Polish space.

An application of Christensen's theorem 6.2.4 shows the following (see [443]):

Proposition 6.2.6 *Let E be a Banach space whose dual E' is separable. Then E is (SWKA) if and only if the closed unit ball $B(E)$ is a Polish space in the weak topology of E.*

Proof Since E' is separable, $B(E)$ is $\sigma(E, E')$-metrizable. Assume E is (SWKA). Since $B(E)$ admits a compact resolution swallowing compact sets in the topology $\sigma(E, E')|B(E)$, we apply Theorem 6.2.4 to complete the proof for this case. Conversely, if $B(E)$ is a Polish space, it is K-analytic and admits a compact resolution swallowing compact sets. □

Since for the Banach space c_0 the closed unit ball of c_0 is not a Polish space in the weak topology, Proposition 6.2.6 shows that c_0 is not (SWKA). Since every closed subspace of a (SWKA) Banach space is (SWKA), and for an infinite metrizable compact space K the Banach space $C_c(K)$ contains an isomorphic copy of c_0, we note the following:

Corollary 6.2.7 *If K is an infinite metrizable compact space, $C_c(K)$ does not admit a compact resolution in the weak topology which swallows weakly compact sets.*

On the other hand, Edgar and Wheeler proved [194, Theorems A, B]:

Theorem 6.2.8 *Let E be a separable Banach space. The following assertions are equivalent:*

(i) *$B(E)$ endowed with the topology $\sigma(E, E')$ is completely metrizable.*
(ii) *$B(E)$ endowed with topology $\sigma(E, E')$ is a Polish space.*
(iii) *For every $\sigma(E, E')$-closed bounded set $B \subset E$ the identity map $(B, \sigma(E, E')) \to E$ has at least one point of continuity.*

(iv) $B(E)$ *endowed with the topology* $\sigma(E, E')$ *is metrizable, and every closed subset of* $B(E)$ *is a Baire space.*

It is well known that every Čech-complete space X, i.e. X is a G_δ-set in some (every) compactification of X, is a Baire space; see [195]. If E is an infinite-dimensional Banach space, then $(E, \sigma(E, E'))$ is not a Baire space, although $B(E)$ is Čech-complete in the weak topology, [194].

We complete this section with the following useful result, which will be used in the sequel, due to Cascales and Orihuela [127, Theorem 1] (see also [130, Theorem 3.11] for a direct proof of that result):

Theorem 6.2.9 (Cascales–Orihuela) *A compact space* K *is metrizable if and only if* $(K \times K) \setminus \Delta$ *has a compact resolution swallowing compact sets, where* $\Delta := \{(x, x) : x \in K\}$.

6.3 Subspaces of Analytic Spaces

In this section we provide some applications of Closed Graph Theorem 5.3.1 to fix analytic subspaces of separable F-spaces. The first version of Corollary 6.3.1 assumes CH. We present also a sketch of another proof of Corollary 6.3.1 without CH and heavily depending on Mycielski's theorem about independent functions.

Corollary 6.3.1 *Let* $E := (E, \xi)$ *be a separable F-space. Let* F *be an analytic subspace of* E. *Then, under CH, the codimension of* F *in* E *is either finite or equals* $2^{\aleph_0}$.

Proof Let G be an algebraic complement of F in E. Assume that the dimension of G is countable. Since G is a countable union of finite-dimensional subspaces G_n and each G_n (as metrizable complete and separable) is analytic, G is analytic. Therefore F and G are analytic, and E endowed with the direct sum topology $\tau := \xi|_F \oplus \xi|_G$ (stronger than ξ) is analytic. Theorem 5.3.1 applies to show $\xi = \tau$. Hence F is closed in E. Corollary 2.2.6 yields that codimension of F is finite. □

Another proof without CH (see [179]): Assume that F has infinite codimension in E; we may assume that F is dense in E. The proof will be completed if we show that there exists a Cantor set $D \subset E$ such that D is linearly independent and $F \cap \operatorname{span} D = \{0\}$. It is enough to show that for distinct points $x_1, \ldots, x_n$ of D one has $\sum_{1 \le i \le n} a_i x_i \notin F$ for each tuple $(a_1, \ldots, a_n) \in \mathbb{K}^n \setminus \{0\}$. For each $n \in \mathbb{N}$ set

$$R_n := \{(x_1, \ldots, x_n) \in E^n : \exists (a_1, \ldots, a_n) \neq 0 \quad \sum_{1 \le i \le n} a_i x_i \in F\}.$$

The proof will be completed if we show that each R_n is of first category in E^n. Indeed, then Mycielski's theorem [458] applies to obtain a Cantor set satisfying the condition above. Observe that R_n is an analytic subset of E^n for each $n \in \mathbb{N}$: Define

$$f : (\mathbb{K}^n \setminus \{0\}) \times E^n \to E$$

by $f((a_1, \ldots, a_n), (x_1, x_2, \ldots x_n)) := \sum_{1 \le i \le n} a_i x_i$. Then f is continuous. Since $R_n = \mathrm{pr}_{E^n} f^{-1}(F)$, it is enough to note that $f^{-1}(F)$ is analytic. As the spaces $\mathbb{K}^n \setminus \{0\})$, E and F are analytic, so $f^{-1}(F)$ is analytic, too. Then [393, S 39.II, Corollary 1] yields that R_n has the Baire property in E^n for each $n \in \mathbb{N}$. Finally, note that each R_n is of first category in E^n. The proof follows by induction: $R_1 = F$ is of first category by Theorem 7.1.2 and density of F in E. Assume that R_n is of first category in E^n for some $n \in \mathbb{N}$. Set

$$A = \{(x_1, \ldots, x_{n+1}) \in E^{n+1} : (x_1, \ldots, x_n) \in R_n\}$$

and

$$B = \{(x_1, \ldots, x_{n+1}) \in E^{n+1} : x_{n+1} \in F + \mathrm{span}\{x_1, \ldots, x_n\}\}.$$

Then $A \cup B$ equals

$$\{(x_1, \ldots, x_{n+1}) \in E^{n+1} : \exists (a_1, \ldots, a_{n+1}) \neq 0, \sum_{1 \le i \le n} a_i x_i + a_{n+1} x_{n+1} \in F\},$$

and the last set is R_{n+1}. Observe that $A = R_n \times E$. Since R_n is of first category in E^n, it follows that A is of first category in E^{n+1}. Note also that $B \subset E^n \times E$, and for each tuple $(x_1, \ldots, x_n) \in E^n$ the vertical section of B equals

$$B(x_1, \ldots, x_n) = F + \mathrm{span}\{x_1, \ldots, x_n\} \subset E.$$

Clearly $B(x_1, \ldots, x_n)$ is analytic and of first category in E. Since E^n is of first category in E^{n+1}, and F is of first category in E, it follows that B has the Baire property in $E^n \times E = E^{n+1}$. Now the fact that B is of first category in E^{n+1} follows from the Kuratowski–Ulam theorem [393]. ■

Clearly every countable-dimensional tvs is analytic. Proposition 6.3.2 is taken from [186], although it was already presented in [179, 506], and [558]; see also Young's theorem in [393] or [275].

Proposition 6.3.2 *If E is an uncountable-dimensional analytic tvs, then the dimension of E is $2^{\aleph_0}$.*

Proof By the assumption there exists a continuous map T from $\mathbb{N}^{\mathbb{N}}$ onto E. It is enough to show that $\mathbb{N}^{\mathbb{N}}$ contains a set C homeomorphic to the Cantor set such that $T(C)$ is linearly independent. Clearly $\dim E \le \mathrm{card} E \le \mathrm{card} \mathbb{N}^{\mathbb{N}} = 2^{\aleph_0}$. Now we prove the reverse inequality. Let $\mathbb{G}$ be the family of open subsets G of $\mathbb{N}^{\mathbb{N}}$ such that

$$\dim T(G) := \dim \mathrm{span} T(G) \le 2^{\aleph_0}.$$

Since $\mathbb{N}^{\mathbb{N}}$ is hereditarily Lindelöf, there exists a countable subfamily $\mathbb{S}$ of $\mathbb{G}$ such that $S := \bigcup \mathbb{S} = \bigcup \mathbb{G}$. Note that $\dim T(S) \leq 2^{\aleph_0}$. Hence for the closed set $D := \mathbb{N}^{\mathbb{N}} \setminus S$ we note the following two conditions:

(i) If H is a non-empty open set in D, then $\dim T(H) > 2^{\aleph_0}$.
(ii) If $H_1, \ldots, H_k$ are non-empty open sets in D, there exist points $d_1 \in H_1, \ldots, d_k \in H_k$, such that the points $T(d_1), \ldots, T(d_k)$ are linearly independent.

Moreover, if $x_1, \ldots, x_k$ are linearly independent points in E, there exist neighbourhoods $U_1, \ldots, U_k$ of $x_1, \ldots, x_k$, respectively, such that each sequence $(y_i) \in \prod_{1 \leq i \leq k} U_i$ is linearly independent.

For each $n \in \mathbb{N}$ set $W_n := \{0, 1\}^n$. By the induction, for each $n \in \mathbb{N}$ and $s \in W_n$ there exist a point $p_s \in D$, a neighbourhood U_s of $x_s := T(p_s)$ in E, and a a closed ball

$$K_s := K_s(p_s, r_s) \subset D$$

with the centre at p_s and radius $r_s < n^{-1}$, such that $T(K_s) \subset U_s$, where U_s and K_s for $s \in W_n$ are pairwise disjoint, every sequence $(y_s) \in \prod_{s \in W_n} U_s$ is linearly independent, and $K_{(s,0)} \cup K_{(s,1)} \subset K_s$. This together implies that the set $C := \bigcap_n \bigcup_{s \in W_n} K_s$ is homeomorphic to the Cantor set, and $T(C)$ is linearly independent. This shows that $\dim E \geq 2^{\aleph_0}$. □

6.4 Trans-Separable Topological Spaces

It turns out that every tvs which admits a resolution consisting of precompact sets is necessarily trans-separable. This fact, due to Robertson [517], will be used in the next section. In this section we collect a couple of results, mostly from [217] and [517], about uniform trans-separable and trans-separable topological vector spaces.

A uniform space X is called *trans-separable* [298, 319], if every uniform cover of X has a countable subcover. Separable uniform spaces and Lindelöf uniform spaces are trans-separable; the converse is not true in general, although every trans-separable pseudometric space is separable. Clearly a uniform space is trans-separable if and only if it is uniformly isomorphic to a subspace of a uniform product of separable pseudometric spaces. This implies that every uniform quasi-Suslin space (see [611, 1.4.2]) is trans-separable. Trans-separable spaces enjoy good permanence properties, for example, the class of trans-separable spaces is hereditary, productive, and closed under uniform continuous images; see [491].

A tvs E is trans-separable if and only if E is isomorphic to a subspace of a product of metrizable, separable tvs. Thus, if E is a lcs, then $(E', \sigma(E', E))$ is trans-separable. A tvs E is trans-separable if and only if for every neighbourhood of zero U in E there exists a countable subset N of E such that $E = N + U$; see, for

example, [294, 390, 492, 517]. It is easy to see also that a tvs E is trans-separable if and only if for each continuous F-seminorm p on E the F-seminormed space (E, p) is separable, or the associated F-normed space $E/\ker p$ is separable.

The concept of trans-separability has been used to study several problems, both from analysis and topology, for example, while studying the metrizability of precompact sets in uniform spaces. We refer the reader to papers [127, 180, 214, 217, 365, 517, 548].

Pfister [492] observed the following:

Proposition 6.4.1 *A lcs E is trans-separable if and only if for every neighbourhood of zero U in E its polar U° is $\sigma(E', E)$-metrizable.*

This fact has been used by Pfister in [492] to show that precompact sets in (DF)-spaces are metrizable. We will provide below much general result of this type. We start with the following:

Proposition 6.4.2 *A completely regular topological Hausdorff space X is realcompact if and only if there exists an admissible uniformity $\mathcal{N}$ on X such that $(X, \mathcal{N})$ is trans-separable and complete.*

Proof If X is realcompact, X is homeomorphic to a closed subset of $\mathbb{R}^{C(X)}$. Then the induced uniformity in X is admissible complete and trans-separable. Conversely, if $\mathcal{N}$ is a trans-separable and complete admissible uniformity on X, then $(X, \mathcal{N})$ is isomorphic to a closed subspace of a product of metrizable, separable uniform spaces. Therefore X is realcompact. □

Corollary 6.4.3 *A completely regular Hausdorff space X is K-analytic if and only if there exists on X an admissible uniformity $\mathcal{N}$ such that $(X, \mathcal{N})$ is complete and admits a compact resolution.*

Proof If X is K-analytic, then X is realcompact, and Proposition 6.4.2 applies. Conversely, by the assumption on X we know that X is trans-separable. By Proposition 6.4.2 the space X is realcompact, and by Proposition 3.2.7 X is K-analytic. □

Next Proposition 6.4.4 is due to Robertson [517].

Proposition 6.4.4 *Let X be a tvs with a precompact resolution. Then X is trans-separable.*

Proof Let $\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a precompact resolution of E. For each α let K_α be the closure of B_α in the completion Y of X. Set $Z = \bigcup_\alpha K_\alpha$. Then $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution in Z. By Proposition 6.2.5 each metrizable space $Z/\ker p$ is separable, where p is a continuous F-seminorm on Z. □

Corollary 6.4.5 (Dieudonné) *A metrizable tvs E is separable and complete if every bounded set in E is relatively compact. Hence every Fréchet–Montel space is separable.*

Proof Let $(U_n)_n$ be a decreasing basis of neighbourhoods of zero in E. For each $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ set $A_\alpha := \bigcap_k n_k U_k$. Then $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bounded resolution on E swallowing bounded sets. Since, by the assumption, every bounded set in E is relatively compact, E has a compact resolution swallowing compact sets. Then E is separable by Proposition 6.4.4, and E is complete by Theorem 6.2.4. □

Corollary 6.4.6 supplements Corollary 3.2.9.

Corollary 6.4.6 *For a complete tvs E the following conditions are equivalent:*

(i) *E has a compact resolution.*
(ii) *E is quasi-Suslin.*
(iii) *E is K-analytic.*

Proof Only (ii) $\Rightarrow$ (iii) needs a proof: Since E is trans-separable, E, as a closed subset of a product of separable and metrizable tvs, is realcompact. Then E is K-analytic by Proposition 3.2.7. □

Since $C_p(X)$ is a (dense) subspace of the product $\mathbb{R}^X$, the space $C_p(X)$ is always trans-separable. For spaces $C_c(X)$ we have the following simple lemma [217]:

Lemma 6.4.7 *The compact sets in a completely regular Hausdorff space X are metrizable if and only if $C_c(X)$ is trans-separable.*

Proof If $C_c(X)$ is trans-separable and K is a compact subset of X, then $C_c(K)$ is a separable Banach space since the restriction map $T : C_c(X) \to C_c(K)$, $T(f) := f|K$ is a continuous surjection. Hence K is a metrizable space. For the converse, assume that $\{K_i : i \in I\}$ is the family of all compact subsets of X, and assume that all K_i are metrizable. Since the map $\varphi : C_c(X) \to \prod_{i\in I} C_c(K_i)$, where $\varphi(f) = \{f|_{K_i} : i \in I\}$ is an isomorphism onto its range, and each $C_c(K_i)$ is separable, the conclusion holds. □

The following result, originally proved in [127], can be shown by using the concept of trans-separability. The proof follows from [448]:

Proposition 6.4.8 *Let $(X, \mathcal{U})$ be a uniform space whose uniformity admits a $\mathcal{U}$-basis $B = \{N_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, i.e. $N_\alpha \subset N_\beta$ if $\beta \le \alpha$. Then the precompact subsets of $(X, \mathcal{U})$ are metrizable in the induced uniformity.*

Proof By E we denote the lcs of bounded real-valued uniformly continuous functions defined on X endowed with the topology of uniform convergence on precompact subsets of X. For each $\alpha = (m, \alpha_1, \alpha_2, \dots, \alpha_n, \dots) \in \mathbb{N}\times \left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}}$ by A_α denote the set of all $f \in E$ such that

$$\|f\|_\infty \le m, |f(s) - f(t)| \le 1/n, \quad (s,t) \in (X \times X) \cap N_{\alpha_n}, n \in \mathbb{N}.$$

Clearly $A_\alpha \subset A_\beta$ if $\alpha \le \beta$ and $E = \bigcup\{A_\alpha : \alpha \in \mathbb{N}\times \left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}}\}$.

Note that each A_α is pointwise compact and, by the equicontinuity, the pointwise topology and the induced topology of E coincide on each A_α. Hence each A_α

is a compact subset of E. By Proposition 6.4.4 E is trans-separable. Thus each equicontinuous subset of the dual E' is $\sigma(E', E)$-metrizable. Assume that K is a precompact subset of X. Set

$$W := \{f \in E : \sup_{x \in K} |f(x)| \leq 1\}.$$

If φ is the evaluation map from X into E', the equality $\varphi(K) = W^\circ \cap \varphi(X)$ yields that K is metrizable. □

The above observations apply to deduce the following theorem from [217].

Theorem 6.4.9 *Compact subsets of a lcs E are metrizable if and only if E' endowed with the topology τ_c of the uniform convergence on compact sets of E is trans-separable.*

Proof Let F be the dual of (E', τ_c). If (E', τ_c) is trans-separable, the τ_c-equicontinuous subsets of $(F, \sigma(F, E'))$ are metrizable by Proposition 6.4.1. Since for a compact set K in E the bipolar $K^{\circ\circ}$ is τ_c-equicontinuous, $K^{\circ\circ}$ is metrizable in the topology $\sigma(F, E')$. Hence $K \subset E$ is $\sigma(F, E')$-metrizable; consequently K is $\sigma(E, E')$-metrizable. This shows that K is metrizable in E. Conversely, assume that all compact sets in E are metrizable. $C_c(E)$ is trans-separable by Lemma 6.4.7. Since (E', τ_c) is a topological subspace of $C_c(E)$, the conclusion follows. □

Since every quasi-Suslin lcs is trans-separable, Theorem 6.4.9 implies Valdivia's [611, Theorem 1.4.3 (27)] stating that, if $(E', \tau_c)$is quasi-Suslin, all compact sets in E are metrizable. We note the following variant for the topology τ_p (see [217]):

Theorem 6.4.10 *Precompact sets are metrizable in a lcs E if and only if E' endowed with the topology τ_p of the uniform convergence on precompact sets of E is trans-separable.*

Proof If (E', τ_p) is trans-separable, by Proposition 6.4.1 all precompact sets in E are metrizable. To prove the converse assume that all precompact sets in E are metrizable. Let $\{P_i : i \in I\}$ be the family of all precompact sets in E. For every $i \in I$ let K_i be the closure of P_i in the completion of E. The map $T : (E', \tau_p) \to \prod_{i \in I} C_c(K_i)$ defined by the form $T(u) = \{v|K_i : i \in I\}$, where v is the continuous linear extension of u to the completion of E, is an isomorphism onto its range. This yields the conclusion. □

Corollary 6.4.11 *The strong dual $(E', \beta(E', E))$ of a lcs E is trans-separable if and only if every bounded set in E is metrizable in $(E, \sigma(E, E'))$. Consequently, the strong dual of a (DF)-space E is separable iff every bounded set in E is weakly metrizable.*

A family $\mathcal{F}$ of functions from a uniform space $(X, \mathcal{N})$ into a uniform space $(Y, \mathcal{M})$ is called *uniformly equicontinuous* [363, Exercise G, Chapter 7], if for each $V \in \mathcal{M}$ there is $U \in \mathcal{N}$ such that $(f(x), f(y)) \in V$ whenever $f \in \mathcal{F}$ and $(x, y) \in U$. A uniform space $(X, \mathcal{N})$ is uniformly isomorphic to a subspace of the uniform product

of the pseudometric spaces $\{(X, d) : d \in \mathcal{P}\}$ endowed with their corresponding pseudometric uniformities $\mathcal{N}_d$. Recall that a topological space X has a *countable tightness*, if for each $A \subset X$ and each $x \in \overline{A}$ there exists a countable set $B \subset A$ such that $x \in \overline{B}$.

In order to prove Theorem 6.4.13 we need the following lemma from [217]:

Lemma 6.4.12 *For a pseudometric space (X, d) the following assertions are equivalent:*

(i) Every pointwise bounded uniformly equicontinuous set of functions on $(X, \mathcal{N}_d)$ is a metrizable relatively compact subset of $C_c(X, d)$.
(ii) Every pointwise bounded uniformly equicontinuous set of functions on $(X, \mathcal{N}_d)$ has countable tightness in $C_c(X, d)$.
(iii) X is separable.

Proof (ii) $\Rightarrow$ (iii): We may assume that the pseudometric space (X, d) is bounded. Let $\mathcal{K}(X)$ be the family of all compact subsets of X. Define $f_A(x) := d(x, A)$ for $x \in X$ and $A \in \mathcal{K}(X)$. Since

$$|f_A(x) - f_A(y)| \leq d(x, y)$$

for every $A \in \mathcal{K}(X)$ and $(x, y) \in X \times X$, the family $\mathcal{H} := \{f_A : A \in \mathcal{K}(X)\}$ of real-valued functions on $(X, \mathcal{N}_d)$ is uniformly equicontinuous. This implies that $\overline{\mathcal{H}}^{\mathbb{R}^X} \subseteq C(X, d)$. Since (X, d) is bounded, one gets $\lambda > 0$ with $d(x, y) \leq \lambda$ for every $(x, y) \in X \times X$. Hence $0 \leq f_A(x) \leq \lambda$ for each $A \in \mathcal{K}(X)$ and $x \in X$. Therefore the family $\overline{\mathcal{H}}^{\mathbb{R}^X}$ is a pointwise bounded uniformly equicontinuous family of functions on (X, d). Let 0 be the null function on X. Then

$$0 \in \overline{\mathcal{H}}^{C_c(X,d)} = \overline{\mathcal{H}}^{\mathbb{R}^X}.$$

By the assumption there is a sequence $\{A_n : n \in \mathbb{N}\}$ in $\mathcal{K}(X)$ such that

$$0 \in \overline{\{f_{A_n} : n \in \mathbb{N}\}}^{C_c(X,d)}.$$

Hence, if $x \in X$ and $\epsilon > 0$, there is $k \in \mathbb{N}$ such that $f_{A_k}(x) < \epsilon$. This proves that $Y := \bigcup_{n=1}^{\infty} A_n$ is a dense subspace of (X, d). Finally, since every compact pseudometric space A_n is separable, it follows that (X, d) is separable.

(iii) $\Rightarrow$ (i): Let D be a countable and dense subset of X and let Z be a closed pointwise bounded uniformly equicontinuous subset of $C_c(X)$. Applying Ascoli's theorem we note that Z is a compact subspace of $C_c(X)$. Set $\delta_x(f) := f(x)$ for all $f \in C(X)$ and for each $x \in X$. Then $\{\delta_x|_Z : x \in D\}$ is a countable family of continuous functions that separates the points of Z. Hence Z, as a subspace of $C_c(X)$, is metrizable. □

Theorem 6.4.13 is related with Lemma 6.4.7.

Theorem 6.4.13 *A uniform space $(X, \mathcal{N})$ is trans-separable if and only if every pointwise bounded uniformly equicontinuous subset of $C_c(X, \tau_{\mathcal{N}})$ is metrizable.*

Proof Assume that every pointwise bounded uniformly equicontinuous subset of $C_c(X)$ is metrizable. Denote by $\mathcal{P}$ the family of all pseudometrics for X generating $\mathcal{N}$. Since the uniformity $\mathcal{N}_d$ on X generated by a pseudometric $d \in \mathcal{P}$ is smaller than $\mathcal{N}$, the above assumption applies to show that a pointwise bounded uniformly equicontinuous set $\mathcal{A}$ of functions on $(X, \mathcal{N}_d)$ has countable tightness in $C_c(X, d)$. Now Lemma 6.4.12 shows that the space (X, d) is separable. Since $(X, \mathcal{N})$ is uniformly isomorphic to a subspace of the uniform product $\prod_{d\in\mathcal{P}}(X, \mathcal{N}_d)$ of the pseudometric spaces $\{(X, d) : d \in \mathcal{P}\}$ endowed with their corresponding pseudometric uniformities, $(X, \mathcal{N})$ is trans-separable.

To prove the converse assume that $(X, \mathcal{N})$ is trans-separable. Let $\mathcal{C}$ be a closed pointwise bounded uniformly equicontinuous subset of $C_c(X, \tau_{\mathcal{N}})$. Since $\sup_{f\in\mathcal{C}} |f(x)| < \infty$ for each $x \in X$, the map

$$(x, y) \to \sup_{f\in\mathcal{C}} |f(x) - f(y)|$$

defines a pseudometric $D(x, y)$ on X. Then $\mathcal{C}$ is a pointwise bounded uniformly equicontinuous subset of $C(X, D)$. On the other hand, for any $\epsilon > 0$ there is a vicinity U in $X \times X$ such that

$$\sup_{f\in\mathcal{C}} |f(x) - f(y)| < \epsilon$$

for every $(x, y) \in U$. Hence $D(x, y) < \epsilon$ for every $(x, y) \in U$, so the identity map from $(X, \mathcal{N})$ onto $(X, \mathcal{N}_D)$ is uniformly continuous. Since $(X, \mathcal{N})$ is trans-separable, (X, D) is separable, and Lemma 6.4.12 applies to show that $\mathcal{C}$ is a compact metrizable subspace of $C_c(X, D)$. Finally, since the map

$$T : C_c(X, D) \to C_c(X, \tau_{\mathcal{N}})$$

defined by $Tf = f \circ \varphi$, where φ is the identity from $(X, \tau_{\mathcal{N}})$ onto (X, D), is injective and continuous, it follows that $\mathcal{C}$ is also a metrizable compact set when considered as a subspace of $C_c(X, \tau_{\mathcal{N}})$. The proof is complete. □

By the Ascoli theorem every closed pointwise bounded uniformly equicontinuous set in $C_c(X)$ is compact. Hence we have:

Corollary 6.4.14 *Let $(X, \mathcal{N})$ be a uniform space. If every compact subset of $C_c(X, \tau_{\mathcal{N}})$ has countable tightness, then $(X, \mathcal{N})$ is trans-separable.*

Proposition 6.4.15 shows that every web-compact space is trans-separable.

Proposition 6.4.15 *Let $(X, \mathcal{N})$ be a uniform space. If the space $C_c(X, \tau_{\mathcal{N}})$ is angelic, $(X, \mathcal{N})$ is trans-separable. In particular, a uniform space X with a precompact resolution is trans-separable.*

Proof If $C_c(X, \tau_{\mathcal{N}})$ is angelic, each compact subset of $C_c(X, \tau_{\mathcal{N}})$ has countable tightness. By Corollary 6.4.14 the uniform space $(X, \mathcal{N})$ is trans-separable. Let $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a precompact resolution in $(X, \mathcal{N})$. Since the space Y covered by the closure of the sets K_α in the uniform completion of X is web-compact, we apply Theorem 4.3.3, and next we apply the first part of the proposition already proved. □

Corollary 6.4.16 *Let X be a metric space. Then $C_p(X)$ is angelic if and only if $C_c(X)$ is angelic if and only if X is separable.*

We show that the converse in Corollary 6.4.14 fails.

Example 6.4.17 There exists a trans-separable space $(X, \mathcal{N})$ such that the space $C_c(X, \tau_{\mathcal{N}})$ contains a compact set K which does not have countable tightness.

Proof Let ζ be the first ordinal of uncountable cardinal, and let φ_ζ denote the locally convex direct sum of $|\zeta| > \aleph_0$ copies of $\mathbb{R}$. Then the dual of φ_ζ is isomorphic to the product $\omega_\zeta = \mathbb{R}^{[0,\zeta)}$ of $|\zeta|$ copies $\mathbb{R}$.

If X denotes the linear space φ_ζ with the weak topology $\sigma(\varphi_\zeta, \omega_\zeta)$, then X is a uniform space under the associated uniformity $\mathcal{N}$. Hence $\tau_{\mathcal{N}} = \sigma(\varphi_\zeta, \omega_\zeta)$ and $X' = \omega_\zeta$. Then $(X, \mathcal{N})$ is trans-separable. Since the topology of uniform convergence on weakly compact subsets of φ_ζ and the product topology of $\mathbb{R}^{[0,\zeta)}$ coincide on ω_ζ, we conclude that ω_ζ is isomorphic to a subspace of $C_c(X, \tau_{\mathcal{N}})$. Consider the compact set $K := [0,1]^{[0,\zeta)}$ of ω_ζ, and let $f : [0,\zeta) \to \mathbb{R}$ be the constant function such that $f(\gamma) = 1$ for each $\gamma \in [0,\zeta)$. Then, if F is a finite subset of $[0,\zeta)$, define $g_F : [0,\zeta) \to \mathbb{R}$ so that $g_F(\gamma) = 1$ if $\gamma \in F$, and $g_F(\gamma) = 0$ if $\gamma \in [0,\zeta) \setminus F$. Let $\mathcal{F}(\zeta)$ be the family of all finite subsets of $[0,\zeta)$. Set

$$A := \{g_F : F \in \mathcal{F}(\zeta)\}.$$

Then $f \in \overline{A}^{\omega_\zeta} \subseteq K$, and uncountable set $\{F_n : n \in \omega\} \subseteq \mathcal{F}(\zeta)$ verifies $f \in \overline{\{g_{F_n} : n \in \mathbb{N}\}}^{\omega_\zeta}$. Indeed, if $\xi \in [0,\zeta) \setminus \bigcup_{n=1}^{\infty} F_n$, then

$$U(f, \xi) := \left\{h \in \mathbb{R}^{[0,\zeta)} : |h(\xi) - 1| < \frac{1}{2}\right\}$$

is a neighbourhood of $f \in \omega_\zeta$ such that $g_{F_n} \notin U(f, \xi)$ for every $n \in \mathbb{N}$. □

6.5 Weakly Analytic Spaces Need Not Be Analytic

Let E be a lcs. By $\mu(E, E')$ and $\beta(E, E')$ we denote the Mackey and the strong topology of E, respectively. By the Mackey and the strong dual of E we understand E' endowed with the Mackey topology $\mu(E', E)$ and $\beta(E', E)$, respectively. The

topology $\mu(E', E)$ is the strongest locally convex topology on E' compatible with the dual pair (E', E); see [322] or [364] for detail.

We have already noticed (see Theorem 12.4.6) that the weak topology $\sigma(E, E')$ of a (WCG) Banach space E is K-analytic, and there exist non-separable (WCG) Banach spaces. The following problem seems to be interesting. When can analyticity or K-analyticity of the weak topology $\sigma(E, E')$ of a dual pair (E, E') be lifted to stronger topologies on E compatible with the dual pair?

Note that, if $(E, \sigma(E, E'))$ is analytic, E endowed with a stronger topology ξ on E is analytic if and only if ξ admits a (relatively countably) compact resolution. Indeed, the analyticity of $\sigma(E, E')$ yields a weaker metric topology on E by Proposition 6.1.4. Hence ξ is also angelic, and Corollary 3.2.9 completes the proof. There exist K-analytic spaces $C_p(X)$ such that the space $C_p(C_p(X))$ is even not a Lindelöf space; see [31, Example 7.14]. This follows from the following fact due to Reznichenko:

Example 6.5.1 There exists a compact space X and $x \in X$ such that $X = \beta(X \setminus \{x\})$ and $C_p(X)$ is K-analytic.

Indeed, since $Y := X \setminus \{x\}$ is pseudocompact not compact, Y is not realcompact. Hence $C_p(C_p(Y))$ is not realcompact, so not Lindelöf. On the other hand, $C_p(Y)$, as a continuous image of the K-analytic space $C_p(X)$, is a K-analytic space.

We will use the following proposition (for the proof we refer the reader to [32, Corollary 0.5.14, Proposition 0.5.12]):

Proposition 6.5.2 *Let $L_p(X)$ be the weak* dual of $C_p(X)$. Then X is a Lindelöf Σ-space (K-analytic space, or analytic space, or separable, or σ-compact) if and only if $L_p(X)$ is a Lindelöf Σ-space (K-analytic space, or analytic space, or separable, or σ-compact).*

In [203] Ferrando proved that the space $L[0, 1]$ endowed with the Mackey topology $\mu(L[0, 1], C[0, 1])$ is a weakly analytic space that is not K-analytic. This result has been extended in [339] to the present form.

Theorem 6.5.3 (Kąkol–López-Pellicer–Śliwa) *For a completely regular Hausdorff space X the Mackey dual of $C_p(X)$ is analytic if and only if X is countable.*

Proof Set $X := (X, \tau)$, where τ is the original topology of X. By $L_\mu(X)$ denote the Mackey dual of $C_p(X)$, i.e. the dual of $C_p(X)$ endowed with the Mackey topology $\mu = \mu(C_p(X)', C_p(X))$. Assume that $L_\mu(X)$ is analytic. Suppose, by a contradiction, that X is uncountable. For $x \in X$ the functional $\delta_x : C_p(X) \to \mathbb{R}$ defined by $\delta_x(f) = f(x)$ is linear and continuous. Denote by $L_p(X)$ the dual of $C_p(X)$ endowed with the *weak** dual topology $\sigma = \sigma(C_p(X)', C_p(X))$. Set $Y = \{\delta_x : x \in X\}$. The map

$$\delta : (X, \tau) \to (Y, \sigma|Y)$$

defined by $x \to \delta_x$ is a homeomorphism and the set Y is closed in $L_p(X)$; see [32, Proposition 0.5.9]. Hence Y is also closed in $L_\mu(X)$. Thus $(Y, \mu|Y)$ is analytic. Let γ be the topology on X such that δ is a homeomorphism between (X, γ) and $(Y, \mu|Y)$. Since (X, γ) is an uncountable analytic space, it contains a set A homeomorphic to the Cantor set; see, for example, [522]. Clearly, $\gamma|A = \tau|A$.

Let $(x_n)_n \subset A$ be a sequence such that $x_n \neq x_m$ for $n \neq m$ that converges to some

$$x_0 \in (A \setminus \{x_n : n \in \mathbb{N}\}).$$

It is easy to see that for every closed subspace G of (X, τ) and every $x \in (X \setminus G)$ there exists $f \in C(X, I)$ with $f(x) = 1$ such that $G \cap \operatorname{supp} f = \emptyset$. Put

$$X_n = \{x_k : k > n\} \cup \{x_0\}$$

for $n \in \mathbb{N}$. Clearly X_n is closed in X and $x_n \notin X_n$ for $n \in \mathbb{N}$. Therefore, we can construct inductively a sequence $(f_n)_n \subset C(X, I)$, such that $f_n(x_n) = 1$ and

$$\operatorname{supp} f_n \cap (X_n \cup \bigcup_{k=1}^{n-1} \operatorname{supp} f_k) = \emptyset.$$

Then $x_0 \notin \bigcup_k \operatorname{supp} f_k$ and

$$\operatorname{supp} f_n \cap \operatorname{supp} f_m = \emptyset$$

for all $n, m \in \mathbb{N}$ with $n \neq m$.

Denote by $C_b(X)$ the Banach space of all bounded real-valued continuous functions on X endowed with the sup norm $\| \cdot \|$. Let $g \in C_b(X)'$. For $k \in \mathbb{N}$ set

$$\alpha_k = |g(f_k)|/g(f_k)$$

if $g(f_k) \neq 0$, and $\alpha_k = 1$, otherwise. Then $|\alpha_k| = 1$ and $\alpha_k g(f_k) = |g(f_k)|$ for $k \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $S_n = \sum_{k=1}^n \alpha_k f_k$. Then $S_n \in C_b(X)$ and $\|S_n\| = 1$. Thus

$$\sum_{k=1}^n |g(f_k)| = |\sum_{k=1}^n \alpha_k g(f_k)| = |g(S_n)| \leq \|g\|$$

for $n \in \mathbb{N}$, so

$$\sum_{k=1}^\infty |g(f_k)| \leq \|g\|.$$

Hence $g(f_k) \to 0$. It follows that the sequence $(f_n)_n$ converges weakly to 0 in $C_b(X)$. Thus the set

$$F_0 = \{0, f_1, -f_1, f_2, -f_2, \ldots\}$$

is weakly compact in $C_b(X)$. By Krein's weak compactness theorem [438, Theorem 2.8.14] the closed convex hull F of F_0 in $C_b(X)$ is weakly compact. Clearly F is the closed absolutely convex hull of the set $\{f_k : k \in \mathbb{N}\}$ in $C_b(X)$. The topology ϱ of the pointwise convergence in $C_b(X)$ is weaker than the weak topology of $C_b(X)$, so F is compact in $(C_b(X), \varrho)$. Hence F is compact in $C_p(X)$, since the injection map $(C_b(X), \varrho) \to C_p(X)$ is continuous. Thus the functional

$$p_F : L_\mu(X) \to [0, \infty),$$

defined by

$$p_F(g) = \sup\{|g(f)| : f \in F\},$$

is a continuous seminorm. Since $(f_n)_n \subset F$, we have

$$p_F(\delta_{x_n}) \geq |f_n(x_n)| = 1$$

for $n \in \mathbb{N}$. It is easy to see that $f(x_0) = 0$ for all $f \in F$, so $p_F(\delta_{x_0}) = 0$. It follows that $\delta_{x_n} \not\to \delta_{x_0}$ in $(Y, \mu|Y)$, so $x_n \not\to x_0$ in (X, γ), a contradiction.

Now assume that X is countable. If $C_p(X)$ is finite-dimensional, the Mackey dual $L_\mu(X)$ of $C_p(X)$ is finite-dimensional, so it is analytic. If $C_p(X)$ is infinite-dimensional, $C_p(X)$ is a metrizable lcs isomorphic to a dense subspace of $\mathbb{R}^{\mathbb{N}}$. Hence $L_\mu(X)$ is algebraically isomorphic to φ, the strong dual of $\mathbb{R}^{\mathbb{N}}$. Since φ endowed with the strongest locally convex topology is the union of an increasing sequence of finite-dimensional Banach spaces, it is an analytic space. It follows that $L_\mu(X)$ is analytic. □

Theorem 6.5.3 and its proof yields also the following:

Corollary 6.5.4 *The strong dual of $C_p(X)$ is analytic if and only if X is countable.*

Recall that $L_p(X)$ is analytic if and only if X is analytic by Proposition 6.5.2. Thus Theorem 6.5.3 provides many concrete non-analytic lcs whose weak topology is analytic.

Corollary 6.5.5 *Let X be an uncountable analytic space. Then the Mackey dual $L_\mu(X)$ of $C_p(X)$ is weakly analytic but not analytic.*

Note that in Corollary 6.5.5 the Mackey dual $L_\mu(X)$ is even not K-analytic. Indeed, since $L_\mu(X)$ is weakly analytic, the *weak** dual of $C_p(X)$ admits a weaker metric topology by Proposition 6.1.4. Assume that $L_\mu(X)$ is K-analytic. Then, by the first part of Proposition 6.1.4 we have that $L_\mu(X)$ is analytic, a contradiction with Corollary 6.5.5.

6.6 More about Analytic Locally Convex Spaces

Corollary 6.5.5 might be a good motivation to study sufficient conditions for a weakly analytic lcs to be analytic in the original topology.

Let E be a vector space, and let $\xi \leq \tau$ be two vector topologies on E. Recall that τ is called *ξ-polar*, if τ admits a basis of ξ-closed neighbourhoods of zero. It is easy to see that, if A is a subset of E that is complete (sequentially complete) in ξ, it is complete (sequentially complete) in τ; see [322, Theorem 3.2.4]. This yields the following:

Corollary 6.6.1 *If τ is ξ-polar, a ξ-complete (sequentially complete) resolution on E is τ-complete (sequentially complete).*

This combined with Proposition 6.1.4 and Corollary 6.2.5 provides:

Corollary 6.6.2 *If τ is a metrizable and separable vector topology on a vector space E such that $\xi \leq \tau$, τ is ξ-polar, and (E, ξ) has a complete resolution, then (E, τ) is analytic.*

Since for every separable lcs E its *weak** dual topology $\sigma(E', E)$ admits a weaker metric topology, we have that for separable E any topology ξ stronger than $\sigma(E', E)$ is analytic if ξ admits a compact resolution. It turns out that the following general fact holds [339]:

Theorem 6.6.3 *A separable tvs $E := (E, \xi)$ having a sequentially complete resolution is analytic if E satisfies one of the following conditions:*

(i) E is covered by a sequence $(S_n)_n$ of absolutely convex metrizable subsets.
(ii) E is a continuous linear image of a separable and metrizable tvs.

Proof Assume that the family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a sequentially complete resolution in the space E.

Assume (i). Then by [322, Theorem 9.2.4] it follows that each $(\overline{S_n}, \xi|\overline{S_n})$ is metrizable, and then it has the complete resolution $\{\overline{S_n} \cap A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Since ξ is separable, (E, ξ) is trans-separable. Then each metrizable and trans-separable subspace $(\overline{S_n}, \xi|\overline{S_n})$ is separable. By Corollary 6.6.2 each space $(\overline{S_n}, \xi|\overline{S_n})$ is analytic, so (E, ξ) is analytic.

Assume (ii). First note that (E, ξ) admits a stronger separable and metrizable vector topology τ. Let τ^ξ be a vector topology on E whose neighbourhoods of zero are composed by the ξ-closures of τ-neighbourhoods of zero. Then $\xi \leq \tau^\xi \leq \tau$, τ^ξ is metrizable, separable, and ξ-polar. The space (E, τ^ξ) admits a complete resolution. Hence, applying Corollary 6.6.2, (E, τ^ξ) and (E, ξ) are analytic. □

Corollary 6.6.4 *Every Montel (DF)-space is analytic.*

Proof Let $(S_n)_n$ be a fundamental sequence of bounded absolutely convex closed subsets of E. The strong dual $(E', \beta(E', E))$ is a Fréchet–Montel space, so $(E', \beta(E', E))$ is separable. Then every bounded set in $(E'', \sigma(E'', E'))$ is metrizable. Hence every bounded closed set in E is metrizable and compact, and Theorem 6.6.3 applies. □

Theorem 5.3.1 and Corollary 6.6.2 supplement Proposition 3.2.13 by noting the following:

Corollary 6.6.5 *Let E be a separable Fréchet space. Then no proper dense finite-codimensional subspace F of E is weakly K-analytic.*

Proof Assume F is weakly K-analytic. By Corollary 6.6.1 the space F has a complete resolution in the relative topology of E. Applying Corollary 6.6.2 we deduce that F is analytic. Let D be a finite-dimensional algebraic complement of F to E. Let $q : E \to E/D$ be the quotient map. Since F is a proper dense subspace of E, $q|F : F \to E/D$ is an injective continuous map that is not isomorphism. Therefore on F there exists a strictly weaker metrizable and complete locally convex topology ξ such that (F, ξ) is isomorphic to E/D. Since F is analytic, we reach a contradiction by applying Theorem 5.3.1 for the identity map $(F, \xi) \to F$. □

6.7 Weakly Compact Density Condition

If E is a Banach space, the Mackey dual $(E', \mu(E', E))$ is not metrizable, except the case when E is reflexive. It is well known that $(E', \mu(E', E))$ is a complete lcs; see, for example, [374]. On the other hand, if B' is the unit ball in the dual E' of E, one may expect that under some conditions $(B', \mu(E', E)|B')$ is metrizable.

In [546] Schlüchtermann and Wheeler introduced the class of *strongly weakly compactly generated* (SWCG) Banach spaces. A Banach space is (SWCG) if the space $(B', \mu(E', E)|B')$ is metrizable; see also [547].

Following Theorem 6.7.1, from [546, Theorem 2.1], shows that every (SWCG) Banach space is (WCG). In [546, Theorem 2.5] it is proved that every (SWCG) Banach space is weakly sequentially complete. Hence the space c_0 being a (WCG) space is not (SWCG).

Theorem 6.7.1 *Let E be a Banach space. Let B, B' be the closed unit ball in E and E', respectively. The following conditions are equivalent:*

(i) $(B', \mu(E', E)|B')$ *is metrizable.*
(ii) There exists a sequence $(K_n)_n$ of weakly compact absolutely convex subsets of E such that for every weakly compact set $L \subset E$ and every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $L \subset K_n + \epsilon B$.
(iii) There exists a weakly compact absolutely convex set $K \subset E$ such that for each weakly compact set $L \subset E$ and every $\epsilon > 0$ there is $n \in \mathbb{N}$ such that $L \subset nK + \epsilon B$.

Proof (i) $\Rightarrow$ (ii): Since $(B', \mu(E', E)|B')$ is metrizable, there exists a sequence $(K_n)_n$ of weakly compact absolutely convex sets in E such that $(K_n^\circ \cap B')_n$ is a countable basis of neighbourhoods of zero in $\mu(E', E)|B'$. Let L be a weakly compact set in E and choose $\epsilon > 0$; we may assume that $\epsilon < 1$. Set $c = \epsilon^{-1}$. Fix $n \in \mathbb{N}$ such that

$$K_n^\circ \cap B' \subset (cL)^\circ \cap B'.$$

Then

$$((cL)^\circ \cap B')^\circ \subset (K_n^\circ \cap B')^\circ.$$

This implies that

$$cL \subset ((cL)^{\circ\circ} \cup B)^{\circ\circ} \subset (K_n^{\circ\circ} \cup B)^{\circ\circ} \subset K_n + B,$$

and this yields $L \subset K_n + \epsilon B$.

(ii) $\Rightarrow$ (iii): Let $(K_n)_n$ be a sequence as assumed in (ii). Let

$$t_n := \sup\{\|x\| : x \in K_n\}$$

for each $n \in \mathbb{N}$. Set

$$K := \left\{ \sum_n 2^{-n} t_n^{-1} x_n : x_n \in K_n \right\}.$$

Note that K is an absolutely convex weakly closed set in E. It is easy to see that K is weakly compact in E, we can use also Grothendieck's characterization of the weak compactness; see [170, p.227]. Note that $K_n \subset 2^n t_n K$.

(iii) $\Rightarrow$ (i): Define a metric

$$d(f, g) := \max_{x \in K} |f(x) - g(x)|$$

for $f, g \in E'$. Clearly the topology ξ generated by this metric is weaker than $\mu(E', E)$. We prove that

$$\mu(E', E)|B' \leq \xi|B'.$$

Let $(f_t)_{t \in T}$ be a net in $(B', \xi|B')$ that converges to f. If L is a weakly compact set in E and $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$L \subset nK + \epsilon(4)^{-1} B.$$

There exists $t_0 \in T$ such that for $t > t_0$ one has

$$|f_t(y) - f(y)| < \epsilon(2n)^{-1},$$

for all $y \in K$. Then $|f_t(x) - f(x)| < \epsilon$ for all $t > t_0$ if $x \in L$, because then there exists $y \in K$ such that $\|x - ny\| \leq \epsilon(4)^{-1}$. This proves that $(f_t)_{t \in T}$ converges to f in the topology $\mu(E', E)$. □

If E is a separable (SWCG) Banach space, $(E', \mu(E', E))$ is separable. Since $E' = \bigcup_n nB'$ and each nB' is metrizable, by Theorem 6.6.3 the space $(E', \mu(E', E))$ is analytic. Therefore we have:

Proposition 6.7.2 *Let E be a (SWCG) Banach space. Then $(E', \mu(E', E))$ is analytic if and only if E is separable.*

Let E be a Banach space and let (S, Σ, μ) be a finite measure space. By $L_1(\mu, E)$ we denote a Banach space of the Bochner integrable functions $f : S \to E$. In [546, Theorem 3.2] Schlüchtermann and Wheeler asked when $L_1(\mu, E)$ is (SWCG). Talagrand [584] (see also Diestel [169]) proved that $L_1(\mu, E)$ is (WCG) if E is a (WCG) Banach space.

If E is a separable Banach space, the Mackey dual $(E', \mu(E', E))$ is separable but $(E', \beta(E', E))$ need not be separable. Clearly $(E', \beta(E', E))$ is analytic if and only if $(E', \beta(E', E))$ is separable. Theorem 6.7.1 and Proposition 6.7.2 may suggest the following question: Let E be a separable Banach space. Is it true that the Mackey dual $(E', \mu(E', E))$ of E is an analytic space?

In this section we provide some sufficient conditions for a separable (LF)-space E to have the Mackey dual $(E', \mu(E', E))$ analytic.

Now we recall the concept of the density condition (dc) introduced by Heinrich in [307]; see also [84]. Let E be a metrizable lcs with a countable basis $(U_n)_n$ of absolutely convex neighbourhoods of zero.

We will say that E satisfies the *density condition* (dc) if there exists a double sequence $(B_{n,k})_{n,k}$ of bounded sets in E such that for each $n \in \mathbb{N}$ and each bounded set $C \subset E$ there is $k \in \mathbb{N}$ such that $C \subset B_{n,k} + U_n$.

For following Proposition 6.7.3 we refer to Bierstedt and Bonet's work [84].

Proposition 6.7.3 *Every Fréchet–Montel space, i.e. a Fréchet space whose every bounded closed set is compact, satisfies the density condition.*

Proof Let $(U_n)_n$ be a countable basis of neighbourhoods of zero in E such that $U_{n+1} + U_{n+1} \subset U_n$ for each $n \in \mathbb{N}$. Since E is Fréchet–Montel, E is separable, Corollary 6.4.5. Let $\{x_k : k \in \mathbb{N}\}$ be a countable dense subset of E. For each $n, k \in \mathbb{N}$ let $B_{n,k}$ be the absolutely convex envelope of the finite set $\{x_1, x_2, \ldots, x_k\}$. Fix $n \in \mathbb{N}$ and a bounded set $C \subset E$. Then (since C is precompact) there exists a finite set $F \subset E$ such that $C \subset F + U_{n+1}$. As the set $\{x_k : k \in \mathbb{N}\}$ is dense in E, there exists $k \in \mathbb{N}$ such that $F \subset B_{n,k} + U_{n+1}$. Then

$$C \subset B_{n,k} + U_{n+1} + U_{n+1} \subset B_{n,k} + U_n.$$

The proof is completed. □

Note that there exist reflexive Fréchet spaces whose strong dual is separable (such spaces are distinguished; see Sect. 6.6) and which do not satisfy the density condition [82].

The next proposition was obtained by Bierstedt and Bonet in [82].

Proposition 6.7.4 *A metrizable lcs E satisfies the density condition if and only if every bounded set in $(E', \beta(E', E))$ is metrizable.*

Proof Let $(U_n)_n$ be a decreasing basis of absolutely convex neighbourhoods of zero for E such that $U_{n+1} + U_{n+1} \subset U_n$ for each $n \in \mathbb{N}$. It is known that the bounded sets in $(E', \beta(E', E))$ are metrizable if and only if each polar U_n° has a countable basis of neighbourhoods of zero in the topology $\beta(E', E)$; see [322, Lemma 9.2.4]. Therefore, we need only to show that for each bounded absolutely convex closed set $C \subset E$, and for each $n \in \mathbb{N}$, there exists a bounded set $B_{n,k} \subset E$ such that

$$B_{n,k}^\circ \cap U_{n+1}^\circ \subset C^\circ.$$

The rest of the proof follows easily from the bipolar theorem [322, Theorem 8.2.1]:

$$C \subset (B_{n,k}^\circ \cap U_{n+1}^\circ)^\circ \subset B_{n,k} + U_{n+1} + U_{n+1} \subset B_{n,k} + U_n.$$

Then

$$B_{n,k}^\circ \cap U_n^\circ \subset 2(B_{n,k} + U_n)^\circ \subset 2C^\circ.$$

□

Let $\mathfrak{K}(E)$ be the family of all absolutely convex $\sigma(E, E')$-compact sets in a lcs E.

Motivated by papers [82, 83, 546], and [84] we will say that a metrizable lcs E with a countable basis $(U_n)_n$ of absolutely convex neigbourhoods of zero in E satisfies the *weakly compact density condition* (wcdc) if there is in $\mathfrak{K}(E)$ a double sequence $(B_{n,k})_{n,k}$ such that for $n \in \mathbb{N}$ and $C \in \mathfrak{K}(E)$ there exists $k \in \mathbb{N}$ such that $C \subset B_{n,k} + U_n$.

If $\mathfrak{K}(E)$ denotes the family of all bounded sets in E, the above condition describes the *density condition* for a metrizable lcs E. For Fréchet–Montel spaces both the conditions (wcdc) and (dc) are equivalent.

We discuss the conditions (wcdc) and (dc) for Köthe echelon spaces. Consider the class of Köthe echelon spaces $\lambda_p := \lambda_p(I, A)$, where $A = (a_n)$ is any Köthe matrix on a countable set I with $1 < p < \infty$, i.e. an increasing sequence of strictly positive functions $(a_n)_n$ on the set I. It is known that the strong dual of λ_p is the analytic (LB)-space with a defining sequence $(\ell_q(I, v_n))_n$ and $q^{-1} + p^{-1} = 1$, $v_n = a_n^{-1}$, $n \in \mathbb{N}$; see [82]. Consider briefly the following particular case λ_1:

(i) *The Mackey dual $(\lambda_1', \mu(\lambda_1', \lambda_1))$ of λ_1 is analytic.*

Indeed, since every weakly compact set in λ_1 is compact, $\mu(\lambda_1', \lambda_1)$ equals the topology $\tau_{pc}(\lambda_1', \lambda_1)$ of the uniform convergence on λ_1-precompact sets. The last topology $\tau_{pc}(\lambda_1', \lambda_1)$ is analytic. Indeed, first observe that $\tau_{pc}(\lambda_1', \lambda_1)$ admits a weaker metric topology. Hence $\tau_{pc}(\lambda_1', \lambda_1)$ is angelic. Since $\tau_{pc}(\lambda_1', \lambda_1)$ admits a relatively countably compact resolution, by the angelicity the space $(\lambda_1', \mu(\lambda_1', \lambda_1))$ has a compact resolution. Now the conclusion follows from Proposition 6.1.4.

Moreover, λ_1 satisfies the (dc) if and only if the Köthe matrix A satisfies the condition (D); see [83, Theorem 4] and [82, Theorem 6].

The space λ_1 satisfies the (wcdc) for any A. Indeed, let $\{x_n : n \in \mathbb{N}\}$ be a dense countable subset in λ_1, and let $(U_n)_n$ be a countable basis of neighbourhoods of zero in λ_1. Then for a weakly compact set $C \subset \lambda_1$ and $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that

$$C \subset \{\sum_{j=1}^{k} a_j x_j : |a_j| \leq 1\} + U_n.$$

(ii) *The strong dual* $(\lambda_1', \beta(\lambda_1', \lambda_1))$ *is analytic if and only if* λ_1 *is Montel.*
Indeed, if λ_1 is Montel, its strong dual is covered by a sequence of absolutely convex compact metrizable sets and Theorem 6.6.3 applies. The converse follows from Dieudonné–Gomes's theorem [439, Theorem 27.9].

(iii) *If* λ_1 *satisfies the (dc) and is not Montel* (see [82, Theorem 4, Corollary 8] describing this case), *then* $(\lambda_1', \beta(\lambda_1', \lambda_1))$ *is not quasi-Suslin.*
Indeed, by the condition (dc) every closed bounded set in the (DF)-space $(\lambda_1', \beta(\lambda_1', \lambda_1))$ is metrizable (and complete by the completeness of $\beta(\lambda_1', \lambda_1)$). Assume $(\lambda_1', \beta(\lambda_1', \lambda_1))$ is quasi-Suslin. Since $(\lambda_1', \mu(\lambda_1', \lambda_1)$ is analytic, it admits a weaker metric topology by Proposition 6.1.4. Hence $(\lambda_1', \beta(\lambda_1', \lambda_1)$ is K-analytic. By Proposition 6.1.4 the space $(\lambda_1', \beta(\lambda_1', \lambda_1))$ is analytic. Hence λ_1 is Montel.

(iv) *There exists a separable reflexive Fréchet space* E *which does not satisfy the (dc) and* $(E', \beta(E', E))$ *is analytic.* Indeed, let $A = (a_n)$ be a Köthe matrix on $\mathbb{N}$ satisfying the condition (ND); see [491]. Then λ_p for $p > 1$ does not satisfy the (dc) [82, p.178]. On the other hand, the strong dual of λ_p is analytic.

Following Proposition 6.7.5 motivates Theorem 6.7.7:

Proposition 6.7.5 *Let* E *be a separable* (LF)*-space. Then the dual* E' *endowed with the topology* $\tau_{pc}(E', E)$ *of the uniform convergence on precompact sets of* E *is an analytic space. Hence* $(E', \sigma(E', E))$ *is also an analytic space.*

Proof Let $(E_n)_n$ be a sequence of Fréchet spaces defining the space E. For each $n \in \mathbb{N}$ let $(U_k^n)_k$ be a decreasing basis of absolutely convex neighbourhoods of zero in E_n. For $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ set $A_\alpha := \bigcap_k n_k (U_{n_k}^k)^\circ$, where $(U_{n_k}^k)^\circ$ is the polar in E' of the set $U_{n_k}^k$. Then $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution in E'. Since each sequence in any A_α is equicontinuous, each set A_α is relatively compact in the topology $\tau_{pc}(E', E)$. By the separability of E there exists on E' a metric topology weaker than the weak topology $\sigma(E', E)$. Hence $(E', \sigma(E', E))$ is angelic, and we apply Theorem 4.1.1 to deduce that $(E', \tau_{pc}(E', E))$ is angelic. Next, we apply Corollary 3.2.9 to get that $(E', \tau_{pc}(E', E))$ is K-analytic. Finally Proposition 6.1.4 is used to show that $(E', \tau_{pc}(E', E))$ is analytic. □

Having in mind this fact, one can ask about conditions under the Mackey dual $(E', \mu(E', E))$ of a separable (LF)-space E is analytic. The following theorem extends Valdivia's [611, Theorem 23, p.77]. We shall need the following lemma due to Valdivia [611, (22), p.76]:

Lemma 6.7.6 *Let E be an (LF)-space with a defining sequence $(E_n)_n$ of separable Fréchet spaces. Let A be a compact absolutely convex set in $\sigma(E, E')$. Then there exists $m \in \mathbb{N}$ such that $A \subset E_m$ and A is bounded in E_m.*

Proof By E_A we denote the linear span of A endowed with the Minkowski functional Banach topology. Since $E_A = \bigcup_n E_A \cap E_n$, by the Baire category theorem there exists $m \in \mathbb{N}$ such that $E_A \cap E_m$ is a dense and Baire subspace in the space E_A. Clearly the inclusion

$$T : E_A \cap E_m \to (E_m, \tau_m)$$

has closed graph, where τ_m is the original Fréchet separable topology of E_m. By Theorem 5.3.1 the map T is continuous. To complete the proof it is enough to observe that $E_A \subset E_m$. □

We are ready to prove the following theorem [339]:

Theorem 6.7.7 *Let E be an (LF)-space and $(E_n)_n$ a defining sequence of E of separable reflexive Fréchet spaces satisfying the (wcdc). Then the Mackey dual $(E', \mu(E', E))$ is an analytic space.*

Proof First assume that E is a metrizable space. Fix $n \in \mathbb{N}$ and set $S_n := U_n^\circ$, where $(U_n)_n$ denotes a decreasing basis of absolutely convex closed neighbourhoods of zero in E. Applying the (wcdc) we deduce that there exists a sequence $(B_{n,k})_k$ in $\mathfrak{K}(E)$ with the desired properties. Since the polars of absolutely convex $\sigma(E, E')$-compact sets A in E compose a basis of neighbourhoods of zero for $(E', \mu(E', E))$, we deduce that for a $\mu(E', E)$-neighbourhood of zero V there is $k \in \mathbb{N}$ such that

$$B_{n,k}^\circ \cap S_n \subset 2V.$$

This yields the metrizablity of $(S_n, \mu(E', E)|S_n)$. The sets $A_\alpha := S_{n_1}$, for $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$, generate a compact resolution in $\sigma(E', E)$, so the assumptions of Theorem 6.6.3 are satisfied. Therefore $(E, \mu(E', E)) = \bigcup_n S_n$ is analytic.

Now assume that E is an (LF)-space and $(E_n)_n$ is a defining sequence of separable reflexive Fréchet spaces satisfying the (wcdc). Since each E_n is a separable and reflexive space, the strong dual $(E_n', \beta(E_n', E_n))$ is analytic (by the previous case), and the projective limit

$$(E', \gamma) := \mathrm{Proj}_n(E_n', \beta(E_n', E_n))$$

is a closed subspace of the analytic space $\prod_n (E_n', \beta(E_n', E_n))$. As closed subspaces of analytic spaces are analytic, (E', γ) is analytic. Since every absolutely convex

$\sigma(E, E')$-compact set is contained and bounded in some E_m by Lemma 6.7.6, we deduce $\mu(E', E) \leq \gamma$. If $j_n : E_n \to E$ is the inclusion map, the dual map

$$j_n' : (E', \mu(E', E)) \to (E_n', \mu(E_n', E_n))$$

is continuous for $n \in \mathbb{N}$. This, combined with the equality $\mu(E_n', E_n) = \beta(E_n', E_n)$ (since E_n are reflexive), yields $\gamma \leq \mu(E', E)$. □

The Mackey dual $(E', \mu(E', E))$ of a Banach space has been studied also in [547] and [367]. In [547] the authors proved that, if E is a separable (SWCG) Banach space, the Mackey dual $(E', \mu(E, E'))$, which is clearly analytic, is an $\aleph_0$-space, i.e. it has a countable pseudobase. Recall that a family $\mathcal{P}$ of subsets of a topological space E is called a *pseudobase*, if for any open set $U \subset E$ and compact $K \subset U$ there exists $P \in \mathcal{P}$ with $K \subset P \subset U$. Every $\aleph_0$-space is separable and Lindelöf [445], [547, Theorem 4.1]. In [367] Kirk studied the Mackey dual for Banach spaces $C(K)$ with compact K. On the other hand, by Batt and Hiermeyer [74, 2.6] (see also [546], [547, p.274], and [547, Theorem 4.2]), there exists a separable Banach space E for which $(E, \sigma(E, E'))$ is not an $\aleph_0$-space. It is known also ([445], [547, Theorem 4.1]) that a regular topological space is both an $\aleph_0$-space and a k-space if and only if it is a quotient of a separable metric space. It seems to be natural to ask about conditions for a Banach space E to have the space $(E, \sigma(E, E'))$ a k-space.

Motivated by the density condition in the sense of Heinrich for Fréchet spaces and by some results of Schlüchtermann and Wheeler for Banach spaces, in [212] we characterized in terms of certain weakly compact resolutions those Fréchet spaces enjoying the property that each bounded subset of its Mackey* dual is metrizable. We also characterized those Köthe echelon Fréchet spaces $\lambda_p(A)$ as well as those Fréchet spaces $C_k(X)$ of real-valued continuous functions equipped with the compact-open topology that have this property.

Recall also that a Hausdorff space X is a *k-space* if a set $A \subset X$ is closed in X if and only if $A \cap K$ is closed in K for each compact set $K \subset X$. We need the following simple fact due to Grothendieck [286, p.134]:

Lemma 6.7.8 *Let E be a Banach space and let $A \subset E'$ be a $\mu(E', E)$-compact set in E'. Then every $\sigma(E, E')$-convergent sequence in E converges uniformly on A.*

Now we prove the following:

Proposition 6.7.9 *If E is a Banach space for which $(E, \sigma(E, E'))$ is a k-space, the space E is finite-dimensional.*

Proof Let γ be the topology on E of the uniform convergence on $\mu(E', E)$-compact sets. Clearly $\sigma(E, E') \leq \gamma$. Since $\sigma(E, E')$ and γ have the same sequentially compact sets by applying Lemma 6.7.8, both the topologies have the same compact sets (note that $\sigma(E', E)$ and γ are angelic). Assume $(E, \sigma(E, E'))$ is a k-space. Then $\sigma(E, E') = \gamma$. Let $(x_n)_n$ be a null sequence in the norm topology of E'. Since $\{0\} \cup \{x_n : n \in \mathbb{N}\}$ is $\mu(E', E)$-compact, the sequence $(x_n)_n$ has a finite-dimensional linear span. This implies that E' (hence also E) is finite-dimensional. □

Krupski and Marciszewski proved [375, Corollary 6.5] (using a different approach) that for an infinite-dimensional Banach space E neither $(E, \sigma(E, E'))$ nor $(E', \sigma(E', E))$ is a k-space. Very recently Ferrando (private communication) proved even the following general fact:

Proposition 6.7.10 (Ferrando) *Let $E = (E, \tau)$ be a lcs. If $(E, \sigma(E, E'))$ is a k-space, then $\tau = \sigma(E, E')$.*

Proof Set $w = \sigma(E, E')$. Let ϱ be the topology on E' of the uniform convergence on the weakly compact sets in E, let F be the dual of $\left(E', \varrho\right)$. Let $\Im$ be the topology on F of uniform convergence on the ϱ-compact sets of E'. Since each w-compact set P in E is ϱ-equicontinuous, $w|P = \Im|P$, [374, 21.6.(2)].

Hence, if $\xi = \Im|E$, both $\xi|P = w|P$, so ξ and w on E have the same compact sets. Thus, if E_w is a k-space, then $\xi = w$. If Q is a ϱ-compact set in E' and Q^0 is the polar of Q in F, then $V = Q^0 \cap E$ is a ξ-neighbourhood of the origin in E. Since $Q \subseteq V^0$, where the latter polar is in E', Q is ξ-equicontinuous. So, the equality $\xi = w$ implies Q is finite-dimensional. Therefore every ϱ-compact set in E' is finite-dimensional.

If U is an absolutely convex τ-neighbourhood of zero in E, the polar U^0 of U in E' is a $\sigma\left(E', E\right)$-compact Banach disc in $\left(E', \sigma\left(E', E\right)\right)$. Since, by the Banach–Mackey theorem, U^0 is also $\beta\left(E', E\right)$-bounded, U^0 is also a Banach disc in $\left(E', \beta\left(E', E\right)\right)$.

Let $E'_{U^0} = \text{span}(U^0)$ be the Banach space with its Minkowski's functional norm topology τ'_{U^0}. If K is compact in E'_{U^0}, then K is also compact in $\beta\left(E', E\right)$ because $\beta(E', E)|\text{span}(U^0) \leq \tau'_{U^0}$. Since $\beta\left(E', E\right) \geq \varrho$, it follows K is finite-dimensional. But a normed space whose all compact sets are finite-dimensional is finite-dimensional. So U^0 is a finite-dimensional set. Hence U is a w-neighbourhood of zero in E, which shows that $\tau = \sigma\left(E, E'\right)$. □

6.8 More Examples of Non-Separable Weakly Analytic Tvs

A tvs E is called *dual-separating*, if its topological dual E' separates points of E, i.e. for each $x \neq 0$ there exists $f \in E'$ such that $f(x) \neq 0$. Next example provides a non-separable non-locally convex dual-separating F-space E such that the weak topology $\sigma(E, E')$ is analytic. Example 6.8.1 uses some arguments from [185].

We need the following simple general fact. If E is a vector space admitting two metrizable vector topologies ξ_1 and ξ_2 such that $\xi_1 \leq \xi_2$, and every ξ_1-bounded set is ξ_2-bounded, then $\xi_1 = \xi_2$. Indeed, take a null sequence $(x_n)_n$ in ξ_1. There exists an unbounded scalar sequence $t_n \nearrow \infty$ such that $t_n x_n \to 0$ in ξ_1. Hence the sequence $(t_n x_n)_n$ is ξ_2-bounded, and then $x_n = t_n^{-1}(t_n x_n) \to 0$ in ξ_2.

Example 6.8.1 There exists a non-locally convex non-separable metrizable and complete tvs $\lambda_0 = (\lambda_0, \xi)$ such that $(\lambda_0, \sigma(\lambda_0, \lambda'_0))$ is isomorphic to a (dense) vector subspace of $\mathbb{R}^{\mathbb{N}}$. Moreover,

(i) $(\lambda_0, \sigma(\lambda_0, \lambda_0'))$ is analytic unordered Baire-like and not Baire.
(ii) $\mathbb{R}^{\mathbb{N}} \setminus (\lambda_0, \sigma(\lambda_0, \lambda_0'))$ is a Baire space.
(iii) $(\lambda_0, \sigma(\lambda_0, \lambda_0'))$ contains a copy of $\mathbb{N}^{\mathbb{N}}$ as a closed subset.

Proof By [185] there exists a non-locally convex non-separable F-space λ_0 with a basis $(U_n)_n$ of balanced neighbourhoods of zero closed in $\mathbb{R}^{\mathbb{N}}$, i.e. λ_0 is the space of all sequences $x = (\varrho_n)$ of real numbers such that $\|tx\| \to 0$ as $t \to 0$, where $\|x\| := \sup \|x\|_n$, $\|x\|_n := n^{-1} \sum_{j=1}^{n} \min\{1, |\varrho_j|\}$. Set

$$U_n := \{x \in \lambda_0 : \|x\| \leq n^{-1}\}$$

for $n \in \mathbb{N}$. The space λ_0 endowed with the topology ξ generated by the F-norm $\|.\|$ is metrizable, complete, and non-separable, and the topological dual of λ_0 is an $\aleph_0$-dimensional vector space; we refer the reader to [185] to check details. We prove the claims (i), (ii), and (iii).

(i) Note that for $m \in \mathbb{N}$ one has $\lambda_0 = \bigcup_n nU_m$, and each nU_m is $\sigma(\lambda_0, \lambda_0')$-analytic (as a complete, metrizable, and separable space). Hence the space $E := (\lambda_0, \sigma(\lambda_0, \lambda_0'))$ is analytic. Since $\sigma(\lambda_0, \lambda_0')$ is metrizable, the topology $\sigma(\lambda_0, \lambda_0')$ equals to the finest locally convex topology ξ_c on λ_0 weaker than ξ. Indeed, the topologies $\sigma(\lambda_0, \lambda_0')$ and ξ_c are metrizable and locally convex, and they generate the same continuous linear functionals. Hence both the metrizable topologies have the same bounded sets, so they coincide.

The space (λ_0, ξ_c) is an unordered Baire-like space [163]. Indeed, if $(A_n)_n$ is sequence of absolutely convex closed subsets in the space (λ_0, ξ_c) covering (λ_0, ξ_c), then each set A_n is closed in the topology ξ. The Baire category theorem applies to find a number $m \in \mathbb{N}$ such that A_m has non-void interior in ξ. Since A_m is an absolutely convex set, using the definition of the topology ξ_c, we deduce that A_m is a ξ_c-neighbourhood of zero. On the other hand, (λ_0, ξ_c) is not Baire. Indeed, otherwise the identity map $I : (\lambda_0, \xi_c) \to (\lambda_0, \xi)$ (which has closed graph) would be continuous by using the closed graph theorem [1] (between Baire tvs and F-spaces), impossible (since (λ_0, ξ) is not locally convex).

(ii) Since $\mathbb{R}^{\mathbb{N}} \setminus E$ is a G_δ-subset of $\mathbb{R}^{\mathbb{N}}$, Proposition 2.1.10 applies to conclude that $\mathbb{R}^{\mathbb{N}} \setminus E$ is a Polish space, hence Baire.
(iii) Note that E cannot be covered by a sequence of bounded sets. Indeed, let F be the closure of E in $\mathbb{R}^{\mathbb{N}}$. Clearly the space F is linearly homeomorphic to the product space $\mathbb{R}^{\mathbb{N}}$. Assume that $E = \bigcup_n S_n$ is the union of a sequence of bounded closed absolutely convex sets. Since E is an unordered Baire-like space, some S_m is a neighbourhood of zero in E. Hence its closure (in the space F) is a neighbourhood of zero in F. Consequently, $\mathbb{R}^{\mathbb{N}}$ is a normed space yielding a contradiction. Hence the space E is not a σ-compact space. Now Theorem 3.5.12 applies to get the conclusion of (iii). □

For the next Example 6.8.9 we need the following definitions. A function f from $[0,1]$ into a vector space E is called *measurable* if all values of f lie in a finite-dimensional subspace $F \subset E$ (depending on f) and f is Lebesque measurable, where F is endowed with the unique Hausdorff vector topology. The set $S(E)$ of equivalent measurable (classes of) functions is a vector space. Since two constant functions agree almost everywhere if and only if they are identical, there is an injective map $t_E : E \to S(E)$ that assigns to every $x \in E$ the constant function $f(t) = x$ for $t \in [0,1]$. A few additional results will be used to present Example 6.8.9. We start with the following one; see, for example, [487, Theorem A].

Proposition 6.8.2 *Every (metrizable) tvs (E,τ) such that $\dim E \geq 2^{\aleph_0}$ is linearly homeomorphic under a linear map t_E to a (metrizable) subspace of a tvs $(S(E), \mu(\tau))$ which does not admit non-zero continuous linear functionals, and such that $\dim E$ equals the codimension of $t_E(E)$ (in $S(E)$), and the density characters of the spaces E and $S(E)$ are the same.*

We will make use of the following observation due to Dierolf [519]; see also [162] and [166].

Lemma 6.8.3 *Let ξ and ϑ be two vector topologies on a vector space E such that $\xi \leq \vartheta$. If F is a vector subspace of E such that $\xi|F = \vartheta|F$ and $\xi/F = \vartheta/F$, then $\xi = \vartheta$.*

Proof Let U be a ϑ-neighbourhood of zero in E. Then there exists a neighbourhood of zero V in ξ such that

$$(V - V) \cap F \subset U.$$

Since $U \cap V$ is a neighbourhood of zero in ϑ, there exists a ϑ-neighbourhood of zero W such that

$$W \subset V, \quad W \subset (U \cap V) + F.$$

Choose arbitrary $w \in W$. Then there exist $x \in U \cap V$ and $y \in F$ such that $w = x + y$, where

$$y := -x + w \in -(U \cap V) + W \subset (V - V) \cap F \subset U.$$

Hence

$$w = x + y \in (U \cap V) + U \subset U + U.$$

This proves that $W \subset U + U$, so $U + U$ is a neighbourhood of zero in ξ. Hence $\xi = \vartheta$. □

Also the following lemma will be used; see [519, Proposition 2.1, Theorem 3.2] and [282].

Lemma 6.8.4 *Let E be a tvs containing a closed vector metrizable (normed) subspace F such that the quotient E/F is also metrizable (normed). Then E is metrizable (normed).*

Proof Let $q : E \to E/F$ be the quotient map. By the assumptions there exists a sequence $(V_n)_n$ of neighbourhoods of zero in E such that for each $n \in \mathbb{N}$ we have $V_{n+1} - V_{n+1} \subset V_n$, $(F \cap V_n)_n$ is a basis of neighbourhoods of zero in F, and $(q(V_n))_n$ is a basis of neighbourhoods of zero in the quotient space E/F. Choose neighbourhoods of zero U and W in E such that $W + W \subset U$. There exist $n, m \in \mathbb{N}$, $m > n$, such that

$$F \cap V_n \subset W, \quad q(V_m) \subset q(W \cap V_{n+1}).$$

Then $V_m \subset W \cap V_{n+1} + F$. This implies that

$$V_m \subset W \cap V_{n+1} + F \cap (V_m - V_{n+1}) \subset$$

$$W + F \cap (V_{n+1} - V_{n+1}) \subset W + F \cap V_n \subset W + W \subset U.$$

This proves that E has a countable basis of neighbourhoods of zero yielding the metrizability of E. The other case we prove similarly. □

Next Lemma 6.8.5 can be found in [520, Proposition 12.20].

Lemma 6.8.5 *Let $F \subset E$ be a separable closed vector subspace of a tvs such that E/F is separable. Then E is separable.*

Proof Let $\{y_n : n \in \mathbb{N}\}$ be a dense subset in E/F. For each $n \in \mathbb{N}$ let $\{x_{m,n} : m \in \mathbb{N}\}$ be a dense subset of $y_n + F$. Then the set $\{x_{m,n} : n, m \in \mathbb{N}\}$ is dense in E. Hence E is separable. □

We need also the following result due to Frolik [256]; see also [520, Proposition 12.21].

Lemma 6.8.6 *Let E be a metrizable and separable tvs containg a closed vector subspace F such that F and E/F are Baire spaces. Then E is a Baire space.*

Proof Let $(U_n)_n$ be a countable basis of open sets in E. Assume that E is covered by a sequence $(A_n)_n$ of nowhere dense closed subsets of E. Let $q : E \to E/F$ be the quotient map. For each $n, m \in \mathbb{N}$ set

$$K_{n,m} := \{y \in E/F : \emptyset \neq U_n \cap q^{-1}(y) \subset A_m\}.$$

Clearly each $K_{n,m}$ has void interior (since $U_n \cap q^{-1}(\operatorname{int} K_{n,m}) \subset A_m$).

We prove that all sets $K_{n,m}$ are nowhere dense in E/F. Clearly $K_{n,m}$ are closed in $q(U_n)$ since

$$q(U_n) \setminus K_{n,m} = \{y \in E/F : U_n \cap q^{-1}(y) \cap [E \setminus A_m] \neq \emptyset\} =$$

$$q\left(U_n \cap (E \setminus A_m)\right).$$

Therefore, $K_{n,m} \subset q(U_n)$ is nowhere dense in the open set $q(U_n)$. This yields that $K_{n,m}$ is nowhere dense in E/F. Set $K := \bigcup_{n,m} K_{n,m}$. Since, by the assumption, the quotient space E/F is a Baire space, there exists $z \in (E/F) \setminus K$. Clearly,

$$q^{-1}(z) = \bigcup_n A_n \cap q^{-1}(z).$$

Since $q^{-1}(z)$ is also a Baire space, there exist $p, r \in \mathbb{N}$ such that

$$\emptyset \neq U_p \cap q^{-1}(z) \subset A_r \cap q^{-1}(z).$$

This implies that $z \in K_{p,r} \subset K$, a contradiction. Hence E is a Baire space. □

We recall the following result; see [97, Ch.III, 3, ex. 9], or [519, Proposition 1.3] and [491, Proposition 2.4.2].

Lemma 6.8.7 *Let E be a tvs containing a complete vector subspace F such that the quotient space E/F is complete. Then E is complete.*

Proof Let H be the completion of E. Assume that $x \in H \setminus E$. Set $G := \text{span}\{E \cup \{x\}\}$. Let $q_x : H \to E/F$ be the unique extension of the quotient map $q : E \to E/F$. Since $q_x(x) \in E/F$, there exists $y \in E$ such that $q_x(x) = q(y)$. Since F is closed in G and $q_x^{-1}(0) \cap G = F$, we deduce that $x - y \in F$, so $x \in E$, a contradiction. This proves that E is complete. □

We need also the following [162, (2), p.194].

Lemma 6.8.8 *Let F be a dense linear subspace of a tvs (E, τ), and let ξ be a vector topology on the quotient space E/F. If η denotes the initial topology on E with respect to the canonical injection $J : E \hookrightarrow (E, \tau)$ and the canonical surjection $Q : E \to (E/F, \xi)$, then*

$$\eta|F = \tau|F, \ \ \eta/F = \xi.$$

Proof Let U be a neighbourhood of zero in $(E/F, \xi)$. Then $F \subset Q^{-1}(U)$. Therefore $\eta|F = \tau|F$. Since

$$Q : (E, \eta) \to (E/F, \xi)$$

is continuous, $\xi \leq \eta/F$. To prove the equality, fix a neighbourhood of zero U in $(E/F, \eta/F)$. Then there exists a neighbourhood of zero V in (E, τ) and a neighbourhood of zero W in $(E/F, \xi)$ such that

$$V \cap Q^{-1}(W) \subset Q^{-1}(U).$$

By density we have $V + F = E$, i.e. $Q(V) = E/F$. Therefore, if $z \in W$, then there exists $v \in V$ such that $Q(v) = z$. Then

$$v \in V \cap Q^{-1}(W) \subset Q^{-1}(U).$$

This implies that $z = Q(v) \in U$. Hence $W \subset U$, and this yields the equality of the topologies. □

Now we are ready to prove the following general example providing many metrizable and separable non-analytic dual-separating tvs whose weak topology is analytic. Clearly, such spaces are necessarily non-locally convex; see [339].

Example 6.8.9 For every infinite-dimensional separable Fréchet space (E, τ) there exist two metrizable non-analytic and weakly analytic vector topologies ξ_1, ξ_2 such that:

(1) $\tau = \inf\{\xi_1, \xi_2\}$.
(2) ξ_1 is Baire and separable and ξ_2 is not separable.
(3) $(E, \tau)' = (E, \xi_1)' = (E, \xi_2)'$, i.e. the three topologies have the same weak topology.

Proof Let $(x_t)_{t\in T}$ be a Hamel basis of E. Consider a partition $(T_n)_n$ of T such that $\operatorname{card} T = \operatorname{card} T_n$ for all $n \in \mathbb{N}$. Set

$$E_n := \operatorname{span}\{x_t : t \in \bigcup_{i=1}^{n} T_i\}.$$

Then the sequence $(E_n)_n$ covers E, and

$$\dim E = \dim E_n = \dim (E/E_n) = 2^{\aleph_0}$$

for $n \in \mathbb{N}$. By the Baire category theorem there exists a dense Baire subspace $F := E_m$ of E. For $0 < p < 1$ set $L^p := (L^p[0, 1], \|.\|_p)$. It is known that the space L^p is a $2^{\aleph_0}$-dimensional metrizable, complete, and separable tvs with trivial topological dual; see, for example, [153].

Let α be a metrizable, complete, and separable vector topology on E/F such that $(E/F, \alpha)$ is linearly homeomorphic to the space L^p, and let ξ_1 be the initial vector topology on E defined in Lemma 6.8.8, i.e.

$$\tau < \xi_1, \quad \xi_1/F = \alpha, \quad \xi_1|F = \tau|F.$$

Then ξ_1 is metrizable and separable, by Lemma 6.8.4 and Proposition 6.8.5. The space (E, ξ_1) is non analytic by the closed graph theorem 5.3.1 applied for the

identity map from (E,τ) onto (E,ξ_1). Note that (E,ξ_1) is a Baire space by Lemma 6.8.6.

Now we construct the topology ξ_2. Since (E,ξ_1) is a Baire space, the same argument as above applies to choose a ξ_1-dense subspace G of E such that

$$\dim G = \dim (E/G) = 2^{\aleph_0}.$$

By Proposition 6.8.2 there exists a $2^{\aleph_0}$-dimensional non-separable metrizable tvs Z without non-zero continuous linear functionals. We proceed as above to define on E a non-separable metrizable vector topology ξ_2 such that

$$\tau < \xi_2, \quad \tau|G = \xi_2|G,$$

and $(E/G, \xi_2/G)$ is linearly homeomorphic to the space Z. Clearly, $\tau \le \inf\{\xi_1,\xi_2\}$ and

$$\tau|G = \inf\{\xi_1,\xi_2\}|G = \xi_2|G.$$

On the other hand, the topologies $\tau/G = \xi_1/G$ are trivial, so the topology τ/G and the topology $\inf\{\xi_1,\xi_2\}/G$ coincide. By Lemma 6.8.3 we note $\tau = \inf\{\xi_1,\xi_2\}$.

Finally, we prove that the topologies τ, ξ_1, and ξ_2 have the same continuous linear functionals on E. This will show that the corresponding weak topologies for the spaces (E,τ), (E,ξ_1), and (E,ξ_2) are the same. Consequently, each E endowed with the weak topology of (E,ξ_i), $i = 1, 2$, respectively, will be analytic.

Indeed, let $f \in (E,\xi_1)'$ be a ξ_1-continuous linear functional on E, and let $h \in (E,\tau)'$ be an extension of $f|F$ in τ. Since $f - h \in (E,\xi_1)'$ and $(f-h)(F) = \{0\}$, we note that the map

$$x + F \to (f-h)(x)$$

belongs to $(E/F, \xi_1/F)'$, so $h(x) = f(x)$ for each $x \in E$, i.e. $f = h \in (E,\tau)'$. The same proof works for the topology ξ_2. □

Since every metrizable K-analytic space is analytic, Example 6.8.9 can be used to deduce the following:

Example 6.8.10 Every infinite-dimensional separable Fréchet space E admits a strictly finer metrizable and separable vector topology ξ with the same continuous linear functionals as the original one of E, and such that (E,ξ) is a Baire space not K-analytic.

In [531] Saxon asked if Theorem 2.3.1 remains true for non-locally convex spaces. We note the following observation by using the argument from Example 6.8.9.

Corollary 6.8.11 *Assume CH. If (E, ξ) is a tvs containing a dense infinite-codimensional subspace F, then E admits a stronger vector topology υ such that $(E, \xi)' = (E, \upsilon)'$, and (E, υ) contains a dense non-Baire hyperplane.*

Proof By the assumption the codimension of F in E is either $2^{\aleph_0}$ or $\aleph_0$. Fix $0 < p < 1$. Then the quotient space E/F admits a finer separable vector topology μ such that, in the first case, $(E/F, \mu)$ is isomorphic to the space L^p, and in the other case $(E/F, \mu)$ is isomorphic to a dense $\aleph_0$-dimensional vector subspace of L^p. As in Example 6.8.9 there exists on E a stronger vector topology υ such that $\xi|F = \upsilon|$ and $\upsilon|F = \mu$. By Remark 2.3.3 it follows that $(E/F, \mu)$ contains a dense non-Baire hyperplane. Then, as we showed in Theorem 2.3.1, the space E contains a dense non-Baire hyperplane. □

Chapter 7
K-Analytic Baire Spaces

Abstract In this chapter, we show that a tvs that is a Baire space and admits a countably compact resolution is metrizable, separable, and complete. We prove that a linear map $T : E \to F$ from an F-space E having a resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ into a tvs F is continuous if each restriction $T|K_\alpha$ is continuous. This theorem (due to Drewnowski) was motivated by Arias de Reyna–Valdivia–Saxon's theorem about non-Baire dense hyperplanes in Banach spaces. We provide a large class of weakly analytic metrizable and separable Baire tvs that are not analytic (clearly such spaces are necessarily not locally convex).

7.1 Baire Tvs with a Bounded Resolution

We know already from Corollary 3.3.13 that a Baire lcs which is a quasi-(LB)-space is a Fréchet space. Tkachuk [592] proved that if $C_p(X)$ is K-analytic and Baire, the space X is countable and discrete. Hence a K-analytic Baire space $C_p(X)$ is a separable Fréchet space. In fact, Tkachuk's theorem follows from the following Theorem 7.1.1 due to De Wilde and Sunyach [159] (see also Valdivia [611, p.64]):

Theorem 7.1.1 (De Wilde–Sunyach) *A Baire K-analytic lcs is a separable Fréchet space.*

Lutzer et al. [424] exhibited a countable space X (having a unique non-isolated point) such that $C_p(X)$ is a separable, metrizable non-complete Baire space that is not K-analytic. In this section we prove Theorem 7.1.2 from [335] extending Theorem 7.1.1.

Theorem 7.1.2 (Kąkol–López-Pellicer) *A Baire tvs F with a relatively countably compact resolution is a separable, metrizable, and complete tvs.*

For the proof we need two additional results from [335].

Proposition 7.1.3 *Every Baire tvs with a bounded resolution is metrizable. Any metrizable tvs has a bounded resolution.*

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Proof Let $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a bounded resolution in E. As usual, for $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ set $C_{n_1,n_2,\ldots,n_k} := \bigcup\{K_\beta : \beta = (m_l), n_j = m_j, j = 1,\ldots,k\}$ and $W_k := C_{n_1,n_2,\ldots,n_k}, k \in \mathbb{N}$. Then for every neighbourhood of zero U in E there exists $k \in \mathbb{N}$ such that

$$W_k \subset 2^k U.$$

Indeed, otherwise there exists a neighbourhood of zero U in E such that for every $k \in \mathbb{N}$ there exists $x_k \in W_k$ such that $2^{-k}x_k \notin U$. Since $x_k \in W_k$ for every $k \in \mathbb{N}$, there exists $\beta_k = (m_n^k)_n \in \mathbb{N}^{\mathbb{N}}$ such that

$$x_k \in K_{\beta_k}, \quad n_j = m_j^k$$

for $j = 1, 2, \ldots, k$. Set $a_n = \max\{m_n^k : k \in \mathbb{N}\}$ for $n \in \mathbb{N}$, and set $\gamma = (a_n)$. Since $\gamma \geq \beta_k$ for every $k \in \mathbb{N}$, then $K_{\beta_k} \subset K_\gamma$. Hence $x_k \in K_\gamma$ for all $k \in \mathbb{N}$. The set K_γ is bounded, so $2^{-k}x_k \to 0$ in E, which provides a contradiction. Since E is a Baire space, and

$$E = \bigcup_{n_1} C_{n_1}, \quad C_{n_1} = \bigcup_{n_2} C_{n_1,n_2}, \ldots,$$

there exist sequences $(n_k) \in \mathbb{N}^{\mathbb{N}}$, $(x_k)_k$ in E, and a sequence $(U_k)_k$ of neighbourhoods of zero in E such that

$$x_k \in \operatorname{int} \overline{W_k}, \quad x_k + U_k \subset \overline{W_k}$$

for all $k \in \mathbb{N}$. Choose arbitrary closed balanced neighbourhoods of zero U and V in E such that $V + V \subset U$. Since there exists $k \in \mathbb{N}$ such that $W_k \subset 2^k V$, we note

$$2^{-k}U_k \subset 2^{-k}\overline{W_k} - 2^{-k}x_k \subset V + V \subset U.$$

This proves that the sequence $(2^{-k}U_k)_k$ forms a countable basis of neighbourhoods of zero in E, so E is metrizable.

Now assume that E is a metrizable tvs. Let $(U_n)_n$ be a countable basis of balanced neighbourhoods of zero for E. For $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ set $K_\alpha := \bigcap_k n_k U_k$. It is easy to see that $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bounded resolution in E. □

It is known that, if F is a Čech-complete space, and E is a completely regular Hausdorff space containing F as a dense subspace, $E \setminus F$ is of first Baire category; see [519, Corollary 13.5]. Making use of Theorem 6.2.4 we note that, if F is a metric space having a compact resolution swallowing compact sets, F is Čech-complete. We provide another applicable result of this type for Baire spaces F admitting a certain resolution.

Proposition 7.1.4 *Let E be a topological space which admits a weaker topology ξ generated by a metric d. Let F be a dense Baire subset of E having a resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ consisting of closed sets in ξ. Then $E \setminus F$ is of first Baire category.*

Proof We claim that

$$\bigcap_k O(C_{n_1,n_2,\ldots,n_k}) \subset F \tag{7.1}$$

for $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$. Indeed, if $z \in \bigcap_k O(C_{n_1,n_2,\ldots,n_k})$, $z \in E$, for every $k \in \mathbb{N}$ there exists

$$x_k \in B(z, k^{-1}) \cap C_{n_1,n_2,\ldots,n_k},$$

where $B(z, k^{-1})$ denotes an open ball in (E, d) with the center at the point z and radius k^{-1}. This implies that the sequence $(x_k)_k$ converges to z in the topology ξ. On the other hand, there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $x_k \in K_\gamma$ for all $k \in \mathbb{N}$. Since K_γ is closed in (F, d), we have $z \in F$. This proves the claim.

Now define the following sets:

$$K_0 := O(F) \setminus \bigcup \{O(C_{n_1}) : n_1 \in \mathbb{N}\},$$

$$K_{n_1,\ldots,n_{k-1}} := O(C_{n_1,\ldots,n_{k-1}}) \setminus \bigcup \{O(C_{n_1,\ldots,n_{k-1},n_k}) : n_k \in \mathbb{N}\}, k \geq 2,$$

and recall that $E = O(F)$.

Applying Proposition 2.1.6 we note that every set of the above type is nowhere dense. Note also

$$E \setminus F \subset K_0 \cup \bigcup K_{n_1,\ldots,n_k}. \tag{7.2}$$

Indeed, to prove this fact, assume that there exists

$$x \notin K_0 \cup \bigcup K_{n_1,\ldots,n_k}.$$

Therefore there exists $n_1 \in \mathbb{N}$ such that $x \in O(C_{n_1})$. Next, we find $n_2 \in \mathbb{N}$ such that $x \in O(C_{n_1,n_2})$. Following this inductive procedure we obtain a sequence $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ such that $x \in O(C_{n_1,n_2,\ldots,n_k})$ for every $k \in \mathbb{N}$. By (7.1) we derive that $x \in F$. This proves the inclusion (7.2). From (7.2) it follows that indeed the space $E \setminus F$ is of first Baire category in E. □

If $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution, the argument is easier: By Corollary 3.2.9 the space F is K-analytic. From [611, Chapt.1, 4.3(19)] we note that F has the Baire property in E. Note also that $E = O(F)$ since all non-void

open sets in E intersect F in a set of second Baire category. Now, from the Baire property of F in E it follows that

$$E \setminus F (= O(F) \setminus F)$$

is of first Baire category in E; see [393, Corollary 2,11.IV] or Proposition 2.1.5.

Now we are ready to prove Theorem 7.1.2.

Proof Since in a tvs relatively countably compact sets are bounded, we apply Proposition 7.1.3 to deduce that F is metrizable. Consequently, metrizable F has a compact resolution, and by Corollary 6.2.5 the space F is analytic, hence separable. Let E be the completion of F. By Proposition 7.1.4 the set $E \setminus F$ is of first Baire category. We prove that $E = F$. If $E \neq F$, then taking $x \in E \setminus F$ we would have $x + F \subset E \setminus F$, and this provides a contradiction since $x + F$ and $E \setminus F$ are of second and first Baire category, respectively. Hence E is separable, metrizable, and complete. □

Applying Proposition 7.1.4, and using a similar argument as in the proof of Theorem 7.1.2 we note the following:

Corollary 7.1.5 *Let E be a metrizable Baire tvs having a complete resolution. Then E is complete.*

The following Corollary 7.1.6 supplements Theorem 7.1.2 and Canela's corresponding result from [114]:

Corollary 7.1.6 *Let X be a paracompact and locally compact space. The following assertions are equivalent:*

(i) $C_p(X)$ *has a bounded resolution.*
(ii) X *is σ-compact.*
(iii) $C_c(X)$ *is a Fréchet space.*

Moreover, if X is metric and locally compact, $C_p(X)$ has a bounded resolution if and only if $C_p(X)$ is analytic.

Proof Since X is paracompact and locally compact, X is a topological direct sum of a disjoint family $\{X_i : i \in I\}$ of locally compact σ-compact spaces. Hence $C_p(X) = \prod_{i \in I} C_p(X_i)$.

(i) $\Rightarrow$ (ii): If $C_p(X)$ has a bounded resolution, the set I is countable. Indeed, otherwise $\prod_{i \in I} C_p(X_i)$ would contain a closed subspace of the type $\mathbb{R}^A$ for some uncountable A. Since $\mathbb{R}^A$ is a Baire space, we apply Theorem 7.1.2 to deduce that $\mathbb{R}^A$ is metrizable. Consequently, the set A is countable. This shows that X is σ-compact.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are obvious. The last part of corollary follows from the previous one, and a known fact stating that a continuous image $C_p(X)$ of a separable Fréchet space $C_c(X)$ is analytic. □

Theorem 7.1.2 fails in general for topological groups. Any non-metrizable compact topological group provides such an example. Nevertheless, Proposition 7.1.4 applies to prove the following variant of Theorem 7.1.2 (see [140, Theorem 5.4]):

Theorem 7.1.7 (Christensen) *A Baire topological group E which is an analytic space is a Polish space.*

Proof Let T be a continuous surjection of $\mathbb{N}^{\mathbb{N}}$ onto E. Let $\mathfrak{T}$ be the set of all $x \in \mathbb{N}^{\mathbb{N}}$ for which there exists a neighbourhood $U(x)$ of x such that $T(U(x))$ is of first Baire category. Note that $\mathfrak{T} \neq \mathbb{N}^{\mathbb{N}}$. Indeed, otherwise (since $\mathbb{N}^{\mathbb{N}}$ is Lindelöf) there exists a sequence $(U(x_n))_n$ of neighbourhoods of the points x_n covering $\mathbb{N}^{\mathbb{N}}$ and such that $T(U(x_n))$ is of first Baire category. Also $\bigcup_n T(U(x_n))$ is of first Baire category. On the other hand,

$$T(\mathbb{N}^{\mathbb{N}}) = E = \bigcup_n T(U(x_n))$$

is the space of second Baire category, a contradiction. This proves that there exists $y \in \mathbb{N}^{\mathbb{N}}$ such that for each neighbourhood $U(y)$ of y the set $T(U(y))$ is of second Baire category. Let $(V_n(y))_n$ be a basis of closed neighbourhoods of y in $\mathbb{N}^{\mathbb{N}}$. Note that

$$\{T(V_n(y))T(V_n(y))^{-1} : n \in \mathbb{N}\}$$

is a basis of neighbourhoods of the unit element e of E. Since $T(V_n(y))$ is of second Baire category and has the Baire property for each $n \in \mathbb{N}$ (note that $T(V_n(y))$ is analytic and therefore has the Baire property), each set $T(V_n(y))T(V_n(y))^{-1}$ is a neighbourhood of e; compare the proof of Proposition 2.1.8. Take neighbourhoods U and V of e in E such that $VV^{-1} \subset U$. By the continuity of T we have

$$T(V_k(y)) \subset VT(y)$$

for some $k \in \mathbb{N}$. Hence

$$T(V_k(y))T(V_k(y))^{-1} \subset VT(y)T(y)^{-1}V^{-1} \subset U.$$

This condition for topological groups yields the metrizability. Since E is Lindelöf, hence separable, there exists a Polish group F which contains E as a dense subset. By Proposition 7.1.4 the set $F \setminus E$ is of first Baire category. Assume that there exists $z \in F \setminus E$. Then $zE \subset F \setminus E$ is also of first Baire category. As E is of second Baire category, the same holds for zE, a contradiction. Hence $E = F$ is a Polish space. □

We complete this section by showing that Proposition 7.1.3 for locally convex spaces can be also deduced from Corollary 3.3.13. Indeed, let E be a Baire lcs

having a bounded resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. We may assume that all bounded sets A_α are absolutely convex. For every $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$ let K_α be the closed absolutely convex hull of A_α in the completion F of E. Let G be the linear span (in F) of $\bigcup\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Fix arbitrary $x \in G$. Then there exist scalars $t_p \in \mathbb{R}$ for $1 \le p \le n$, $\alpha_p \in \mathbb{N}^\mathbb{N}$, and $x_{\alpha_p} \in K_{\alpha_p}$ for $1 \le p \le n$, such that

$$x = \sum_{p=1}^{n} t_p x_{\alpha_p} \in |t| n K_\gamma,$$

where

$$|t| = \max\{|t_p| : p = 1, 2, \dots n\},$$

and $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\alpha_p \le \gamma$ for every $p = 1, 2, \dots n$. Choose $m_1 \in \mathbb{N}$, $\beta = (m_k) \in \mathbb{N}^\mathbb{N}$, such that $n|t| \le m_1$ and $\gamma \le \beta$. Then $x \in m_1 K_\beta$. Therefore

$$G = \bigcup\{n_1 K_\alpha : \alpha = (n_k) \in \mathbb{N}^\mathbb{N}\}.$$

Hence G is a Baire locally convex space having a resolution of Banach discs. By Corollary 3.3.13 the space G is metrizable and complete. Hence E is metrizable.

7.2 Continuous Maps on Spaces with Resolutions

A possible approach to prove a deep Theorem 2.3.1 for separable infinite-dimensional Fréchet spaces might be related with the following problem posed in [335]:

Problem 7.2.1 *Let E be an infinite-dimensional separable Fréchet space. Does there exist on E a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, and a discontinuous linear functional λ on E that is $\lambda|K_\alpha$ continuous for each $\alpha \in \mathbb{N}^\mathbb{N}$?*

We know already that a separable Fréchet space E admits always a compact resolution (even swallowing compact sets). Assume for a moment that Problem 7.2.1 has a positive answer. Let $H := \ker \lambda$ be the (dense) hyperplane of E defined by λ. Note that by the assumption the family $\{K_\alpha \cap H : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution in H. Then, by Theorem 7.1.2 the hyperplane H cannot be a Baire space. This approach motivates Problem 7.2.1.

It turns out that the answer to Problem 7.2.1 has a negative solution, recently proved by Drewnowski [180]. In this section we prove a couple of results (mostly due to Drewnowski [180]). Theorem 7.2.2 extends earlier Corollary 5.3.3, proved by using the closed graph theorem 5.3.1. Recall that an F-space is a metrizable and complete tvs.

Theorem 7.2.2 (Drewnowski) *Let E be an F-space having a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Let $f : E \to F$ be a linear map from E into a tvs F such that each restriction $f|A_\alpha$ is continuous. Then f is continuous on E.*

For the proof we need the following three additional lemmas from [180]:

Lemma 7.2.3 *Let E be a topological space having a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Let $f : E \to F$ be a map from E into a regular topological space F. If each $f|A_\alpha$ is continuous, f is continuous on the closure K_α of each A_α.*

Proof Fix A_α and $x \in K_\alpha$. There exists $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $A_\alpha \subset A_\beta$ and $x \in A_\beta$. Since $f|A_\beta$ is continuous at x, the map $f|(A_\alpha \cup \{x\})$ is also continuous at x. This is just enough to claim that $f|K_\alpha$ is continuous; see [195, Exerc.3.2(A)(b)]. □

Next Lemma 7.2.4 is easy to check; the proof is left to the reader.

Lemma 7.2.4 *Let E and F be topological spaces and $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ a resolution on E. Assume that $f : E \to F$ is a map continuous on each A_α. Then:*

(i) If the resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is compact, $\{f(A_\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution on the range space $f(E)$.
(ii) If the resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is compact (closed), for every closed set $C \subset F$ the family $\{f^{-1}(C) \cap A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact (closed) resolution on the subset $f^{-1}(C)$ of E.

Lemma 7.2.5 *Assume that a linear map $f : E \to F$ is continuous on a closed set $X \subset E$ and on a compact set $K \subset E$. Let A be a compact set of non-zero scalars. Then f is continuous on the closed set $AX + K$.*

Proof Let $(z_t)_{t\in T}$ be a net in $AX + K$ converging to $z \in E$. For each $t \in T$ there exist $x_t \in X$, $a_t \in A$, and $y_t \in K$ such that $z_t = a_t x_t + y_t$. By the compactness of A and K we may assume that $a_t \to a \in A$, $y_t \to y \in K$. Then $x_t \to a^{-1}(z - y) := x \in X$. Hence $z \in AX + K$. Finally, we note that

$$f(z_t) = a_t f(x_t) + f(y_t) \to af(x) + f(y) = f(z).$$

Hence f is continuous at the point z. □

Now we are ready to prove Theorem 7.2.2.

Proof Applying Lemma 7.2.3 we may assume that each set A_α is closed in E. Observe that the kernel $N := \{x \in E : f(x) = 0\}$ is a closed subspace of E. Indeed, without loss of generality we may assume that N is dense in E. We check that $N = E$ by showing that $f(E) \subset V_0$ for each closed balanced neighbourhood of zero V_0 in F.

Let $(V_n)_n$ be a sequence of closed balanced neighbourhoods of zero in F such that

$$V_{n+1} + V_{n+1} \subset \operatorname{int} V_n$$

for each $n \in \mathbb{N} \cup \{0\}$. Set $U_n := f^{-1}(V_n)$. Clearly $U = \bigcap_{n\geq 0} U_n$ is a linear subspace of E. Since $N \subset U$, U is dense, and from Lemma 7.2.4 (ii) it follows that each U_n admits a closed resolution. Clearly $f^{-1}(\text{int } V_n)$ is a dense algebraically open subset of E. Then, for every non-empty open subset G of E the intersection $G \cap f^{-1}(\text{int } V_n)$ is of second Baire category in E. Therefore $f^{-1}(\text{int } V_n)$ and U_n are Baire and dense subspaces of E. Applying Proposition 7.1.4 we note that each $E \setminus U_n$ is of first Baire category. This shows that

$$E \setminus U = E \setminus \bigcap_n U_n = \bigcup_n E \setminus U_n$$

is of first Baire category. If $y \in E \setminus U$, then $y + U \subset E \setminus U$, and this shows that the Fréchet space $E = (E \setminus U) \cup U$ is of first Baire category, a contradiction. Hence $f(E) \subset V_0$. This proves that the kernel of f is closed.

Next, we prove that the map f is continuous. Assume first that F is an F-space, and define the linear map

$$T : E \times F \to F, \quad T(x, y) := y - f(x).$$

Consider the resolution

$$\{A_\alpha \times F : \alpha \in \mathbb{N}^{\mathbb{N}}\}$$

on the F-space $E \times F$. Since $\ker(T) = \text{Graph}(f)$ is a closed subspace in $E \times F$ by the last observation, we use the closed graph theorem between F-spaces [1, Theorem 8.6] to deduce that T is continuous. Hence f is continuous as well.

Finally, every tvs is a subspace of a product $\prod_{i\in I} F_i$ of F-spaces; see [1, Theorem 3.5]. Using the previous case for f composed with each of the canonical projection $\prod_{i\in I} F_i \to F_i$ which is clearly continuous for each $i \in I$, we deduce that f is continuous. This completes the proof. □

Theorem 7.2.2 applies to prove the following corollaries (see [180]):

Corollary 7.2.6 *Let E be an F-space, and let E_0 be a subset of E admitting a closed resolution. Let $(E_j)_{1\leq j\leq n}$ be a finite family of subsets of E admitting a compact resolution. If $E = E_0 + \sum_{1\leq j\leq n} E_j$, and a linear map $f : E \to F$ to a tvs F is continuous on each set E_j for $0 \leq j \leq n$, then f is continuous on E.*

Proof For each $0 \leq j \leq n$ let $\{K_\alpha^j : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a resolution on the set E_j. Set

$$K_\alpha := K_\alpha^0 + K_\alpha^1 + \cdots + K_\alpha^n.$$

Then $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a closed resolution on E, and by Lemma 7.2.5 the map f is continuous on each set K_α. Now Theorem 7.2.2 applies for the continuity of f. □

Corollary 7.2.7 *Let E be an F-space equal to an algebraic direct sum of vector subspaces $(E_j)_{0\le j\le n}$. Assume that E_0 admits a closed resolution while the remaining subspaces admit a compact resolution. Then the associated projections $p_j : E \to E_j$ are continuous. Hence all subspaces E_j are closed in E.*

Corollary 7.2.8 *If a linear subspace F of an F-space E has countable codimension in E and admits a closed resolution, F is closed and has finite codimension.*

Proof Let G be a countable-dimensional vector subspace of E such that $E = E \oplus G$ (algebraically). Clearly G admits a compact resolution as an analytic space. We apply Corollary 7.2.7 to deduce that both the spaces F and G are closed in E. By Corollary 2.2.6 we conclude that G is finite-dimensional. □

The proofs of the next two simple corollaries are left to the reader.

Corollary 7.2.9 *Let E be a separable F-space having a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Then for every (sequentially) complete tvs F, the space $L(E, F)$ endowed with the topology of uniform convergence on sets K_α is (sequentially) complete.*

Corollary 7.2.10 *Let E be an F-space having a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Then the original topology of E coincides with the finest vector topology ξ for which all inclusions $j_\alpha : A_\alpha \to (E, \xi)$ are continuous.*

Corollary 7.2.10 shows that for a separable F-space E does not exist a strictly finer vector topology ξ such that (E, ξ) has a compact resolution. This observation follows also from Theorem 5.3.1. A simple application of Theorem 7.2.2 says that a linear map $f : E \to F$ from an F-space E to tvs F which is continuous on an absorbing subset $A \subset E$ is continuous on the whole space E. Indeed, set $A_n := nA$, $n \in \mathbb{N}$, and then $A_\alpha := A_{n_1}$ for $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$.

We complete this section with the following observation (see [180]):

Proposition 7.2.11 *Let E be a metrizable lcs covered by an increasing sequence $(K_n)_n$ of compact sets. Assume that every linear functional on E that is continuous on each set K_n is continuous. Then E is finite-dimensional.*

Proof By E'_K we denote the set of all linear functionals f on E which are continuous on each set K_n. Since $E' = E'_K$, the space E' endowed with the topology ξ_K of the uniform convergence on the sets $(K_n)_n$ is a Fréchet space. Clearly $\sigma(E', E) \le \xi_K$. The space E is metrizable; hence there exists a countable basis $(U_n)_n$ of closed absolutely convex neighbourhoods of zero of E. Since each polar U_n° of U_n is $\sigma(E', E)$-compact, it is closed in ξ_K. Since $E' = \bigcup_n U_n^\circ$, the Baire category theorem applies to get $m \in \mathbb{N}$ such that $tK_m^\circ \subset U_n^\circ$ for some $t > 0$ and $m \in \mathbb{N}$. This implies that $U_n \subset t^{-1}\overline{\mathrm{ac}}K_m$, which implies that E has a precompact neighbourhood of zero. Hence E is finite-dimensional; see, for example, [322, Theorem 3.5.6]. □

Chapter 8
A Three-Space Property for Analytic Spaces

Abstract In this chapter, we show that analyticity is not a three-space property. We prove, however, that a metrizable tvs E is analytic if it contains a complete locally convex analytic subspace F such that the quotient E/F is analytic. We reprove (using Corson's example) that the Lindelöf property is not a three-space property.

8.1 An Example of Corson

By a *three-space property* (for topological vector spaces) we understand the following [519]: Suppose E is a tvs and $F \subset E$ is a closed vector subspace of E such that F and the quotient E/F have certain property $\mathcal{P}$. Does E have property $\mathcal{P}$? Several important topological properties of tvs are a three-space property like the separability, local boundedness, completeness, barrelledness, etc.; see [520] and [519] and Lemmas 6.8.4, 6.8.5, 6.8.6, 6.8.7.

In this section we provide an interesting example, originally due to Corson [147, Example 2], showing that the Lindelöf property is not a three-space property. We shall need several additional partial results of own interest; see also [198, Theorem 12.43].

Let us briefly recall the main idea related with his example. Let E be the subspace of the space $\ell^\infty[0,1]$ formed by all bounded real-valued functions on $[0,1]$ that are right continuous and have finite left limits. The space $(E, \sigma(E, E'))$ is not normal. Hence E is not weakly Lindelöf. The space E contains a subspace isomorphic with $C[0,1]$ and the quotient $E/C[0,1]$ is isomorphic to the (WCG) Banach space $c_0[0,1]$, so the weak topology of $c_0[0,1]$ is K-analytic.

Now we start with some facts in order to present Corson's example mentioned above. We need the following simple fact (see [195, Problem 2.7.12(b)]):

Lemma 8.1.1 *Let I be a set. Let U and V be disjoint open subset in $\mathbb{K}^I$. Then there exists a countable set $J \subset I$ such that $p_J(U) \cap p_J(V) = \emptyset$, where p_J is the projection of $\mathbb{K}^I$ onto $\mathbb{K}^J$.*

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We need also the following lemma due to Corson and Isbell (see [148, Theorem 2.1]):

Lemma 8.1.2 *Let I be a set. Let $W \subset \mathbb{K}^I$ be a dense subset. For every continuous function $f : W \to \mathbb{K}$ there exist a countable $J \subset I$ and a continuous function $g : p_J(W) \to \mathbb{K}$ such that $f = g \circ p_J$.*

Proof For each rational number r set

$$U_r := f^{-1}(r, \infty), \;\; V_r := f^{-1}(-\infty, r).$$

Choose open disjoint sets U_r^1 and V_r^1 in $\mathbb{K}^I$ such that $U_r = U_r^1 \cap W$ and $V_r = V_r^1 \cap W$. Lemma 8.1.1 provides countable $J_r \subset I$ such that

$$p_{J_r}(U_r^1) \cap p_{J_r}(V_r^1) = \emptyset.$$

Set $I_0 := \bigcup_r J_r$. Note that, if $x = (x_i)_{i \in I} \in W$, $y = (y_i)_{i \in I} \in W$, and $x_i = y_i$ for $i \in I_0$, then $f(x) = f(y)$. □

This applies to get the following next interesting [147, Example 2]:

Lemma 8.1.3 *Let E be a lcs. Let $\mathfrak{F}$ be the set of all continuous real-valued function on $(E, \sigma(E, E'))$. Then $|\mathfrak{F}| \leq 2^{\aleph_0 \dim\,(E')}$.*

Proof Clearly $(E'^*, \sigma(E'^*, E')$ is linearly homeomorphic with $\mathbb{K}^I$, where I is a set with $|I| = \dim E'$ and E'^* denotes the algebraic dual of E. Note that E is identified with a dense subspace of E'^*. By Lemma 8.1.2 we have $|\mathfrak{F}| \leq 2^{\aleph_0 \dim\,(E')\,\mathfrak{m}}$, where $\mathfrak{m} = 2^{\aleph_0}$ denotes the cardinality of continuous real-valued functions on separable $\mathbb{R}^{\aleph_0}$. □

By $c_0(I)$ we denote the Banach space (for $|I| = 2^{\aleph_0}$) of functions $f : I \to \mathbb{K}$ such that for each $\epsilon > 0$ there exists a finite set $J \subset I$ such that $|f(i)| < \epsilon$ for all $i \in I \setminus J$. Note that $c_0(I)$ is a (WCG) Banach space, i.e. $c_0(I)$ contains a weakly compact set whose linear span is dense in $c_0(I)$. Indeed, let $A := (e_t)_{t \in I}$ be the family of the standard unit vectors in c_0. Observe that the set $\{e_t \cup \{0\} : t \in I\}$ is weakly compact in c_0. Indeed, each sequence $(e_n)_n$ in A of distinct elements converges pointwise to zero; hence $(e_n)_n$ converges to zero weakly in c_0. This means, by applying the classical Eberlein–Šmulian theorem, that A is weakly compact in c_0. Since $c_0(I)$ is a (WCG) Banach space, we refer the reader to Theorem 12.4.6 (and Proposition 3.1.5; see also [147, Theorem 1]).

Lemma 8.1.4 *The space $c_0(I)$ is weakly K-analytic, hence weakly Lindelöf.*

Now we are ready to prove the following theorem due to Corson [147, Example 2] (see also [172]):

Theorem 8.1.5 (Corson) *There exists a Banach space E such that E is not normal in the weak topology and contains a closed subspace F such that F and E/F are weakly Lindelöf.*

Proof Since every regular Lindelöf space is normal by Lemma 6.1.3, the space E is not weakly Lindelöf. Let E be the subspace of $\ell^\infty[0,1]$ formed by all bounded real-valued functions on $[0,1]$ that are right continuous and have finite left limits. Endow E with the norm $\|x\| := \sup_{t\in[0,1]} |x(t)|$ for all $x \in E$. Let F be the subspace of E of continuous functions; therefore F is isometric to the space $C[0,1]$. Clearly F in the weak topology is analytic, hence Lindelöf. We show that the quotient space E/F is linearly homeomorphic to the space $c_0[0,1]$. Set $I := [0,1]$, and define continuous linear functionals on E by the formulas

$$g_t(x) := x(t),\ \ h_t(x) := \lim_{u\to t^-} x(u),$$

for each $t \in I$. Set $f_t := g_t - h_t$. We show that the map T defined by $T(x) := (f_t(x))_{t\in I}$ is a continuous linear surjection from E onto $c_0(I)$. Note that T is linear, $T(E) \subset c_0(I)$ and $\ker T = F$. Clearly

$$|f_t(x)| = |x(t) - \lim_{u\to t^-} x(u)| \le 2\|x\|.$$

This yields $\|T(x)\| \le 2\|x\|$, so T is continuous. Now we show that T is a surjection. Fix arbitrary $(y_t)_{t\in I}$ in $c_0(I)$. We construct a sequence $(x_n)_{n\in\mathbb{N}\cup\{0,-1\}}$ in E which uniformly converges. Define finite sets for $n \ge 1$

$$I_0 := \{t \in I : 1 \le |y_t|\},\ \ I_n := \{t \in I : 2^{-n} \le |y_t| < 2^{1-n}\}.$$

Define $x_{-1}(t) = 0$ for each $t \in I$. Next, we construct the sequence $(x_n)_n$ as follows:

$$x_n(t) = x_{n-1}(t),\ \ \lim_{u\to t^-} x_n(u) = \lim_{u\to t^-} x_{n-1}(u),\ \ t \in \bigcup_{k<n} I_k,$$

$$x_n(t) = x_{n-1}(t) + 2^{-1}y_t,\ \lim_{u\to t^-} x_n(u) = x_{n-1}(t) - 2^{-1}y_t,\ t \in I_n,$$

and

$$|x_n(t) - x_{n-1}(t)| \le 2^{1-n}.$$

Note that for $m > n$

$$|x_m(t) - x_n(t)| \le \sum_{k=n}^{m-1} 2^{-k} < 2^{-n+1}.$$

This shows that $(x_n)_n$ is a Cauchy sequence in E, so its limit $x \in E$ satisfies $T(x) = (y_t)_{t\in I}$. By the closed graph theorem we deduce that E/F is linearly homeomorphic to the space $c_0(I)$.

Now we prove that E is not normal in its weak topology. Set

$$x_u(t) := \begin{cases} 0 & t < u \\ 1 & t \geq u. \end{cases}$$

Note that the set $A := \{x_t : t \in I\}$ is discrete in the weak topology $\sigma(E, E')$ of E. Indeed, this follows from the equalities:

$$(g_t + h_t)(x_u) := \begin{cases} 0 & u > t \\ 1 & u = t \\ 0 & u < t. \end{cases}$$

Note also that A is closed. On the other hand, it is known that on a closed discrete set of cardinality $\mathfrak{c} := 2^{\aleph_0}$ there exist $2^{\mathfrak{c}}$ continuous real-valued functions. Assume that $(E, \sigma(E, E'))$ is normal. Every such function from A can be extended by Tietze–Urysohn's theorem to the space $(E, \sigma(E, E'))$. To get a contradiction, it is enough to show that there exist only $2^{\aleph_0}$ continuous real-valued functions on the space. This is the case, if we realize that $\dim E' \leq 2^{\aleph_0}$, by using Lemma 8.1.3. The last inequality follows from the well-known inequalities $\dim F' \leq 2^{\aleph_0}$ and $\dim (E/F)' \leq 2^{\aleph_0}$. □

We shall come back to Corson's example in the context of separable complementation property; see Theorem 23.3.1.

8.2 A Positive Result and a Counterexample

Corson's example shows also that the K-analyticity is not a three-space property, since E is not weakly Lindelöf and the spaces $C[0, 1]$ and $E/C[0, 1]$ (as isomorphic to a (WCG) Banach space) are weakly K-analytic. On the other hand, the space $c_0[0, 1]$ is not separable, so this example does not cover the case for $\mathcal{P}$=*analytic*.

If E is a tvs containing a vector subspace F such that F and E/F are separable Fréchet spaces, E is a separable Fréchet space. This is clear, since the separability, metrizability, and completeness are a three-space property; see Lemmas 6.8.4, 6.8.5, 6.8.7. Let us consider two other cases: Let F and E/F be analytic, and let E be metrizable. Assume that F (or E/F) is complete. Is E analytic?

In order to provide a positive result concerning a three-space property for the analyticity, we need additional facts; see [397, Theorem 1], [411], and [412].

Proposition 8.2.1 (Labuda–Lipecki) *Assume that $E := (E, \|.\|)$ is a separable infinite-dimensional Banach space. Then there exists a linearly independent sequence $(y_n)_n$ in E such that: $\sum_n \|y_n\|$ converges; $\operatorname{span}\{y_n : n \in \mathbb{N}\}$ is dense in E; if $(t_n) \in \ell^\infty$ and $\sum_n t_n y_n = 0$, then $t_n = 0$ for all $n \in \mathbb{N}$.*

Proof First observe that, if X is a proper subspace of E, the space $E \setminus X$ is dense in E. Indeed, fix $\epsilon > 0$ and take $x \in E \setminus X$ with $\|x\| < \epsilon$. Then $\|(x + y) - y\| < \epsilon$ and $x + y \in E \setminus X$ provided $y \in X$. This will be used to see that E contains a dense linearly independent sequence $(x_n)_n$. Indeed, let $\{z_n : n \in \mathbb{N}\}$ be a dense subset of E. By the above remark $(E \setminus X) \cap \{z_n : n \in \mathbb{N}\}$ is dense in E if X is any finite-dimensional subspace of E. Then, we can construct by a simple induction a sequence $(z_{n_k})_k$ with

$$z_{n_{k+1}} \notin \text{span}\{z_{n_1}, z_{n_2}, \ldots, z_{n_k}\},$$

and such that $\|z_{n_k} - z_k\| \leq k^{-1}$ for all $k \in \mathbb{N}$. The set $\{z_{n_k} : k \in \mathbb{N}\}$ is dense in E (by the last inequality) and linearly independent. Set $x_k := z_{n_k}$ for all $k \in \mathbb{N}$. Take numbers $\beta_n > 0$ such that $\|\beta x_n\| \leq 2^{-n}$ for $|\beta| \leq \beta_n$ and all $n \in \mathbb{N}$. Then, the series $\sum_n \beta_n x_n$ is bounded multiplier convergent, i.e. for each bounded sequence $(t_n)_n \in \ell^\infty$ the series $\sum_n t_n \beta_n x_n$ converges in E. For each $n \in \mathbb{N}$ set $y_n := \beta_n x_n$. We show that $(y_n)_n$ contains a subsequence $(y_{n_k})_k$ as desired. Put

$$K_n := \{\sum_{i=0}^{n} s_i y_i : |s_i| \leq 1,\ i \leq n-1,\ 2^{-1} \leq |s_n| \leq 1\},$$

where $n \in \mathbb{N}$ and $y_0 := 0$. Note that the sets K_n are compact and $0 \notin K_n$. Since $E \setminus K_n$ is an open neighbourhood of zero for each $n \in \mathbb{N}$, there exists $m > n$ such that

$$\{\sum_{i=m}^{\infty} t_i y_i : |t_i| \leq 1, i \geq m\} \subset E \setminus K_n.$$

Hence, we can choose a sequence $n_1 < n_2 < \ldots$ such that

$$\sum_{i=k+1}^{\infty} t_i y_{n_i} \in E \setminus K_{n_k},$$

if $|t_k| \leq 1$ and $k \in \mathbb{N}$. Moreover, we select the sequence $y_{n_k} = \beta_{n_k} x_{n_k}$, $k \in \mathbb{N}$, to have

$$\|x_{n_k} - x_k\| \leq k^{-1}$$

for all $k \in \mathbb{N}$. Since $(x_k)_k$ was dense in E, $(x_{n_k})_k$ is also dense in E. Therefore the linear span of $(y_{n_k})_k$ is dense in E. Assume that $\|(t_i)\|_\infty = 1$ in ℓ^∞. Then there exists $j \geq 1$ such that $|t_j| \geq 2^{-1}$. Finally, having in mind the sequence $(n_k)_k$, we deduce that $\sum_i t_i y_{n_i}$ is different from zero. □

Now we are ready to prove the following theorem [339, Theorem 12]:

Theorem 8.2.2 *(i) Let E be a metrizable tvs containing a closed subspace F such that F and E/F are analytic. If F is complete and locally convex, then E is analytic.*

(ii) There is a separable normed space E which is not analytic and contains a closed analytic subspace F such that E/F is a separable Banach space.

Proof Proof of (i) is due to Drewnowski (private communication). Let G be the completion of E. Let $Q : G \to G/F$ be the quotient map. By a result of Michael [444] (see also [79, Proposition 7.1], or [79, Corollary 7.1] for Fréchet spaces), there is a continuous map $g : G/F \to G$ such that $Q \circ g$ is the identity map on G/F, i.e.

$$g(x + F) \in x + F$$

for each $x \in G$. Since the quotient $E/F \subset G/F$ is analytic, the quotient space E/F admits a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Next, assume that $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution on F. We show that the compact sets

$$M_\alpha := g(K_\alpha) + A_\alpha$$

form a compact resolution on E for $\alpha \in \mathbb{N}^\mathbb{N}$. Indeed, first observe that $g(K_\alpha) \subset E$, so then, each compact set M_α is contained in E. Fix $x \in E$. Since $g(x+F) \in x+F$, there exists $y \in F$ such that $g(x + F) + y = x$. Then $x + F \subset K_\alpha$ and $y \in A_\alpha$ for some $\alpha \in \mathbb{N}^\mathbb{N}$. This shows that $x \in M_\alpha$. Consequently, $E = \bigcup\{M_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. We proved that the space E (metrizable and separable) has a compact resolution. Now we apply Corollary 6.2.5 to deduce that E is analytic.

Now we prove (ii): Fix an infinite-dimensional separable Banach space $E := (E, \|.\|)$. By Proposition 8.2.1, we obtain a sequence $(y_n)_n$ in E such that $\sum \|y_n\| < \infty$, $\text{span}\{y_n : n \in \mathbb{N}\}$ is dense in E, and if $(t_n) \in \ell^\infty$ with $\sum_n t_n y_n = 0$, then $(t_n) = 0$. Define a compact injective map

$$T : \ell^1 \to E, \quad T(x) := \sum_n x_n y_n,$$

where $x = (x_n) \in \ell^1$. Clearly the image $F := T(\ell^1)$ is a dense subspace of E, and since the map T is compact, F is a proper subspace of E.

On the other hand, F is not barrelled. Indeed, assume that F is a barrelled subspace of E. By Propositions 2.4.1 and 2.4.2 F is Baire-like.

Since T is an injective continuous map, F admits a strictly finer topology γ such that (F, γ) is linearly homeomorphic with the Banach space ℓ^1. The identity map from Baire-like F onto (F, γ) has closed graph, so by Theorem 2.4.15 this map is continuous. Hence γ equals the original topology of F, a contradiction.

Note that $\dim(E/F) = 2^{\aleph_0}$. Indeed, since $\dim E = 2^{\aleph_0}$, we apply Theorem 2.4.11 to get the claim. Let τ be a normed topology defined by the norm $\|.\|$, and let $q : E \to E/F$ be the quotient map. Since the quotient topology of E/F is

trivial and $\dim \ell^1 = \dim E/F$, the space E/F admits a stronger separable Banach topology α such that $(E/F, \alpha)$ is isomorphic to ℓ^1. Therefore the assumptions of Lemma 6.8.8 are satisfied. Hence there exists on E a coarsest vector topology ξ such that $\tau < \xi$, $\xi/F = \alpha$, and $\xi|F = \tau|F$.

Note that the sets $U \cap q^{-1}(V)$, where U and V run over τ- and α-neighbourhoods of zero, respectively, form a basis of neighbourhoods of zero for ξ. Since the separability and the property of being a normed space are a three-space property (see Propositions 6.8.4 and 6.8.5), the topology ξ is normed and separable.

Finally, since, by Theorem 5.3.1, every linear map with closed graph from a Banach space into an analytic space is continuous, we deduce that the space (E, ξ) is not analytic. Its closed subspace F is analytic, and the quotient space $(E/F, \alpha)$ is isomorphic to ℓ^1. Hence α is the topology of a separable Banach space. □

Applying the argument used in the proof of Theorem 8.2.2, we note the following result:

Proposition 8.2.3 *Let Y be a (WCG) Banach space with $\dim Y = 2^{\aleph_0}$; for example, let $Y := c_0[0, 1]$. Then there exists a lcs E such that E is not a Lindelöf Σ-space, E contains a closed subspace F which is a continuous image of ℓ^1, and E/F is linearly homeomorphic to the K-analytic space Y endowed with the weak topology. Consequently, the Lindelöf Σ property is not a three-space property.*

Proof Let (E, ξ) be an infinite-dimensional separable Banach space, and let $T : \ell^1 \to E$ be an injective compact map such that $T(\ell^1)$ is a proper dense subspace of E; see the proof of Theorem 8.2.2. Note that a similar argument that was used above yields that $\dim (E/F) = 2^{\aleph_0}$, where $F = T(\ell^1)$. Then the quotient space E/F, whose topology ξ/F is trivial, admits a locally convex non-separable K-analytic topology γ such that $\xi \leq \gamma$, and $(E/F, \gamma)$ is linearly homeomorphic to $(Y, \sigma(Y, Y'))$. By Lemma 8.1.4 the space $(Y, \sigma(Y, Y'))$ is K-analytic.

Applying the procedure as in the proof of Theorem 8.2.2 we obtain a locally convex topology θ on E such that

$$\xi \leq \theta, \ \ \xi|F = \theta|F, \ \ \theta/F = \gamma.$$

Since $(E/F, \gamma)$ is non-separable, (E, θ) is non-separable.

Finally, note that $E = (E, \theta)$ is not a Lindelöf Σ-space. Indeed, assume that (E, θ) is a Lindelöf Σ-space. Since ξ is a separable, metrizable topology and $\xi \leq \theta$, by Lemma 3.1.13 we conclude that (E, θ) is separable, a contradiction. □

We complete this section with the following:

Problem 8.2.4 *Is a metrizable tvs E analytic if E contains a complete analytic vector subspace F such that E/F is analytic?*

Also the following question seems to be still open:

Problem 8.2.5 *Does there exist a weakly analytic (DF)-space that is not analytic?*

Chapter 9
K-Analytic and Analytic Spaces $C_p(X)$

Abstract This chapter deals with K-analytic and analytic spaces $C_p(X)$. Some results due to Talagrand, Tkachuk, Velichko, and Canela are presented. A remarkable theorem of Christensen stating that a metrizable and separable space X is σ-compact if and only if $C_p(X)$ is analytic is proved. We show that the analyticity of $C_p(X)$ for any X implies that X is σ-compact (Calbrix's theorem). We show that $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$ if and only if $C_p(X)$ admits a bounded resolution. We also gather several equivalent conditions for spaces $C_p(X)$ to be Lindelöf spaces over locally compact groups X.

9.1 A Theorem of Talagrand for Spaces $C_p(X)$

We recall a couple of facts about Eberlein compact sets. Let X be a (Hausdorff) compact space. Then

$$X \hookrightarrow I^{\Gamma}, \tag{9.1}$$

i.e. X is embedded into a cube I^{Γ}, where $I = [-1, 1]$, and Γ denotes a set of continuous functions on X. A compact space X is called *Eberlein compact* [629], if this embedding can be chosen to take values in the Banach space $c_0(\Gamma)$ endowed with the pointwise convergence topology τ_p; see Theorem 24.17.1.

The following result, see [32, Proposition IV.1.6], and [198], [24], characterizes Eberlein compact spaces. More results about Eberlein compact sets will be discussed in Chapter 24. Recall that a topological space X is called *σ-compact* if X is covered by a sequence of compact sets.

Theorem 9.1.1 *The following assertions are equivalent for compact X.*

(i) *X is Eberlein compact.*
(ii) *There exists a compact space Y such that X is homeomorphic to a subspace of $C_p(Y)$.*
(iii) *There exists a compact (resp. σ-compact) space $Z \subset C_p(X)$ separating points of X.*

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(iv) *X is homeomorphic to a weakly compact subset in $c_0(\Gamma)$.*
(v) *X is homeomorphic to a weakly compact subset of a Banach space.*

For the proof we refer the reader to Section 24.17 or to the books [32], [198]. In [477, Corollary 2.5] Okunev supplemented Theorem 12.3.5 by showing that:

(o) If X is a σ-compact space, and there exists a compact space Y such that X is homeomorphic to a subspace of $C_p(Y)$ and the space $C_p(X)$ is K-analytic.

This fact will be used below. The following result is due to Arkhangel'skii [27] and Talagrand [579].

Proposition 9.1.2 *(i) A compact space X is Eberlein compact if and only if $C_p(X)$ has a dense σ-compact subspace.*
(ii) If X is Eberlein compact, then $C_p(X)$ is an intersection of some countable family of σ-compact subspaces of $\mathbb{R}^X$.

A compact space X is called a *Corson compact* space if the embedding (9.1) can be chosen to take values in the $\Sigma(\Gamma)$ space, where

$$\Sigma(\Gamma) := \{x \in I^\Gamma : |\{t \in I : x(t) \neq 0\}| \leq \aleph_0\}.$$

Clearly every compact metric space is Eberlein compact. The one-point compactification of an uncountable discrete space provides a concrete Fréchet–Urysohn non-metrizable Eberlein compact space; see [32, III.3.3].

A compact space X for which $C_p(X)$ is K-analytic is called *Talagrand compact*. A compact space X is *Gul'ko compact*, if $C_p(X)$ is a Lindelöf Σ-space. Note that Eberlein compact $\Rightarrow$ Talagrand compact $\Rightarrow$ Gul'ko compact $\Rightarrow$ Corson compact, and none of the reverse implication holds in general. In [441, Theorems 3.1, 3.2, 3.3] Mercourakis characterized Talagrand (Gul'ko) compact spaces; see also Sokolov [571]. More facts concerning such compacts sets will be discussed in Chap. 24.

Talagrand in his remarkable paper [579] proved the following:

Theorem 9.1.3 (Talagrand) *(i) If X is compact, then $C_p(X)$ is K-analytic if and only if $C_p(X)$ has a compact resolution if and only if $C(X)$ is weakly K-analytic.*

(ii) If X is compact and $C_p(X)$ contains a K-analytic subspace separating the points of X, then $C_p(X)$ is K-analytic.

Proof We prove only a part of (i). The remaining parts will be proved below in more general cases. Assume that $C_p(X)$ is K-analytic. Then $C_p(X)$ admits a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Let B be the closed unit ball in $C(X)$. Note that $\{K_\alpha \cap B : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution in the weak topology σ of $C(X)$. Indeed, since $C_p(X)$ is angelic by Theorem 4.3.3, it is enough to show that every sequence $(f_n)_n$ in B which converges in $C_p(X)$ converges in the weak topology of $C(X)$. This is the case by using Lebesgue's classical dominating theorem; see [322]. Since B in the topology σ is angelic, we apply Corollary 3.2.9 to get the K-analyticity of $(B, \sigma|B)$. Clearly $C(X) = \bigcup_n nB$, so the space $C(X)$ is weakly K-analytic. □

In [592] Tkachuk extended the first part of Theorem 9.1.3 (i) to any completely regular Hausdorff space X, and Canela [115] extended the other part of (i) for locally compact paracompact spaces X.

Theorem 9.1.4 (Tkachuk) *For a completely regular Hausdorff space X the space $C_p(X)$ is K-analytic if and only if $C_p(X)$ has a compact resolution.*

We extend Theorems 9.1.3 and 9.1.4. Recall that a (topological) space X is called *hemicompact*, if X is covered by a sequence $(K_n)_n$ of compact subsets such that every compact subset of X is contained in some set K_m. A space X is called a *k_R-space* if every real-valued map defined on X that is continuous on each compact subset of X is continuous.

Theorem 9.1.5 *Let X be a hemicompact k_R-space. If $C_p(X)$ contains a subset S having a compact resolution and S separates points of X, the space $C_p(X)$ has a compact resolution. Hence $C_p(X)$ is K-analytic.*

Proof Let Y be the algebra generated by S and the constant functions. Set

$$B = \{x \in Y : |x(t)| \leq 1, t \in X\}.$$

First assume that X is compact. Define the map $\varphi : Y \times B^{\mathbb{N}} \to C_p(X)$ by

$$\varphi\left(x, (x_n)_n\right) = x + \sum_{n=1}^{\infty} 2^{-n} x_n.$$

Since for each $t \in X$, $p \in \mathbb{N}$ one has

$$\left| \sum_{n=p+1}^{\infty} 2^{-n} x_n(t) \right| \leq 2^{-p},$$

the map φ is continuous. By Proposition 3.2.4 the sets Y and B have a compact resolution, so $\varphi\left(Y \times B^{\mathbb{N}}\right)$ has a compact resolution as well. By the Stone–Weierstrass theorem Y is uniformly dense in $C_p(X)$. Hence

$$\varphi\left(Y \times B^{\mathbb{N}}\right) = C_p(X).$$

This implies that $C_p(X)$ has a compact resolution. Now assume that X is a hemicompact k_R-space space. Let $(K_n)_n$ be a fundamental (increasing) sequence of compact subsets of X. Note that for any $n \in \mathbb{N}$ the set $S_n = \{f | K_n : f \in S\}$ has a compact resolution, belongs to $C_p(K_n)$, and separates points of K_n. The first part applies to claim that each $C_p(K_n)$ has a compact resolution. Hence $\prod_n^{\infty} C_p(K_n)$ has the same property. On the other hand, since X is a hemicompact k_R-space, the space $C_p(X)$ is a closed subspace of $\prod_n^{\infty} C_p(K_n)$, so $C_p(X)$ has a compact resolution again by Proposition 3.2.4. □

The assumptions in Theorem 9.1.5 are essential. If X is not hemicompact, the result fails. This follows from the following:

Example 9.1.6 Let X be a discrete space of cardinality $2^{\aleph_0}$. Then $C_p(X) = \mathbb{R}^X$ is not K-analytic and has a countable dense subset. Also, there exists a hemicompact space X that is not a k_R-space and $C_p(X)$ is not K-analytic.

The first claim is clear by Theorem 7.1.1. For the other one, set $X = \mathbb{N} \cup \{\xi\}$ for $\xi \in \beta\mathbb{N} \setminus \mathbb{N}$. Since every compact subset of X is finite, the space X is hemicompact. By Lutzer and McCoy, see [424], the space $C_p(X)$ is a Baire space. Moreover, $C_p(X)$ is metrizable and not complete (since X is not discrete; therefore X is not a k_R-space). The space $C_p(X)$ is not K-analytic; otherwise $C_p(X)$ would be a separable metrizable and complete space by Theorem 5.3.1 or Theorem 7.1.1.

By a σ-product of a family $\{X_t : t \in T\}$ of topological spaces with a basis at the point $b \in \prod_{t\in T} X_t$ we mean the subspace σ_b of $\prod_{t\in T} X_t$ defined by

$$\sigma_b := \{x = (x_t) \in \prod_{t\in T} X_t : |t : x_t \neq b_t| < \infty\}. \tag{9.2}$$

If in the above definition *finite* is replaced by *countable*, σ_b is called a Σ-product at point b and is denoted by Σ_b. Recall, that according to Noble [470], every Σ-product is Fréchet–Urysohn provided each space X_t is first-countable. Clearly, if each X_t is a vector space, Σ_0 is a vector subspace of $\prod_{t\in T} X_t$.

We complete this section with the following result due to Okunev [477, Theorem 2.6].

Theorem 9.1.7 (Okunev) *Let X be a σ-product of a family $\{X_t : t \in T\}$ of Eberlein compact spaces. Then $C_p(X)$ is K-analytic.*

Proof It is easy to see that the σ-product of compact spaces is a σ-compact space. Since every space X_t is Eberlein compact, by Theorem 9.1.1 there exists a compact space Y_t such that X_t is homeomorphic to a subspace of $C_p(Y_t)$. We assume that the t-th coordinate of the base point of the σ-product coincides with the zero function on T_t. Let $Y := \{\infty\} \cup \bigcup_{t\in T} Y_t$ be the one-point compactification of the direct sum $\bigoplus_{t\in T} Y_t$. Then the map $j : X \to C_p(Y)$ defined by $j(x)(y) := x_t(y)$, $y \in Y_t$, $j(x)(\infty) := 0$ for all $x \in X$, is a homeomorphism onto the range. Now, applying Okunev's result (o) mentioned above, we note that $C_p(X)$ is K-analytic as claimed. □

9.2 Theorems of Christensen and Calbrix for $C_p(X)$

All topological spaces in this section are completely regular and Hausdorff. If $\xi \in \beta\mathbb{N} \setminus \mathbb{N}$ and $X = \mathbb{N} \cup \{\xi\}$, the space $C_p(X)$ is metrizable and not K-analytic; nevertheless X is σ-compact; see Example 9.1.6. The following general question was posed by Christensen [140].

Problem 9.2.1 (Christensen) *Let $C_p(X)$ be an analytic space. Is X a σ-compact space?*

A space X is called *cosmic* if X admits a countable network, i.e. a sequence $(U_n)_n$ of subsets of X such that for each $x \in X$ and every open subset G of X with $x \in G$ there exists U_n such that $x \in U_n \subset G$. X is cosmic if and only if X is a continuous image of a metric separable space; see [21], and see also Proposition 3.1.8. By Proposition 6.1.7 X is analytic if and only if X is cosmic and K-analytic. Cosmic spaces look as a natural generalization of analyticity. Nevertheless, analytic spaces need not be metrizable, but any analytic space is always a cosmic space.

We start with the following result due to Christensen [140].

Theorem 9.2.2 (Christensen) *A metric and separable space X is σ-compact if and only if $C_p(X)$ is analytic.*

Proof Assume that $C_p(X)$ is an analytic space. The idea of the proof of this part uses the argument from [36, Theorem 2.3]. Let (Y, d) be a compact metric space in which X is densely embedded. Then $Z = Y \setminus X$ is separable and metrizable. First we show that Z is a Polish space. Set

$$\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}.$$

As there exists a (strictly increasing) homeomorphism of $\mathbb{R}$ onto $\mathbb{R}_+$, then $C_p^+(X) := C_p(X) \cap \mathbb{R}_+^X$ is an analytic subspace of $C_p(X)$. Hence there exists a continuous map

$$\varphi : \mathbb{N}^{\mathbb{N}} \to C_p(X)$$

such that $\varphi(\mathbb{N}^{\mathbb{N}}) = C^+(X)$. Since for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ the set $\{\beta \in \mathbb{N}^{\mathbb{N}} : \beta \leq \alpha\}$ is compact in $\mathbb{N}^{\mathbb{N}}$, the function $\Gamma : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}_+^X$ defined by

$$\Gamma(\alpha) := \inf\{\varphi(\beta) : \beta \leq \alpha\},$$

for all $\alpha \in \mathbb{N}^{\mathbb{N}}$, is well-defined. Note that $\Gamma(\beta) \leq \Gamma(\alpha)$ if $\alpha \leq \beta$, and if $f \in C^+(X)$, then $f \geq \Gamma(\alpha)$ for some $\alpha \in \mathbb{N}^{\mathbb{N}}$. Then $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, where $K_\alpha := \bigcap_{x \in X} Y \setminus B_x$ and

$$B_x := \{y \in Y : d(x, y) < \Gamma(\alpha)(x)\},$$

is a compact resolution on Z that swallows compact sets in Z: Let A be compact in Z. Then the continuous map $f : Y \to \mathbb{R}$ defined by $f(y) := d(y, A)$ restricted to X belongs to $C^+(X)$, so $f|X \geq \Gamma(\alpha)$ for some α. Hence $A \subset K_\alpha$. Now Theorem 6.2.4 applies to deduce that Z is a Polish space. Therefore Z is a G_δ subset of Y and hence $Y \setminus Z = X$ is a F_σ subspace of the space Y. Consequently, X is σ-compact and X is analytic.

For the converse we prove more than listed above: Let $A \subset X$ be a dense subset of X. Let $(K_n)_n$ be an increasing sequence of compact sets in X covering X. We show, following Okunev's argument from the proof of [476, Theorem 2.1], that the space $C_p(X|A) := \{f \in C_p(A) : f \text{ admits a continuous extension over } X\}$ is analytic. Note that the restriction map from $C_p(X) \to C_p(X|A)$ is injective. Fix a metric d on X compatible with the topology of X. Set

$$M_{kln} := \{f \in [0,1]^A : |f(x) - f(y)| \le k^{-1}, \text{ if}$$

$$d(x,z) < n^{-1}, d(y,z) < n^{-1}, \text{ for some } z \in K_l\},$$

where $k, l, n \in \mathbb{N}$. Note that

$$C_p(X|A, [0,1]) = \bigcap_k \bigcap_l \bigcup_n M_{kln}.$$

Indeed, assume $f \in \bigcap_k \bigcap_l \bigcup_n M_{kln}$. We need only to prove that f is continuous at any point $z \in X$. There exists $l \in \mathbb{N}$ such that $z \in K_l$. Clearly, the oscillation of f on A near the point z is arbitrary small since $f \in \bigcap_k \bigcap_l \bigcup_n M_{kln}$. Then

$$\bigcap_k \bigcap_l \bigcup_n M_{kln} \subset C_p(X|A, [0,1]).$$

For the converse inclusion, assume that $f \in C_p(X|A, [0,1])$. Let f_1 be a continuous extension of f to the space X. Now fix $k, l \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that $|f_1(x) - f_1(y)| \le k^{-1}$ if $d(x,z) < n^{-1}$ and $d(y,z) < n^{-1}$ for some $z \in K_l$. This proves that $f \in M_{kln}$.

Next, we prove that each set M_{kln} is compact. Indeed, observe that $[0,1]^A \setminus M_{kln}$ is the set of all $f \in [0,1]^A$ with $|f(x) - f(y)| > k^{-1}$ for some $x \in A$ and $y \in A$ such that $d(x,z) < n^{-1}, d(y,z) < n^{-1}$ for some $z \in K_l$. The set $[0,1]^A \setminus M_{kln}$ is open since the evaluation map $x \to e_x$, $e_x(f) := f(x)$, $x \in A$, is continuous on $[0,1]^A$. Hence M_{kln} is closed in the compact space $[0,1]^A$. This shows that $C_p(X|A, [0,1])$ is a K-analytic space. Let $\overline{\mathbb{R}}$ be the two-point compactification of $\mathbb{R}$ homeomorphic to [0.1]. Then, $C_p(X|A, \overline{\mathbb{R}})$ is K-analytic, as homeomorphic to $C_p(X|A, [0,1])$. Set

$$S := \bigcap_l \bigcup_n \{f \in \overline{\mathbb{R}}^A : |f(x)| \le n, \ x \in A \cap K_l\}.$$

Clearly S is a K-analytic subset in $\overline{\mathbb{R}}^A$ (as a $F_{\sigma\delta}$-set). Since $C_p(X|A) \subset S \subset \overline{\mathbb{R}}^A$, we have $C_p(X|A) = S \cap C_p(X|A, \overline{\mathbb{R}})$. Therefore $C_p(X|A)$ is K-analytic. As A is metrizable and separable, the space $C_p(A)$ is cosmic. Hence $C_p(X|A)$ is cosmic, as well. $C_p(X|A)$ is analytic by Proposition 6.1.7. □

In [113] Calbrix answered the Problem 9.2.1 by proving the following:

Theorem 9.2.3 (Calbrix) *If the space $C_p(X)$ is analytic, then X is σ-compact.*

Theorem 9.2.3 will be deduced from some results presented below. Calbrix's Theorem 9.2.3 was extended by Arkhangel'skii and Calbrix in [36]. Very recently Ferrando and Kąkol [209] showed Arkhangel'skii–Calbrix's theorem for spaces $C_p(X)$ having a bounded resolution; see Proposition 9.2.12.

The following example provides σ-compact spaces X such that $C_p(X)$ is not Lindelöf; see [477, Example 2.7].

Example 9.2.4 (Okunev) There exists a topological space X that is a countable union of Eberlein compact spaces and $C_p(X)$ is not Lindelöf.

Proof Let Z be a σ-product of uncountable copies of the compact space $[0, 1]$ with zero base point. Let p be the point whose all coordinates equal 1. Set $X := Z \cup \{p\}$. Clearly X is the countable union of Eberlein compact sets. Now define a function $f : X \to [0, 1]$ by $f(z) = 0$ for each $z \in Z$ and $f(p) = 1$. Since p does not belong to the closure of any countable subset of Z, the function f satisfies the following condition:

(*) for any countable set $A \subset X$ the restriction $f|A$ admits a continuous extension to the whole space X.

Assume that $C_p(X)$ is Lindelöf. Let $\mathcal{A} := \{A \subset X \text{ countable}\}$, and let

$$G_A := \{g \in C(X) : g|A = f|A\}$$

for each $A \in \mathcal{A}$. Let $\mathcal{D} := \{G_A : A \in \mathcal{A}\}$. Clearly $\mathcal{D}$ has the countable intersection property. Since each G_A is closed in $C_p(X)$, the condition (*) combined with the Lindelöf property of $C_p(X)$ implies that $\bigcap_{A\in\mathcal{A}} G_A$ is non-empty. Now choose $h \in \bigcap_{A\in\mathcal{A}} G_A$. Then $h = f$, a contradiction since f is discontinuous. □

We need the following two facts, the first one from [436, Corollary 4.1.3], the other one from [32, Corollary IV.9.9].

Proposition 9.2.5 *$C_p(X)$ is cosmic if and only if X is cosmic.*

Proposition 9.2.6 *If X is a separable space that is not a cosmic space, $\upsilon C_p(X)$ is not a Lindelöf Σ-space.*

A space X is called *projectively σ-compact* if every separable metrizable space Y that is a continuous image of X is σ-compact.

Every *σ-bounded* space X, i.e. X is a countable union of topologically bounded subsets of X, is projectively σ-compact. Indeed, this follows from the fact that any σ-bounded paracompact topological space is σ-compact. The following result of Okunev [476] is essential in the proof of Calbrix's Theorem 9.2.3.

Proposition 9.2.7 *A cosmic projectively σ-compact space X is σ-compact.*

Proof Since a cosmic space that is projectively analytic is analytic, see [477, Theorem 1.3], X is analytic, and then it is normal: see Lemma 6.1.3. Assume X

is not σ-compact. Then X contains a homeomorphic copy F of $\mathbb{N}^{\mathbb{N}}$ as a closed subspace; see Corollary 3.6.9. There exists a homeomorphism r between the space F and the space of the irrational numbers. Let $f : X \to \mathbb{R}$ be a continuous extension of r from X into $\mathbb{R}$. Fix a continuous map $g : X \to \mathbb{R}$ such that $F = g^{-1}(0)$. Define $h : X \to \mathbb{R}^2$, $h(x) := (f(x), g(x))$ for all $x \in X$. Note that the set $\{(x, y) \in h(X) : y = 0\}$ is closed in the range $h(X)$ and is homeomorphic to the space $\mathbb{N}^{\mathbb{N}}$. Since $\mathbb{N}^{\mathbb{N}}$ is not σ-compact, $h(X)$ is not σ-compact, a contradiction. □

The following theorem was obtained by Arkhangel'skii and Calbrix [36]; the proof uses the argument presented in the proof of Theorem 9.2.2.

Theorem 9.2.8 (Arhangel'skii–Calbrix) *If $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$, i.e. there exists a K-analytic space A such that $C_p(X) \subseteq A \subseteq \mathbb{R}^X$, then X is projectively σ-compact.*

Proof Assume $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$. Let Y be a separable metrizable space, and let $f : X \to Y$ be a continuous map from X onto Y. We show that Y is σ-compact. The dual map $f^\sharp(g) := g \circ f$ of $\mathbb{R}^Y \to \mathbb{R}^X$, is a (linear) homeomorphism of $\mathbb{R}^Y$ onto the range, $f^\sharp(C_p(Y)) \subset C_p(X)$, and $f^\sharp(\mathbb{R}^Y)$ is closed in $\mathbb{R}^X$. Let S be a K-analytic subspace of $\mathbb{R}^X$ containing $C_p(X)$. Then

$$f^\sharp(C_p(Y)) \subset C_p(X) \subset S$$

and $S \cap f^\sharp(\mathbb{R}^Y)$ as closed in S is also K-analytic. Since

$$f^\sharp(C_p(Y)) \subset S \cap f^\sharp(\mathbb{R}^Y),$$

we deduce that $C_p(Y)$ is K-analytic-framed in $\mathbb{R}^Y$. Set

$$C_p^+(Y) := C_p(Y) \cap \mathbb{R}_+^Y.$$

It is easy to see (by using the exponential function) that $C_p^+(Y)$ is K-analytic-framed in $\mathbb{R}_+^Y$. Therefore, there exist a K-analytic space $T \subset \mathbb{R}_+^Y$ with $C_p^+(X) \subset T$, and a (usco) compact-valued map φ from $\mathbb{N}^{\mathbb{N}}$ covering T. We complete the proof similarly as in the proof of Theorem 9.2.2. □

The proof of Arkhangel'skii–Calbrix's theorem depends on a result of Christensen about fundamental compact resolutions in metric spaces and Okunev's about projectively σ-compact spaces X. They asked if the same holds when X is just a Lindelöf space. The answer to this question could already be found in Leiderman's [401] and also recalled in [65, Remark 3.9]. We shall discuss again this example in a slightly stronger form.

As usual, for a topological space X, by βX and υX we denote the Čech–Stone and the realcompactification of X, respectively. By $f^\upsilon : \upsilon X \to \mathbb{R}$ we denote the unique extension of $f \in C(X)$. We need the following simple:

Lemma 9.2.9 *(a) The map $\Phi : C_p(X) \to C_p(\upsilon X)$ defined by $f \to f^\upsilon$ is linear, injective, and surjective. The converse map Φ^{-1} is continuous.*
(b) If $(g_n) \subset C_p(X)$, and $y \in \upsilon X$, there exists $x \in X$ such that $g_n^\upsilon(y) = g_n(x)$ for every $n \in \mathbb{N}$.
(c) If a subset A of $C_p(X)$ is countable, the map $\Phi|_A$ is continuous.

Proof (a) is clear.
(b) Set $f_n(x) := g_n(x) - g_n^\upsilon(y)$ for all $x \in X$ and $n \in \mathbb{N}$. Then each f_n is continuous, and $f_n^\upsilon(y) = 0$ for all $n \in \mathbb{N}$. Set

$$g(x) := \sum_n \min\{2^{-n}, |f_n(x)|\}$$

for all $x \in X$. Clearly $g \in C(X)$ and $g^\upsilon(y) = 0$. Assume that $g(x) \neq 0$ for each $x \in X$. Let $z(x) := g^{-1}(x)$ for all $x \in X$. Then $z^\upsilon(x)g^\upsilon(x) = 1$ for each $x \in X$. Hence $z^\upsilon(y)g^\upsilon(y) = 1$ by the density of X in υX, a contradiction. Hence the conclusion holds.
(c) Assume $(f_\gamma) \subset A$, $f_0 \in A$ and $f_\gamma \to f_0$ in $C_p(X)$. Let $y \in \upsilon X$. By (b) there is $x \in X$ with $f(x) = f^\upsilon(y)$ for every $f \in A$. Then $f_\gamma(x) \to f_0(x)$. Hence

$$f_\gamma^\upsilon(y) \to f^\upsilon(y).$$

Consequently, $f_\gamma^\upsilon \to f^\upsilon$ in $C_p(\upsilon X)$, so the map $\Phi|_A$ is continuous. □

The following lemma we can find in [128].

Lemma 9.2.10 *The space $C_p(X)$ is angelic if and only if $C_p(\upsilon X)$ is angelic.*

Proof If $C_p(X)$ is angelic, $C_p(\upsilon X)$ is angelic by Lemma 9.2.9 and Theorem 4.1.1 (angelic lemma). To prove the converse, assume that $C_p(\upsilon X)$ is angelic. If $A \subset C(X)$ is a relatively countably compact set in $C_p(X)$, the set $A^\upsilon := \{f^\upsilon : f \in A\}$ is relatively countably compact in $C_p(\upsilon X)$ because (by Lemma 9.2.9) the restriction map is a homeomorphism on any countable subset of $C_p(\upsilon X)$. By hypothesis each relatively countably compact set A in $C_p(\upsilon X)$ is relatively compact, and the closure of A^υ equals the sequential closure. The same conclusion holds for A in $C_p(X)$, which proves that $C_p(X)$ is angelic. □

For Proposition 9.2.12 we need the following result due to Arkhangel'skii (i) and Okunev (ii); see [32, Proposition IV.9.4], or [30, 477].

Proposition 9.2.11 *(i) If υX is a Lindelöf Σ-space, there exists a Lindelöf Σ-space Z such that $C_p(X) \subset Z \subset \mathbb{R}^X$.*
(ii) If there exists a Lindelöf Σ-space Z such that $C_p(X) \subset Z \subset \mathbb{R}^X$, υX is a Lindelöf Σ-space.

There exists a Lindelöf space X that is not K-analytic and $C_p(X)$ is K-analytic; see [401, 477]. For such X, by Proposition 9.2.11 (ii), the space υX is a Lindelöf Σ-space. This follows also from the following fact: If $C_p(X)$ is a Lindelöf Σ-space,

υX is a Lindelöf Σ-space. Indeed, since $C_p(X)$ is a Lindelöf Σ-space, applying Proposition 9.2.11 (i) we note that there exists a Lindelöf Σ-space Z such that

$$C_p(C_p(X)) \subset Z \subset \mathbb{R}^{C_p(X)}.$$

Clearly X is included in $C_p(C_p(X))$. Let Y be the closure of X in Z. Then Y is a Lindelöf Σ-space. Since every continuous map on X can be extended to a continuous map over $\mathbb{R}^{C_p(X)}$, every continuous map on X can be extended to a continuous map to Y. Hence $\upsilon X = Y$ is a Lindelöf Σ-space.

Proposition 9.2.12 was proved in [209].

Proposition 9.2.12 (Ferrando–Kąkol) *The following conditions are equivalent:*

(i) $C_p(X)$ *admits a bounded resolution.*
(ii) $C_p(X)$ *is K-analytic-framed in* $\mathbb{R}^X$ *and* $C_p(X)$ *is angelic.*
(iii) $C_p(X)$ *is K-analytic-framed in* $\mathbb{R}^X$.
(iv) For any tvs Y *containing* $C_p(X)$ *there exists a space* Z *such that* $C_p(X) \subseteq Z \subseteq Y$ *and* Z *admits a resolution consisting of* Y*-bounded sets.*

Proof First we note the following fact: Let X be non-empty and Z be a subspace of $\mathbb{R}^X$. If Z has a countable network modulo a cover $\mathcal{B}$ of Z by pointwise bounded subsets, then $Y = \bigcup\{\overline{B} : B \in \mathcal{B}\}$, closures in $\mathbb{R}^X$, is a Lindelöf Σ-space such that $Z \subseteq Y \subseteq \mathbb{R}^X$. Indeed, let $\mathcal{N} = \{T_n : n \in \mathbb{N}\}$ be a countable network modulo a cover $\mathcal{B}$ of Z consisting of pointwise bounded sets. Set $\mathcal{N}_1 = \{\overline{T}_n : n \in \mathbb{N}\}$, $\mathcal{B}_1 = \{\overline{B} : B \in \mathcal{B}\}$, closures in $\mathbb{R}^X$, and $Y = \cup\mathcal{B}_1$. Let us show that $\mathcal{N}_1$ is a network in Y modulo the compact cover $\mathcal{B}_1$ of Y. In fact, if U is a neighbourhood in $\mathbb{R}^X$ of $\overline{B}$, use $\overline{B}$ compactness to get a closed neighbourhood V of $\overline{B}$ in $\mathbb{R}^X$ contained in U. Since $\mathcal{N}$ is a network modulo $\mathcal{B}$ in Z there is $n \in \mathbb{N}$ with $B \subseteq T_n \subseteq V \cap Z$, which implies that $\overline{B} \subseteq \overline{T}_n \subseteq U$. According to Proposition 3.1.8 Y is a Lindelöf Σ-space such that $Z \subseteq Y \subseteq \mathbb{R}^X$.

Now we prove $(i) \Rightarrow (ii)$: Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a resolution for $C_p(X)$ of bounded sets, denote by B_α the closure of A_α in $\mathbb{R}^X$, and put $Z = \bigcup\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Clearly each B_α is a compact subset of $\mathbb{R}^X$ and Z is a quasi-Suslin space (Proposition 3.2.5) such that $C_p(X) \subseteq Z \subseteq \mathbb{R}^X$. As each quasi-Suslin space Z has a countable network modulo a *resolution* $\mathcal{B}$ of Z consisting of countably compact sets and every countable compact subset of $\mathbb{R}^X$ is pointwise bounded, the above beginning remark assures that $Y = \bigcup\{\overline{B} : B \in \mathcal{B}\}$ is a Lindelöf Σ-space, hence Lindelöf, such that $Z \subseteq Y \subseteq \mathbb{R}^X$. As each set $\overline{B}$ with $B \in \mathcal{B}$ is compact, and $\{\overline{B} : B \in \mathcal{B}\}$ is a resolution for Y, again Y is a quasi-Suslin space. Since every Lindelöf quasi-Suslin space is K-analytic and $C_p(X) \subseteq Y \subseteq \mathbb{R}^X$, it turns out that $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$.

This shows that $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$. By Proposition 9.2.11 we note that υX is a Lindelöf Σ-space. Finally, as each Lindelöf Σ-space is web-compact by Example 4.3.2 (2), we apply Theorem 4.3.3 to deduce that $C_p(\upsilon X)$ is angelic. Hence $C_p(X)$ is angelic by Lemma 9.2.10.

(ii) $\Rightarrow$ (iii) is clear.
(iii) $\Rightarrow$ (iv): If L is a space having a compact resolution $\left\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$, and

$$C_p(X) \subseteq L \subseteq \mathbb{R}^X,$$

then

$$\left\{A_\alpha \cap C_p(X) : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$$

is a bounded resolution in $Z := C_p(X)$ consisting of bounded sets in any tvs Y topologically containing $C_p(X)$.

The implication (iv) $\Rightarrow$ (i) is obvious. □

The equivalence between conditions (i), (iii), and (iv) below was proved in [36, Theorem 2.4].

Corollary 9.2.13 *Let X be a cosmic space. The following assertions are equivalent:*

(i) *X is σ-compact.*
(ii) *$C_p(X)$ has a bounded resolution.*
(iii) *$C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$.*
(iv) *$C_p(X)$ is analytic-framed in $\mathbb{R}^X$.*

Proof (i) $\Rightarrow$(ii): Note that if X is σ-compact, $C_p(X)$ has a bounded resolution. Indeed, if X is covered by a sequence $(C_n)_n$ of compact sets, then $\left\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ with

$$A_\alpha = \{f \in C(X) : \sup_{x \in C_n} |f(x)| \leq a_n,\ n \in \mathbb{N}\},$$

where $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$ is a bounded resolution for $C_p(X)$.
(ii) $\Rightarrow$ (iii): This follows from Proposition 9.2.12.
(iv) $\Rightarrow$ (iii): Obvious.
(iii) $\Rightarrow$ (i): This follows from Theorem 9.2.8 and Okunev's Proposition 9.2.7.
(i) $\Rightarrow$ (iv): Since X is cosmic and covered by a sequence $(K_n)_n$ of compact sets, the topological sum $Y := \bigoplus_n K_n$ provides a metrizable (recall that cosmic compact is metrizable) separable σ-compact space. By Theorem 9.2.2 the space $C_p(Y)$ is analytic. If $f : Y \to X$ is a canonical map from Y onto X, its dual map $f^\sharp(g) := g \circ f$ of $\mathbb{R}^X \to \mathbb{R}^Y$ is a homeomorphism of $\mathbb{R}^X$ onto the range, $f^\sharp(C_p(X)) \subset C_p(Y)$, and $f^\sharp(\mathbb{R}^X)$ is closed in $\mathbb{R}^Y$. Hence

$$f^\sharp(C_p(X)) \subset f^\sharp(\mathbb{R}^X) \cap C_p(Y)$$

and $f^\sharp(\mathbb{R}^X) \cap C_p(Y)$ is analytic (as closed in $C_p(Y)$). □

The following corollary extends Corollary 7.1.6.

Corollary 9.2.14 *For a metric space X the following assertions are equivalent:*

(i) X *is* σ*-compact.*
(ii) $C_p(X)$ *has a bounded resolution.*
(iii) $C_p(X)$ *is analytic.*

Proof (i) $\Rightarrow$ (ii): Proved in Corollary 9.2.13.
(ii) $\Rightarrow$ (iii): Since $C_p(X)$ has a bounded resolution, by Proposition 9.2.12 the space $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$ and $C_p(X)$ is angelic. A metric space X for which $C_p(X)$ is angelic is separable by Corollary 6.4.16. Hence X is a cosmic space and Corollary 9.2.13 applies to get that X is σ-compact. Now Christensen's Theorem 9.2.2 applies to deduce that $C_p(X)$ is analytic.
(iii) $\Rightarrow$ (i): By Calbrix's Theorem 9.2.3. □

From Corollary 9.2.13 it follows the following:

Corollary 9.2.15 *The space $C_p(\mathbb{N}^{\mathbb{N}})$ does not admit a bounded resolution.*

The following observation is due to Christensen [140]: If X is a k_R-space, there is no continuous surjection from X onto $C_c(X)$. Consequently, the space $C_c(\mathbb{N}^{\mathbb{N}})$ is not analytic. Indeed, assume that there is a continuous surjection φ from X onto $C_c(X)$. Define a real-valued function f by $f(x) = 1+\varphi(x)(x)$ for all $x \in X$. Since X is a k_R-space and f is continuous on each compact subset of X, the map f is continuous. Fix $x \in X$ such that $\varphi(x) = f$. Then $1 + \varphi(x)(x) = f(x) = \varphi(x)(x)$, a contradiction.

Corollary 9.2.16 *If $C_p(X)$ has a bounded resolution, a compact set $A \subset C_p(X)$ is metrizable if and only if A is contained in a separable subset of $C_p(X)$.*

Proof Assume a compact set $A \subset C_p(X)$ is contained in some separable subset of $C_p(X)$. Set $A^\upsilon := \{f^\upsilon : f \in A\}$. Then A^υ is contained in a separable subset of $C_p(\upsilon X)$; see Lemma 9.2.9(c). Since υX is web-compact (the proof of Proposition 9.2.12 (i) $\Rightarrow$ (ii) showed that υX is a Lindelöf Σ-space), we apply Corollary 4.4.3 to deduce that A^υ is metrizable. Since A^υ and A are homeomorphic, the conclusion follows. □

Now we are ready to prove Calbrix's Theorem 9.2.3 as a simple consequence of above results.

Proof If $C_p(X)$ is analytic, then $C_p(X)$ is a cosmic space. Hence X is cosmic by Proposition 9.2.5. Next, by Proposition 9.2.12 the space $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$. This, together with Theorem 9.2.8, implies that X is projectively σ-compact. Proposition 9.2.7 yields that X is σ-compact. □

Theorem 9.2.17 extends Tkachuk's Theorem 9.1.4.

Theorem 9.2.17 *Let ξ be a regular topology on $C(X)$ stronger than the pointwise one. The following assertions are equivalent.*

(i) $(C(X), \xi)$ *is K-analytic.*
(ii) $(C(X), \xi)$ *is quasi-Suslin.*
(iii) $(C(X), \xi)$ *admits a relatively countably compact resolution.*

Proof Since, by Proposition 9.2.12, each condition listed above implies the angelicity of $C_p(X)$, Theorem 4.1.1 yields the angelicity of $(C(X), \xi)$, and then Corollary 3.2.9 applies. □

Theorem 9.2.17 fails in general for spaces of the form $C_p(X, Y)$ if Y is an arbitrary metric space. Let $Y := [-1, 1]$ be endowed with the natural topology from $\mathbb{R}$. The following example is due to Tkachuk [592].

Example 9.2.18 Let X be the Lindelöfication of the discrete space ω_1. Under CH the space $C_p(X, [-1, 1])$ is not Lindelöf and admits a compact resolution.

Proof Set $\mathbb{I} := [-1, 1]$ and $X := \omega_1 \cup \{x\}$, where ω_1 consists of isolated points, and U_x is an open neighbourhood of x in X if $|X \setminus U_x|$ is countable. Since $C_p(X, \mathbb{I})$ is countably compact and not compact [590], the space $C_p(X, \mathbb{I})$ is not Lindelöf. Note also that $C_p(X, \mathbb{I}) = Y \times \mathbb{I}$, where

$$Y := \{f \in C(X, \mathbb{I}) : f(x) = 0\}$$

$$= \{f \in \mathbb{I}^X : f(x) = 0 \wedge \exists t < \omega_1 : f(r) = 0 \ \forall\, r \geq t\}.$$

By Proposition 3.2.4 it is enough to show that Y has a compact resolution. Assume CH. Then $Y = \{f_t : t < \omega_1\}$, and there exists a set

$$Z := \{\alpha_t : t < \omega_1\} \subset \mathbb{N}^{\mathbb{N}}$$

such that $\alpha_t \leq^* \alpha_r$ if $t < r$ for $t, r < \omega_1$, and Z is cofinal in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. If $\alpha \in \mathbb{N}^{\mathbb{N}}$, the set

$$N_\alpha := \{t < \omega_1 : \alpha_t \leq^* \alpha\}$$

is countable. Then

$$K_\alpha := \{f_t : t \in N_\alpha\}$$

is relatively compact in Y, since the closure of any countable subset of Y is compact and $K_\alpha \subset K_\beta$ if $\alpha \leq \beta$ for $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$. If $f \in Y$, there exists $t < \omega_1$ such that $f = f_t \in \overline{K_{\alpha_t}}$. □

The space $C_p(X, \mathbb{I})$ is not angelic. Note that the space $C_p(X)$ for X from Example 9.2.18 is not K-analytic. This can also be deduced from the following general fact extending Theorem 9.2.17.

Proposition 9.2.19 *Assume that $C_p(X)$ is K-analytic. Let Z be a metric space. Let ξ be a regular topology on the space $C(X, Z)$ which is stronger than the pointwise topology of $C(X, Z)$. The following assertions are equivalent:*

(i) $(C(X, Z), \xi)$ *is K-analytic.*
(ii) $(C(X, Z), \xi)$ *admits a compact resolution.*
(iii) $(C(X, Z), \xi)$ *admits a relatively countably compact resolution.*

Proof Since $C_p(X)$ is K-analytic, by Proposition 9.2.12 the space $C_p(X)$ is angelic. By Theorem 4.1.5 the space $C_p(X, Z)$ is also angelic. Now we follow like as in Theorem 9.2.17. □

The following Corollary 9.2.20 is from [128].

Corollary 9.2.20 *Let X be a web-compact space. Then $C_p(X)$ is K-analytic if and only if $C_p(\upsilon X)$ is K-analytic.*

Proof By Theorem 4.3.3 the space $C_p(X)$ is angelic. Assume that $C_p(\upsilon X)$ is K-analytic. Since the restriction map from $C_p(\upsilon X)$ onto $C_p(X)$ is a continuous bijection, $C_p(X)$ is K-analytic. Assume $C_p(X)$ is a K-analytic space. Then $C_p(X)$ admits a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

Since $C_p(\upsilon X)$ is angelic by Lemma 9.2.10, the following family $\{\overline{K_\alpha^\upsilon} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution in $C_p(\upsilon X)$, where $K_\alpha^\upsilon := \{f^\upsilon : f \in K_\alpha\}$. Indeed, every set K_α^υ is relatively countably compact; see the proof of Lemma 9.2.10. Since $C_p(\upsilon X)$ is angelic, the set K_α^υ is relatively compact for each $\alpha \in \mathbb{N}^{\mathbb{N}}$. Now Theorem 9.2.17 applies to deduce that the space $C_p(\upsilon X)$ is K-analytic. □

We conclude this section with the following application of Theorem 9.2.3.

Proposition 9.2.21 *$C_p(X)$ is an analytic space if and only if $C_p(X)$ admits a stronger metrizable and analytic vector topology ξ such that ξ is polar with respect to the topology of $C_p(X)$.*

Proof Assume that $C_p(X)$ is analytic. The spaces $C_p(X)$ and $C_c(X)$ are separable; see [322, Theorem 2.10.3]. By Theorem 9.2.3 the space X is σ-compact. Let $(K_n)_n$ be an increasing sequence of compact sets in X covering X. For each $k \in \mathbb{N}$ set

$$V_k := \{f \in C(X) : \sum_n 2^{-n} \min\{1, \sup_{x \in K_n} |f(x)|\} \leq 2^{-k}\}.$$

It is easy to see that

$$V_{k+1} + V_{k+1} \subset V_k,$$

and V_k is balanced and absorbing for each $k \in \mathbb{N}$. Let τ_c and τ_p denote the original topologies of the spaces $C_c(X)$ and $C_p(X)$, respectively. Let ξ be a vector topology on $C(X)$ whose basis of neighbourhoods of zero are formed by the sets V_k, $k \in \mathbb{N}$. Clearly $\xi \leq \tau_c$. Note also that $\tau_p \leq \xi$. Indeed, let $y \in X$, $0 < \epsilon < 1$, and set

$$U := \{f \in C(X) : |f(y)| < \epsilon\}.$$

Then U is a τ_p-neighbourhood of zero. There exists $m \in \mathbb{N}$ such that $y \in K_m$. One proves that $V_p \subset \epsilon^{-1}2^{-p}U$. This shows our conjecture. Therefore ξ is a metrizable and separable vector topology, stronger than τ_p. Moreover, ξ admits a basis of neighbourhoods of zero consisting of τ_p-closed sets. By Corollary 6.6.1 we obtain that ξ admits a complete resolution. Corollary 6.6.2 proves that $(C(X), \xi)$ is analytic. The converse implication is clear. □

Finally, note that if X is a σ-compact space, then $X \times Y$ is Lindelöf for each Lindelöf space Y. What about the converse implication? Alster [6, Theorem] provided under Martin's axiom an interesting sufficient condition for a metrizable analytic space E to be σ-compact; for the proof we refer to [6].

Theorem 9.2.22 *Under Martin's axiom if X is metrizable and analytic and such that the product X with every Lindelöf space is Lindelöf, then X is σ-compact.*

An interesting approach (in terms of the reminder $\beta X \setminus X$) to determine if a topological group X is σ-compact was provided in [35] and [34].

9.3 Around Arkhangel'skii–Calbrix's Theorem and Nice Framings

In this section we show that $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$ if and only if X admits a *nice framing* (Theorem 9.3.7) if and only if X has a *fundamental resolution of functions* (Theorem 9.3.10, Corollary 9.3.8).

The latter concept will be directly used to construct a (usco) map from $\mathbb{N}^{\mathbb{N}}$ into the compact sets of Z with $C_p(X) \subseteq Z \subseteq \mathbb{R}^X$ showing the K-analyticity of Z. Examples illustrating results will be also presented and we discuss some consequences of Theorem 9.3.7 for obtaining σ-compactness of X.

Recall first that X is *projectively analytic* if each continuous metrizable and separable image of X is analytic. The space X is said to have the *Discrete Countable Chain Condition* (DCCC) if every discrete family of open sets in X is countable, what is equivalent to say that each continuous metrizable image of X is separable. The following result was proved in [225].

Theorem 9.3.1 (Ferrando–Kąkol–Śliwa) *If an infinite Tychonoff space X is projectively analytic, then X has the DCCC.*

Proof Assume that there exists a continuous surjective map $h : X \to Z$ and Z is metrizable but not separable. Next choose a closed discrete set D in Z with $|D| = \aleph_1$. Such set exists since $d(Z) = w(Z) = e(Z)$, where $e(Z)$ means the extent of Z; see [195].

Observe that there exists a continuous injective map $f : D \to Y$ onto a metrizable and separable space Y, which is not analytic. Indeed, such Y we obtain as follows: Under CH we know that $\mathbb{R}$ contains $2^{\mathfrak{c}}$ subsets of cardinality continuum, but only $\mathfrak{c}$ analytic subsets. So, one of those $2^{\mathfrak{c}}$ subsets Y is not analytic. Under

$\neg$CH, take a subset $Y \subseteq \mathbb{R}$ of cardinality $\aleph_1$. Then it is not analytic. Indeed, every uncountable analytic subset of $\mathbb{R}$ contains a copy of the Cantor set and hence has cardinality $\mathfrak{c}$.

The map f admits a (canonical) extension $Pf : PD \to PY$ to spaces of finitely supported maps, where PY is the space of finitely supported probability measures endowed with the weak* topology determined by the subspace $C^b(X)$ of $C(Y)$ consisting of bounded functions.

It turns out that PY is a separable and metrizable convex set by the Prokhorov–Wasserman–Kantorovich metric; see, for example, [52, Lemma 4.3]. As it follows from the proof of [32, 0.5.9 Proposition], the $y \mapsto \delta_y$ copy of Y in $L(Y)$ (the dual of $C_p(Y)$) is closed in $L(Y)$ when the latter linear space is provided with the weak topology of the dual pair $\langle L(Y), C^b(Y)\rangle$.

Hence we deduce that the space Y is closed in PY. Since PY is a convex metrizable subset of a lcs, $f : D \to Y \subseteq PY$ admits a continuous extension $\bar{f} : Z \to PY$ by the Dugundji extension theorem [190, page 185]. On the other hand, $\bar{f}(Z)$ in PY is not analytic since it contains a closed subset Y which is not analytic. Then $\bar{f} \circ h$ has a non-analytic (separable) metrizable image, a contradiction. □

A Tychonoff space X is called *strongly projectively σ-compact* if every continuous metrizable image of X is σ-compact.

Corollary 9.3.2 *Let X be an infinite Tychonoff space. Then X is projectively σ-compact if and only if X is strongly projectively σ-compact.*

Corollary 9.3.3 *A metrizable space X is analytic if and only if every continuous metrizable and separable image of X is analytic.*

Corollary 9.3.4 *A paracompact Čech-complete space X is σ-compact if and only if $C_p(X)$ has a bounded resolution.*

Indeed, if $C_p(X)$ has a bounded resolution, X is strongly projectively σ-compact. Since X is mapped onto a completely metrizable space Y by a perfect map T, the space Y is σ-compact. Hence X is σ-compact (since T is perfect). The converse implication is clear.

Problem 9.3.5 *Characterize Tychonoff spaces X such that $C_p(X)$ has a bounded resolution.*

According to [206] or [208] a family $\{U_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ of closed subsets of X will be called a *framing* if:

(i) For each $\alpha \in \mathbb{N}^{\mathbb{N}}$ the *layer* $\{U_{\alpha,n} : n \in \mathbb{N}\}$ is an increasing covering of X.
(ii) For every $n \in \mathbb{N}$ one has that $U_{\beta,n} \subseteq U_{\alpha,n}$, $\alpha \leq \beta$.

We need the following obvious fact mentioned in [206].

Lemma 9.3.6 *A set $A \subseteq C_p(X)$ is bounded if and only if there is an increasing covering $\{V_n : n \in \mathbb{N}\}$ of X by closed sets such that $\sup_{f\in A} |f(x)| \leq n$, $x \in V_n$.*

Proof Let A be a pointwise bounded set in $C_p(X)$. For each $n \in \mathbb{N}$ define

$$V_n = \{x \in X : \sup_{f \in A} |f(x)| \leq n\}.$$

Obviously the sets V_n are closed and compose an increasing sequence covering X as desired. Conversely, if for a subset $A \subset C_p(X)$ such sequence $(V_n)_n$ exists, the set A is pointwise bounded. □

The following essential results can be found in [225].

Theorem 9.3.7 *The space $C_p(X)$ has a bounded resolution if and only if there exists a framing $\{U_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ in X enjoying the property that if $f \in C(X)$ there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $|f(x)| \leq n$ for each $x \in U_{\gamma,n}$ and $n \in \mathbb{N}$.*

Proof Assume there exists a framing $\{U_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ of the aforementioned characteristics. Then the sets

$$A_\alpha := \left\{ f \in C(X) : \sup_{x \in U_{\alpha,n}} |f(x)| \leq n \ \forall n \in \mathbb{N} \right\}$$

compose a bounded resolution for $C(X)$. Indeed, each set A_α is pointwise bounded by virtue of Lemma 9.3.6, since $\{U_{\alpha,n} : n \in \mathbb{N}\}$ is an increasing covering of X by closed sets such that $\sup_{f \in A_\alpha} |f(x)| \leq n$ for all $x \in U_{\alpha,n}$.

Moreover, $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$ since $U_{\beta,n} \subseteq U_{\alpha,n}$. If $f \in C(X)$, by the statement of the theorem there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $|f(x)| \leq n$ for each $x \in U_{\gamma,n}$ and all $n \in \mathbb{N}$. Hence $f \in A_\gamma$, so $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ covers $C(X)$.

For the converse assume that $C_p(X)$ has a bounded resolution $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. If

$$V_{\alpha,n} = \{x \in X : \sup_{f \in B_\alpha} |f(x)| \leq n\},$$

then $\{V_{\alpha,n} : n \in \mathbb{N}\}$ is an increasing covering of X by closed sets for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ with $V_{\beta,n} \subseteq V_{\alpha,n}$ whenever $\alpha \leq \beta$, $n \in \mathbb{N}$. If $f \in C(X)$ there is $\delta \in \mathbb{N}^{\mathbb{N}}$ such that $f \in B_\delta$. Hence $|f(x)| \leq n$ for each $x \in V_{\delta,n}$ $n \in \mathbb{N}$, so

$$\{V_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$$

is a framing as claimed. □

We shall say that X has a *nice framing* if X admits a framing as stated in Theorem 9.3.7.

The following concept (also fixed in [65] for uniform spaces X with uniformly continuous functions f_α) will be used in the sequel.

A Tychonoff space X admits a *fundamental resolution of functions* if there exists on X a family of non-negative real-valued functions $\{f_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that $f_\alpha \leq f_\beta$ for $\alpha \leq \beta$ and for each $f \in C(X)$ there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ with $|f| \leq f_\alpha$.

Corollary 9.3.8 *Let X be a Tychonoff space. The following assertions are equivalent:*

(i) X has a fundamental resolution of functions.
(ii) $C_p(X)$ has a bounded resolution.
(iii) X has a nice framing.

Proof If $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bounded resolution on $C_p(X)$, then functions

$$f_\alpha(x) = \sup\{|f(x)| : f \in A_\alpha\}$$

form a fundamental resolution of functions, and if (f_α) is a fundamental resolution of functions, sets $A_\alpha = \{f \in C_p(X) : |f| \leq f_\alpha\}$ form a bounded resolution on $C_p(X)$. The last claim we obtain from Theorem 9.3.7. □

We apply this concept to provide a short proof of Corollary 9.2.13 (i) $\Leftrightarrow$ (iii) if X is metrizable (or cosmic). The main idea remains similar to that one as presented in [65, Proof of Theorem 2.2].

Theorem 9.3.9 *A metrizable space X is σ-compact if and only if X admits a nice framing. The same statement holds if X is cosmic.*

Proof Assume first that X is metrizable and separable, with a nice framing. Let $\{f_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a fundamental resolution of functions on X (we apply Corollary 9.3.8).

Let $\overline{X}$ be its metric compactification (see [272] for necessary details). For $\alpha \in \mathbb{N}^{\mathbb{N}}$ set

$$K_\alpha = \bigcap_{y \in X} (\overline{X} \setminus B(y, \exp(-f_\alpha(y)),$$

where $B(y, r)$ is the open ball at y and radius r.

Clearly each K_α is a compact subset of $\overline{X} \setminus X$ and $K_\alpha \subseteq K_\beta$, if $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ with $\alpha \leq \beta$.

Let $K \subseteq \overline{X} \setminus X$ be compact. For $h(y) = |\ln d(K, y)|$, $y \in X$, there exists $\sigma \in \mathbb{N}^{\mathbb{N}}$ with $h \leq f_\sigma$. Hence $d(K, y) \geq \exp(-f_\sigma(y))$, and then

$$K \subseteq \overline{X} \setminus B(y, \exp(-f_\sigma(y)))$$

for every $y \in X$; so $K \subseteq K_\sigma$. Thus

$$\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$$

is a compact resolution swallowing compact sets for the metrizable and separable space $\overline{X} \setminus X$. By Christensen's theorem $\overline{X} \setminus X$ is Polish, so X is σ-compact.

Assume now that X is metrizable and contains a nice framing. By Corollary 9.3.8 the space $C_p(X)$ has a bounded resolution. Assume that X is mapped onto a metrizable and separable space Y by a continuous map. Since $C_p(Y)$ is isomorphic to a subspace of $C_p(X)$, the space $C_p(Y)$ has a bounded resolution; consequently the

metrizable and separable space Y admits a nice framing. By the first case we derive that Y is σ-compact. We apply Corollary 9.3.2 to conclude that X is σ-compact.

The converse follows from the fact that if $\{B_n\}_{n=1}^{\infty}$ is a sequence of functionally bounded sets covering X, the sets

$$A_\alpha = \{f \in C(X) : \sup_{x \in B_n} |f(x)| \leq \alpha(n)\ \forall n \in \mathbb{N}\}$$

for $\alpha \in \mathbb{N}^{\mathbb{N}}$ compose a bounded resolution for $C_p(X)$.

Finally, assume that X is cosmic with a nice framing, and let Y be a continuous metrizable and separable image of X.

By the previous argument Y is σ-compact. So, according to Proposition 9.2.7, the space X is σ-compact. The converse is clear. □

Theorem 9.3.10 *If X has a nice framing, $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$ and angelic.*

We need the following two simple technical lemmas.

Lemma 9.3.11 *Each increasing function $\varphi : \mathbb{N}^{\mathbb{N}} \to [0, \infty)$ is bounded on some non-empty open subset of $\mathbb{N}^{\mathbb{N}}$.*

Proof Suppose, by contrary, that φ is unbounded on every non-empty open subset of $\mathbb{N}^{\mathbb{N}}$. Let

$$\beta^1 = (\beta_n^1) \in \mathbb{N}^{\mathbb{N}}$$

with $\varphi(\beta^1) \geq 1$. Let

$$\gamma^1 \in \{\beta_1^1\} \times \mathbb{N}^{\mathbb{N}}$$

with $\varphi(\gamma^1) \geq 2$. Put

$$\beta^2 = (\beta_n^2) = \max\{\beta^1, \gamma^1\};$$

then

$$\varphi(\beta^2) \geq 2, \beta^2 \geq \beta^1$$

and $\beta_1^2 = \beta_1^1$. Let

$$\gamma^2 \in \{(\beta_1^1, \beta_2^2)\} \times \mathbb{N}^{\mathbb{N}}$$

with $\varphi(\gamma^2) \geq 3$. Put again

$$\beta^3 = (\beta_n^3) = \max\{\beta^2, \gamma^2\}.$$

Then we have $\varphi(\beta^3) \geq 3$, $\beta^3 \geq \beta^2$ and $\beta_1^3 = \beta_1^1, \beta_2^3 = \beta_2^2$.

Following this procedure we obtain an element $\beta = (\beta_n^n) \in \mathbb{N}^{\mathbb{N}}$ and an increasing sequence $(\beta^k) \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\varphi(\beta^k) \geq k$ and $\beta_i^k = \beta_i^i$ for all $1 \leq i \leq k, k \in \mathbb{N}$. Then $\beta^k \leq \beta$ and

$$k \leq \varphi(\beta^k) \leq \varphi(\beta) < +\infty$$

for any $k \in \mathbb{N}$, a contradiction. □

Lemma 9.3.12 *(1) Each increasing function $\varphi : \mathbb{N}^{\mathbb{N}} \to [0, \infty)$ is locally bounded, i.e. each point $x \in \mathbb{N}^{\mathbb{N}}$ has an open neighbourhood U such that $\varphi(U)$ is bounded. (2) For every locally bounded function $\varphi : \mathbb{N}^{\mathbb{N}} \to [0, \infty)$ there exists a locally constant function $g : \mathbb{N}^{\mathbb{N}} \to [0, \infty)$ with $g \geq \varphi$; in particular, g is continuous.*

Proof (1) Assume the claim fails. Then there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that φ is unbounded on

$$\{(\alpha_1, \ldots, \alpha_m)\} \times \mathbb{N}^{\mathbb{N}}$$

for every $m \in \mathbb{N}$.

Hence for every $\beta \geq \alpha$ the function φ is unbounded on

$$\{(\beta_1, \ldots, \beta_m)\} \times \mathbb{N}^{\mathbb{N}}$$

for every $m \in \mathbb{N}$.

Set

$$\psi : \mathbb{N}^{\mathbb{N}} \to [0, +\infty), \psi((\beta_n)) = \varphi((\beta_n + \alpha_n)).$$

Then ψ is increasing and unbounded on any non-empty open subset of $\mathbb{N}^{\mathbb{N}}$. Indeed, let $\beta = (\beta_n) \in \mathbb{N}^{\mathbb{N}}$ and $m \in \mathbb{N}$.

Let $\gamma_i = \beta_i + \alpha_i$ for $1 \leq i \leq m$ and $A = \{(\lambda_i) \in \mathbb{N}^{\mathbb{N}} : \lambda_i > \alpha_{i+m}, i \in \mathbb{N}\}$. Then

$$\psi(\{(\beta_1, \ldots, \beta_m)\} \times \mathbb{N}^{\mathbb{N}} = \varphi(\{(\gamma_1, \ldots, \gamma_m)\} \times A)$$

and for any $(\lambda_i') \in \mathbb{N}^{\mathbb{N}}$ we have

$$\varphi((\gamma_1, \ldots, \gamma_m, \lambda_1', \lambda_2', \ldots)) \leq \varphi((\gamma_1, \ldots, \gamma_m, \lambda_1' + \alpha_{m+1}, \lambda_2' + \alpha_{m+1}, \ldots))$$

and

$$(\gamma_1, \ldots, \gamma_m, \lambda_1' + \alpha_{m+1}, \lambda_2' + \alpha_{m+1}, \ldots) \in \{(\gamma_1, \ldots, \gamma_m)\} \times A.$$

Thus $\psi(\{(\beta_1, \ldots, \beta_m)\} \times \mathbb{N}^{\mathbb{N}}) = \varphi(\{(\gamma_1, \ldots, \gamma_m)\} \times A)$ is unbounded, since $\varphi(\{(\gamma_1, \ldots, \gamma_m)\} \times \mathbb{N}^{\mathbb{N}})$ is unbounded. This shows that ψ is unbounded on any non-empty open subset of $\mathbb{N}^{\mathbb{N}}$, a contradiction with Lemma 9.3.11.

(2) For $\alpha \in \mathbb{N}^{\mathbb{N}}$ let $m(\alpha)$ be the least integer such that φ is bounded on

$$\{(\alpha_1, \ldots, \alpha_{m(\alpha)})\} \times \mathbb{N}^{\mathbb{N}}.$$

Put

$$V_\alpha = \{(\alpha_1, \ldots, \alpha_{m(\alpha)})\} \times \mathbb{N}^{\mathbb{N}}$$

for any $\alpha \in \mathbb{N}^{\mathbb{N}}$.

Clearly, $\bigcup\{V_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} = \mathbb{N}^{\mathbb{N}}$. For all $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ we have $V_\alpha = V_\beta$ or $V_\alpha \cap V_\beta = \emptyset$.

Indeed, if $m(\alpha) = m(\beta)$ and $\alpha_i = \beta_i$ for $1 \leq i \leq m(\alpha)$, then $V_\alpha = V_\beta$.

If $m(\alpha) = m(\beta)$ and $\alpha_i \neq \beta_i$ for some $1 \leq i \leq m(\alpha)$, then $V_\alpha \cap V_\beta = \emptyset$.

If $m(\alpha) \neq m(\beta)$, then $\alpha_i \neq \beta_i$ for some $1 \leq i \leq \min\{m(\alpha), m(\beta)\}$ and $V_\alpha \cap V_\beta = \emptyset$.

Thus for some $W \subseteq \mathbb{N}^{\mathbb{N}}$ the family $\{V_\alpha : \alpha \in W\}$ is a partition of $\mathbb{N}^{\mathbb{N}}$ on non-empty clopen subsets such that φ is bounded on V_α for every $\alpha \in W$.

Let $t_\alpha = \sup \varphi(V_\alpha)$ for $\alpha \in W$. Let

$$g : \mathbb{N}^{\mathbb{N}} \to [0, +\infty)$$

be the function such that $g(\beta) = t_\alpha$ for any $\beta \in V_\alpha, \alpha \in W$. Then $g \geq \varphi$ and g is locally constant, so it is continuous. □

Proof of Theorem 9.3.10 By Corollary 9.3.8 fix a fundamental resolution of functions $\{f_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ for X. Let $x \in X$. Then

$$\varphi_x : \mathbb{N}^{\mathbb{N}} \to [0, +\infty), \alpha \to f_\alpha(x)$$

is increasing.

By Lemma 9.3.12 there exists a locally constant function

$$g_x : \mathbb{N}^{\mathbb{N}} \to [0, +\infty)$$

with $g_x \geq \varphi_x$. Let

$$g : \mathbb{N}^{\mathbb{N}} \times X \to [0, +\infty), g(\alpha, x) = g_x(\alpha).$$

Clearly, for any $x \in X$ the function

$$\mathbb{N}^{\mathbb{N}} \to [0, +\infty), \alpha \to g(\alpha, x)$$

is locally constant.

Moreover, for any function $f \in C_p(X)$ there is an $\alpha \in \mathbb{N}^{\mathbb{N}}$ with

$$|f(x)| \leq f_\alpha(x) = \varphi_x(\alpha) \leq g_x(\alpha) = g(\alpha, x)$$

for every $x \in X$.

For any $\alpha \in \mathbb{N}^{\mathbb{N}}$ the set

$$F_\alpha = \prod_{x \in X} [-g(\alpha, x), g(\alpha, x)]$$

in $\mathbb{R}^X$ is compact. Put

$$Z = \bigcup \{F_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}.$$

Then $C_p(X) \subseteq Z \subseteq \mathbb{R}^X$. Using the continuity of g with respect to the first variable it is easy to see that $F : \alpha \mapsto F_\alpha$ is an upper semicontinuous (usco) set-valued map from $\mathbb{N}^{\mathbb{N}}$ with compact values in Z.

Thus $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$. □

Corollary 9.3.13 *If $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, X has a nice framing if and only if Y has a nice framing.*

The following problems have been posed in [36].

Problem 9.3.14 ([36]) *Let X be a Tychonoff space.*

(i) Is X σ-compact if X is Lindelöf and $C_p(X)$ is K-analytic-framed in $\mathbb{R}^X$?
(ii) Let $C_p(X)$ be K-analytic-framed in $\mathbb{R}^X$. Is X a σ-bounded space?
(iii) Let X be a Lindelöf space such that $C_p(X)$ is K-analytic. Is X a σ-compact space?

Example 9.3.16, due to Leiderman [401], shows that the above problems have negative solutions. Note also that later on, Banakh and Leiderman recalled this again in [65, Proposition 3.8, Remark 3.9]. The present version of Example 9.3.16 provides a slightly stronger claim than the original one from [401].

We need the following simple:

Proposition 9.3.15 *A Tychonoff space X is pseudocompact if and only if there exists a σ-compact space K with $C_p(X) \subseteq K \subseteq \mathbb{R}^X$.*

Proof If X is pseudocompact and $S = \{f \in C(X) : |f(x)| \leq 1, x \in X\}$, the sequence $\{nS\}_{n=1}^{\infty}$ covers $C_p(X)$. Hence, the closure of nS in $\mathbb{R}^X$ provides a sequence of compact sets in $\mathbb{R}^X$ whose union K contains $C(X)$.

Conversely, if the conclusion holds, the space $C_p(X)$ is covered by a sequence of bounded sets. If X is not pseudocompact, $C_p(X)$ contains a complemented copy of $\mathbb{R}^{\mathbb{N}}$, which is not covered by a sequence of functionally bounded sets. □

We are ready to present the following:

Example 9.3.16 There is a Lindelöf Σ-space X with a unique non-isolated point and such that:

(i) $C_p(X)$ is K-analytic.
(ii) X lacks a compact resolution, so X is not σ-compact and does not have countable functional boundedness.
(iii) Every continuous metrizable image of X is countable.
(iv) X has a nice framing; no nice framing has a layer with functionally bounded sets.

Remark 9.3.17 Leiderman's example [401] is based on Talagrand's paper [582] who constructed a space X with a unique non-isolated point which is a Lindelöf Σ-space but is not K-analytic. Leiderman proved that $C_p(X)$ is K-analytic. The part (iii) follows from the fact stating that: Every disjoint covering of X by G_δ-sets is countable. Item (ii) follows from the observation that X is K-analytic if and only if X is a μ-space and X has a compact resolution; see [330, Lemma 2.3].

Note that if X is both separable and is a continuous image of a metrizable space, the conclusion in (1) of Problem 9.3.14 still may fail.

Example 9.3.18 There exists a separable pseudocompact Tychonoff space X not being a μ-space and:

(i) X is a continuous compact-covering image of a metric space.
(ii) X does not admit a compact resolution; in particular X is not σ-compact.
(iii) There exists a σ-compact space L such that $C_p(X) \subseteq L \subseteq \mathbb{R}^X$ but $C_p(X)$ is not K-analytic. Hence X admits a nice framing.
(iv) $C_p(X)$ admits a quotient map onto the σ-compact subspace $(\ell_\infty)_p = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : \sup_n |x_n| < \infty\}$ of $\mathbb{R}^{\mathbb{N}}$, but $C_p(X)$ is not projectively σ-compact.

Proof Denote the family of all infinite subsets of a countable set X by $[X]^{\mathbb{N}}$. Set $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. For each $A \in [\mathbb{N}]^{\mathbb{N}}$, choose an ultrafilter $u_A \in \mathbb{N}^*$ in the closure of A in $\beta\mathbb{N}$. Let

$$X = \mathbb{N} \cup \{u_A : A \in [\mathbb{N}]^{\mathbb{N}}\}$$

be topologized as a subspace of $\beta\mathbb{N}$.

(i) It is known (Haydon [302]) that X is pseudocompact (separable) with cardinality continuum and all compact subspaces of X are finite. Clearly, X is a continuous compact-covering image of a metrizable space.
(ii) Assume X admits a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Since X is uncountable, some K_α is infinite by Proposition 3.2.1, a contradiction.
(iii) By Proposition 9.3.15 the space $C_p(X)$ has the first property.

Now assume $C_p(X)$ is K-analytic. Then the Banach space $C^b(X)$ of continuous bounded real-valued functions on X equipped with the Banach topology ξ generated by $\|f\| = \sup_{x\in X}|f(x)|$ is weakly K-analytic, i.e. the weak topology σ of $C^b(X)$

is K-analytic; see, for example, [219, Corollary 3.4]. Hence the weak topology of $C^b(X)$ admits a compact resolution.

Since X is separable, $C_p(X)$ admits a weaker metrizable topology. But then σ is analytic by Proposition 6.1.4. Hence $C^b(X) = C(\beta X)$ is separable, impossible as βX is non-metrizable. Note also that X is not a μ-space: Otherwise $C_p(X) = C_c(X)$ is barrelled by Proposition 2.4.16, so by the closed graph theorem the identity map $I : C_c(X) \to (C(X), \xi)$ is continuous; hence X is compact. This provides a contradiction.

(iv) Since X is pseudocompact containing $\mathbb{N}$, C^*-embedded into X, we apply [61, Theorem 1] to get a quotient map from $C_p(X)$ onto the subspace $(\ell_\infty)_p$ of $\mathbb{R}^{\mathbb{N}}$. Clearly $(\ell_\infty)_p$ is covered by the sequence $[-n, n]^{\mathbb{N}}$ of compact sets. By construction of X it is clear that $C_p(X)$ is not projectively σ-compact. □

9.4 More about Bounded Resolutions for $C_p(X)$

A topological space Y has *tightness* $\mathfrak{m}$ (we denote $\mathrm{t}(Y) \le \mathfrak{m}$) if for each set $A \subset Y$, and every $x \in \overline{A}$, there exists a subset $B \subset A$ of cardinality $\mathfrak{m}$ such that $x \in \overline{B}$. If this holds for $\mathfrak{m} = \aleph_0$, we say, as usual, that Y has *countable tightness*. Recall the following theorem from [510] (see also [32, Theorem II.1.1]).

Proposition 9.4.1 (Arkhangel'skii–Pytkeev) $\mathrm{t}(C_p(X)) \le \mathfrak{m}$ *if and only if* $\ell(X^n) \le \mathfrak{m}$ *for each* $n \in \mathbb{N}$, *where* $\ell(X)$ *denotes the Lindelöf number of* X. *Hence, if* X *is a Lindelöf* Σ*-space, the space* $C_p(X)$ *has countable tightness.*

We recall also the following result due to Asanov [40]; see also [32, Theorem I.4.1].

Theorem 9.4.2 (Asanov) *For every completely regular Hausdorff space* X *we have* $\mathrm{t}(X^n) \le \ell(C_p(X))$ *for each* $n \in \mathbb{N}$.

Hence, if $C_p(X)$ is Lindelöf, every finite product of X has countable tightness.

A family $\mathcal{U}$ of non-empty open subsets of a topological space X is called a π*-base* if for every non-empty open set V in X there exists $U \in \mathcal{U}$ such that $U \subset V$. A family $\mathcal{U}$ of non-empty open subsets in X is called a *local* π*-base at* x if for each neighbourhood V of x there exists $U \in \mathcal{U}$ such that $U \subset V$. Set

$$\pi\chi(x, X) := \min\{|\mathcal{U}| : \mathcal{U} \text{ is a local } \pi\text{-base of } x\} + \aleph_0.$$

Let $\pi\chi(X) := \sup_{x \in X} \pi\chi(x, X)$.

The following result of Shapirovsky [555] has numerous consequences.

Proposition 9.4.3 *If* X *is a compact space such that* $\mathrm{t}(X) \le \mathfrak{m}$, *the space* X *has a* π*-base such that* $\pi\chi(X) \le \mathfrak{m}$. *Consequently, every compact space with countable tightness has a countable local* π*-base at each point* x.

Proposition 9.4.3 is used to prove following Tkachuk's [593, Proposition 2.1]; see also [594] and [588].

Proposition 9.4.4 *A topological group G which embeds (as a topological space) into a compact space K of countable tightness is metrizable.*

Proof We may assume that G is dense in K and $t(K) \leq \aleph_0$. By Proposition 9.4.3 we have $\pi\chi(K) \leq \aleph_0$. Then $\pi\chi(x, G) = \pi\chi(x, K) \leq \aleph_0$ for each $x \in G$. Since G is a topological group, $\chi(x, G) = \pi\chi(x, G) \leq \aleph_0$ for any $x \in G$, where $\chi(x, G)$ denotes the character of x; see, for example, [39, Proposition 5.2.6]. This clearly implies that G is metrizable. □

To prove Velichko's Theorem 9.4.8 we need three lemmas; see [593].

Lemma 9.4.5 *Let $(F_n)_n$ be an increasing sequence of subsets of X. Let $y \in X \setminus \bigcup_n F_n$, and let Z be a compact subset of $C_p(X)$ contained in $\{f \in C_p(X) : f(y) = 0\}$. If there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $f_n \in Z$ such that $|f_n(x)| \geqslant \varepsilon$ for each $x \in F_n$, then $y \notin \overline{\bigcup_n F_n}$.*

Proof Let f be an adherent point of the sequence $(f_n)_n$. Then $|f(x)| \geqslant \varepsilon$ for each $x \in \bigcup_n F_n$. By the continuity of f, if $z \in \overline{\bigcup_n F_n}$, then $|f(z)| \geqslant \varepsilon$. Since $f(y) = 0$, we conclude $y \notin \overline{\bigcup_n F_n}$. □

Lemma 9.4.6 *Let $(F_n)_n$ be a sequence of closed subsets of X. Assume $C_p(X)$ is σ-compact. Then $\bigcup_n F_n$ is a closed subset of X. Hence, if $C_p(X)$ is σ-compact, the space X is a P-space, i.e. every G_δ-set in X is open.*

Proof Without loss of generality we may assume that $(F_n)_n$ is increasing. Choose

$$y \in X \setminus \bigcup_n F_n.$$

Let $(W_n)_n$ be an increasing sequence of compact sets covering $C_p(X)$. If $Z_m := \{f \in W_m : f(y) = 0\}$, then

$$\{f \in C_p(X) : f(y) = 0\} = \bigcup_m Z_m. \tag{9.3}$$

Assume that for each $m \in \mathbb{N}$ the set Z_m does not verify Lemma 9.3.5. Then for $\varepsilon = 2^{-m}$ we find n_m such that for each $f \in Z_m$ there exists $y_{f,m} \in F_{n_m}$ such that

$$\left|f(y_{f,m})\right| < 2^{-m}. \tag{9.4}$$

Let $C_m := F_{n_m}$ and let $g_m : X \to [0, 2^{-m}]$ be a continuous function such that $g_m(y) = 0$ and $g_m(C_m) = \{2^{-m}\}$. Define a continuous function $g : X \to [0, 1]$ by $g(x) := \Sigma\{g_n(x) : n \in \mathbb{N}\}$. Then we have

$$g(y) = 0. \tag{9.5}$$

Note that $g \notin Z_m$ for each $m \in \mathbb{N}$. Indeed, otherwise

$$g(y_{g,m}) \geqslant g_m(y_{g,m}) = 2^{-m}. \tag{9.6}$$

Therefore

$$g \notin \bigcup_m Z_m. \tag{9.7}$$

This finally yields a contradiction. Therefore there exists m such that $Z := Z_m$ verifies Lemma 9.4.5 for $\varepsilon = 2^{-m}$. Then $y \notin \overline{\bigcup_n F_n}$, proving that $\bigcup_n F_n$ is a closed subset of X. □

Lemma 9.4.7 *Let $\{x_n : n \in \mathbb{N}\}$ be an infinite set in a P-space X. Then for a sequence $(\alpha_n)_n$ of real numbers there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x_n) = \alpha_n$ for $n \in \mathbb{N}$.*

Proof Clearly all subsets of $\{x_n : n \in \mathbb{N}\}$ are closed. Therefore, by an easy induction, there exists a sequence of open sets $(U_n)_n$ such that each set U_n is an open neighbourhood of x_n and $\{\overline{U_n} : n \in \mathbb{N}\}$ is a family of pairwise disjoint subsets of X. The family $\{\overline{U_n} : n \in \mathbb{N}\}$ is discrete. Indeed, if $z \notin \bigcup_n \overline{U_n}$, then

$$W := X \setminus \bigcup_n \overline{U_n}$$

is an open neighbourhood of z such that $W \cap \overline{U_n} = \emptyset$ for $n \in \mathbb{N}$. Moreover, if $z \in \overline{U_p}$, then $z \notin \bigcup_{n \neq p} \overline{U_n}$ and $W := X \setminus \bigcup_{n \neq p} \overline{U_n}$ is an open neighbourhood of z such that $W \cap \overline{U_n} = \emptyset$ for $n \in \mathbb{N} \setminus \{p\}$. Since for each $n \in \mathbb{N}$ there exists a continuous function $\varphi_n : X \to \mathbb{R}$ such that $\varphi_n(x_n) = \alpha_n$, and $\varphi_n(X \setminus U_n) = \{0\}$, $|\varphi_n(x)| \leq |\alpha_n|$ for each $x \in X$, the continuous function $f : X \to \mathbb{R}$, defined by

$$f(x) := \sum_n \varphi_n(x),$$

satisfies $f(x_n) = \alpha_n$ for each $n \in \mathbb{N}$. □

We are ready to prove Theorem 9.4.8; see [32, Theorem I.2.1]. Recall that a subset F of X is topologically bounded if $f(F)$ is bounded for each $f \in C_p(X)$.

Theorem 9.4.8 (Velichko) *The space $C_p(X)$ is σ-compact if and only if X is finite.*

Proof Let $(W_n)_n$ be an increasing sequence of compact subsets of $C_p(X)$ covering $C_p(X)$. By Lemmas 9.4.7 and 9.4.6 every topologically bounded subset of X is finite. By Proposition 2.4.18 the space $C_p(X)$ is barrelled. Since the absolutely convex closed envelope B_n of each W_n is a bounded set, we apply Proposition 2.4.8 to deduce that

$$\mathbb{R}^X = \overline{C_p(X)} = \overline{\bigcup_n B_n} = \bigcup_n \overline{B_n},$$

where the closure is taken in $\mathbb{R}^X$. By the Baire category theorem there exists $m \in \mathbb{N}$ such that $\overline{B_m}$ is a neighbourhood of zero in $\mathbb{R}^X$; hence X is finite. □

Theorem 9.4.8 was extended by Tkachuk and Shakhmatov [598] for σ-countably compact spaces $C_p(X)$, i.e. $C_p(X)$ is covered by a sequence of countably compact sets; see also [32, Theorem I.2.2]. Note however, according to Proposition 9.2.12, that $C_p(X)$ having a bounded resolution is angelic. Since for angelic spaces (relatively) countable compact sets and (relatively) compact sets are the same, we see that Tkachuk–Shakhmatov's result follows from Velichko's theorem. In fact we have the following:

Theorem 9.4.9 *The following assertions are equivalent for X.*

(i) *X is finite.*
(ii) *$C_p(X)$ is σ-compact.*
(iii) *$C_p(X)$ is σ-relatively countably compact.*
(iv) *$C_p(X)$ is σ-countably compact.*

A similar result was obtained in [590] for σ-bounded spaces $C_p(X)$, i.e. $C_p(X)$ is covered by a sequence of topologically bounded sets. The proof of Proposition 9.4.10 uses Corollary 9.2.15 and the argument from [36, Proposition 3.1].

Proposition 9.4.10 (Tkachuk) *If $C_p(X)$ is σ-bounded, the space X is pseudocompact and every countable set in X is discrete. Consequently, every compact set in X is finite.*

Proof Assume X is not pseudocompact. Hence X contains a copy of the discrete space $\mathbb{N}$, also named by $\mathbb{N}$, and the continuous map $T : C_p(X) \to C_p(\mathbb{N})$ defined by $T(f) := f|\mathbb{N}$ is surjective. Therefore $C_p(\mathbb{N})$ is σ-bounded, so $C_p(C_p(\mathbb{N}))$ has a bounded resolution. Since $C_p(\mathbb{N}^{\mathbb{N}}) \subset C_p(C_p(\mathbb{N}))$, it follows that $C_p(\mathbb{N}^{\mathbb{N}})$ has a bounded resolution, a contradiction with Corollary 9.2.15. Hence X is pseudocompact. Now we prove that every countable set in X is discrete. Let $D \subset X$ be a countable subset of X. By $C_p(D|X)$ we denote the subspace of $C_p(D)$ that is the image of $C_p(X)$ under the restriction map. As $C_p(D)$ is separable and metrizable, $C_p(D|X)$ is a cosmic space. Since $C_p(D)$ is σ-bounded, the space $C_p(D|X)$ is σ-bounded, and then $C_p(C_p(D|X))$ has a bounded resolution. By Corollary 9.2.13 $C_p(D|X)$ is σ-compact. Using a similar argument as in the proof of Lemma 9.4.6, we deduce that D is a P-space. Since D is countable, D must be discrete. □

Proposition 9.4.10 shows that a realcompact space X is finite if and only if $C_p(X)$ is σ-bounded. Indeed, since every realcompact pseudocompact space is compact, we may apply Proposition 9.4.10. Note that, by [28, Proposition 9.31], see also [36, Remark], there exists an infinite space X such that $C_p(X)$ is σ-bounded.

Corollary 9.4.11 *Let $C_p(X)$ be a σ-bounded space. Then X is countable if and only if X is a K-analytic space.*

Proof If X is K-analytic, the space X admits a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Since, by Proposition 9.4.10, every compact set in X is finite, the space $X = \bigcup_\alpha K_\alpha$ is countable by Proposition 3.2.1. The converse is obvious. □

Using Theorems 9.2.3 and 9.4.8 we have: If $C_p(C_p(X))$ is analytic, then X is finite. Indeed, the analyticity of $C_p(C_p(X))$ implies that $C_p(X)$ is σ-compact, and then Theorem 9.4.8 applies. We have even more: If $C_p(C_p(X))$ is K-analytic, the space X is finite by [32, IV.9.21]. We note also the following:

Corollary 9.4.12 *For a realcompact space X the space $C_p\left(C_p(X)\right)$ has a bounded resolution if and only if X is finite. Hence, if $C_p(C_p(X))$ is K-analytic, the space X is finite.*

Proof Assume that $C_p\left(C_p(X)\right)$ has a bounded resolution. Then by Corollary 9.2.13 the space $C_p\left(C_p(X)\right)$ is K-analytic-framed in $\mathbb{R}^{C(X)}$. Hence there is a K-analytic space Y such that

$$C_p\left(C_p(X)\right) \subseteq Y \subseteq \mathbb{R}^{C(X)}.$$

Then every compact subset of X is finite; see [36, Corollary 3.4]. Since

$$X \subseteq Y \subseteq \mathbb{R}^{C(X)}$$

and X is realcompact, X is a closed subspace of Y. Hence X is a K-analytic space whose compact sets are finite. Thus X is countable by Proposition 3.2.1. Consequently, $C_p(X)$ is a separable metric space, hence a cosmic space. Applying Corollary 9.2.13 we derive that $C_p(X)$ is σ-compact. By Theorem 9.4.8 X is finite. Conversely, if X is finite, $C_p\left(C_p(X)\right)$ has a bounded resolution. □

Corollary 9.4.13 provides another Velichko-type result; this follows also from Proposition 9.4.10.

Corollary 9.4.13 *A realcompact space X is finite if and only if $C_p(X)$ is σ-bounded.*

Proof If $C_p(X)$ is covered by a sequence of topologically bounded sets, $C_p\left(C_p(X)\right)$ has a bounded resolution; see the proof of Corollary 9.2.13, (i) $\Rightarrow$ (ii). Now Corollary 9.4.12 applies. □

It is known that $C_p(X)$ is a Fréchet space if and only if X is countable and discrete. Applying Theorem 6.2.4 we characterize Fréchet spaces $C_p(X)$ as spaces having a special resolution. We need the following two results, the first one due to Argyros–Negrepontis [19], see also [31, Theorem 7.25], and the other one follows from [592].

Proposition 9.4.14 *If X is a Gul'ko compact space, the density, weight, and the Suslin number of X coincide.*

Proposition 9.4.15 *If $C_p(X)$ has a compact resolution swallowing compact sets, the same property holds for $C_p(\upsilon X)$.*

Proof Let $T : C_p(\upsilon X) \to C_p(X)$ be the restriction map from $C_p(\upsilon X)$ onto $C_p(X)$. It is enough to show that for every compact set $K \subset C_p(X)$ the set $T^{-1}(K)$ is compact in $C_p(\upsilon X)$. Let $A \subset T^{-1}(K)$ be a countable subset. $T|A : A \to T(A)$ is a homeomorphism by Lemma 9.2.9. Therefore A has a cluster point in $W := T^{-1}(K)$, so W is countably compact. By Proposition 9.2.12 the space $C_p(X)$ is angelic. Since, by Lemma 9.2.10, the space $C_p(\upsilon X)$ is angelic (in angelic spaces countably compact sets are compact), we obtain that W is compact. □

Now we prove the following result [592, Theorem 3.7].

Theorem 9.4.16 (Tkachuk) *If $C_p(X)$ admits a compact resolution swallowing compact sets, X is countable and discrete.*

Proof Since $C_p(X)$ has a compact resolution, $C_p(X)$ is K-analytic, $X \subset C_p(C_p(X))$ is angelic by Theorem 4.3.3. We claim that every compact set in X is finite. Assume that K is an infinite compact subset of X. As X is angelic, there exists a sequence $\{x_n : n \in \mathbb{N}\}$ of different elements in K which converges to $x \in K$. Since every compact countable subset of K is metrizable by Proposition 9.4.14, the compact set $K_0 := \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is metrizable. There exists a continuous linear extender $T : C_p(K_0) \to C_p(X)$ such that $T(f)|K_0 = f$ for each $f \in C(K_0)$; see [31, Proposition 4.1] (or [37, 195, 552, 623] if X is metrizable). Hence $C_p(K_0)$ embeds in $C_p(X)$ as a closed subset. Consequently, $C_p(K_0)$ admits a compact resolution swallowing compact sets. Now Theorem 6.2.4 applies to deduce that $C_p(K_0)$ is a Polish space, and then K_0 is discrete, hence finite, a contradiction. We proved that every compact set in X is finite.

Since, by Proposition 9.4.15, the space $C_p(\upsilon X)$ admits a compact resolution swallowing compact sets, every compact set in υX is finite. As υX is a Lindelöf Σ-space by Proposition 9.2.11, we use Corollary 3.1.11 to get that υX is countable. Hence $C_p(X)$ is a separable and metrizable lcs admitting a compact resolution swallowing compact sets. Theorem 6.2.4 applies to deduce that $C_p(X)$ is a Polish space; hence X is countable and discrete; see [32, Corollary I.3.3]. □

The above proof uses some extension of the Tietze–Urysohn theorem. We refer the reader to articles [306, 627] and [305] for more information. For example, in [306, Corollary J] it is shown that there exist compact separable spaces X having closed subspaces Y which contain uncountable disjoint collections of relatively open sets and there exist no continuous extenders from $C_p(Y)$ into $C_p(X)$.

A lcs E is called *web-bounded* if E admits a family $\{C_\alpha : \alpha \in \Sigma\}$ of subsets of E covering E for some non-empty subset $\Sigma \subset \mathbb{N}^\mathbb{N}$ such that, if $\alpha = (n_k) \in \Sigma$ and $x_k \in C_{n_1,n_2,\ldots,n_k}$, the sequence $(x_k)_k$ is bounded in E. Since $C_\alpha \subset C_{n_1,n_2,\ldots,n_k}$, each set C_α is bounded (so the family $\{C_\alpha : \alpha \in \Sigma\}$ is bounded). Indeed, it is enough

to see that for each neighbourhood of zero U in E there exists $k \in \mathbb{N}$ such that $C_{n_1,n_2,\ldots,n_k} \subset kU$.

A topological space X is called *almost quasi-Suslin* if there exist a non-empty set $\Omega \subset \mathbb{N}^{\mathbb{N}}$ and a map T from Ω into subsets of E such that, if $\alpha_k \to \alpha$ in Ω and $x_k \in T(\alpha_k)$, then $(x_k)_k$ has an adherent point belonging to $T(\alpha)$. If the same holds for $\Omega = \mathbb{N}^{\mathbb{N}}$, the space X is known already as a quasi-Suslin space. The following simple observation can be found in [204].

Proposition 9.4.17 *If X is almost quasi-Suslin(quasi-Suslin), the space υX is Lindelöf Σ (K-analytic). Hence X is a Lindelöf Σ-space if and only if X is Lindelöf and almost quasi-Suslin.*

Proof We prove only the case if X is an almost quasi-Suslin space. Since every $T(\alpha)$ is countably compact, its closure $\overline{T(\alpha)}$ in υX is compact. It is easy to see that the map $\alpha \to \overline{T(\alpha)}$ is (usco), so $Z := \bigcup_{\alpha\in\Sigma} \overline{T(\alpha)}$ is a Lindelöf Σ-space. Since $X \subset Z \subset \upsilon X$, the space $Z = \upsilon Z = \upsilon X$ is a Lindelöf Σ-space. □

A topological space X is called *web-bounding* if $X = \overline{\bigcup\{A_\alpha : \alpha \in \Sigma\}}$ for some non-empty subset $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and, if $\alpha = (n_k) \in \Sigma$ and $x_k \in C_{n_1,n_2,\ldots,n_k}$ for all $k \in \mathbb{N}$, the set $\{x_k : k \in \mathbb{N}\}$ is topologically bounded; see [481]. If $X = \bigcup\{A_\alpha : \alpha \in \Sigma\}$, X will be called *strongly web-bounding*.

We need the following result of Nagami [460] which supplements Proposition 3.1.8 above; see also [32, Proposition IV.9.2] or [594, Theorem 2.1].

Proposition 9.4.18 *A topological space X is a Lindelöf Σ-space if and only if in some (hence in any) compactification bX of X there is a countable family $\mathcal{F}$ of compact sets such that, if $x \in X$ and $y \in bX \setminus X$ there exists $B \in \mathcal{F}$ for which $x \in B$ and $y \notin B$.*

In [88] Blasco proved that for a separable space X its realcompactification υX is a Lindelöf space if and only if every base in X is complete. Spaces X for which υX is a Lindelöf space were called *pseudo-Lindelöf*; see also [314].

Next Theorem 9.4.19 characterizes those spaces X for which the realcompactification υX is a Lindelöf Σ-space. We are ready to prove the following theorem [336].

Theorem 9.4.19 *(A) For a topological space X the following assertions are equivalent:*

(i) υX *is a Lindelöf Σ-space.*
(ii) X *is strongly web-bounding.*
(iii) $C_p(X)$ *is web-bounded.*
(iv) $L_p(X)$ *is web-bounded.*
(v) $C_p(X)$ *is a dense subspace of a lcs which is a Lindelöf Σ space.*

(B) For a topological space X the following conditions are equivalent:

(i) $C_p(X)$ *is web-bounded and X is realcompact.*
(ii) $L_p(X)$ *is a Lindelöf Σ-space.*

Proof (A) (ii) $\Rightarrow$ (i): Assume first that X is strongly web-bounding and let $\{A_\alpha : \alpha \in \Sigma\}$ be a covering of X verifying the web-bounding condition. Then, for each $f \in C(X)$ and each $\alpha = (n_k)_{k\in\mathbb{N}} \in \Sigma$ there exists $k \in \mathbb{N}$ such that $f(C_{n_1 n_2 \cdots n_k}) \subset [-k, k]$. The set

$$Z := \{f \in \mathbb{R}^X : \forall \alpha = (n_i) \in \Sigma, \exists k \in \mathbb{N}, f(C_{n_1 n_2 \cdots n_k}) \subset [-k, k]\}$$

verifies the condition

$$C_p(X) \subset Z \subset \mathbb{R}^X. \tag{9.8}$$

Endow Z with the topology induced by $\mathbb{R}^X$. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ be the two-point compactification of $\mathbb{R}$ homeomorphic to $[0, 1]$. Then $\overline{\mathbb{R}}^X$ is a compactification of Z. For each $\alpha = (n_i)_{i\in\mathbb{N}} \in \Sigma$ and $k \in \mathbb{N}$ let $F_{\alpha|k} = F_{n_1 n_2 \cdots n_k}$ be the closure in $\overline{\mathbb{R}}^X$ of the set

$$\{f \in \mathbb{R}^X : f(C_{n_1 n_2 \cdots n_k}) \subset [-k, k]\}.$$

The family

$$S := \{F_{\alpha|k}, \alpha \in \Sigma, k \in \mathbb{N}\}$$

is a countable family of compact subsets of $\overline{R}^X$. Clearly

$$\overline{\mathbb{R}}^X \backslash Z = \left(\overline{\mathbb{R}}^X \backslash \mathbb{R}^X\right) \cup \left(\mathbb{R}^X \backslash Z\right).$$

Take arbitrary

$$g \in \overline{Z}^{\overline{\mathbb{R}}^X} \backslash Z.$$

If $g \in \overline{\mathbb{R}}^X \backslash \mathbb{R}^X$, there exists $a \in X$ such that $g(a) \in \{-\infty, +\infty\}$. There exists $\alpha = (n_i) \in \Sigma$ such that $a \in A_\alpha$. Then, from

$$g(C_{n_1 n_2 \cdots n_k}) \cap \{-\infty, +\infty\} \neq \emptyset,$$

it follows that $g \notin F_{\alpha|k}$ for each $k \in \mathbb{N}$.

If $g \in \mathbb{R}^X \backslash Z$, there exists $\alpha = (n_i) \in \Sigma$ such that for each $k \in \mathbb{N}$ we have

$$g(C_{n_1 n_2 \cdots n_k}) \nsubseteq [-k, k].$$

Also we have that $g \notin F_{\alpha|k}$ for each $k \in \mathbb{N}$. Therefore, if $f \in Z$ and $g \in \overline{\mathbb{R}}^X \backslash Z$, there exists $\alpha = (n_i) \in \Sigma$ such that $g \notin F_{\alpha|k}$ for each $k \in \mathbb{N}$. From the definition of Z it follows that for this α there exists $k \in \mathbb{N}$ such that $f \in F_{\alpha|k}$. Therefore,

by Proposition 9.4.18, we deduce that Z is a Lindelöf Σ-space. Finally, we apply Proposition 9.2.11 to get that υX is a Lindelöf Σ-space.

(i) $\Rightarrow$ (ii): Assume υX is a Lindelöf Σ-space. Then there exist $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and a (usco) compact-valued map T from Σ into υX covering υX. Set $A_\alpha := T(\alpha) \cap X$ for $\alpha = (n_k) \in \Sigma$. Take a sequence $f_k \in C_{n_1,n_2,\ldots,n_k}$. There exists a sequence $(\alpha_k)_k$ in Σ that converges to α such that $f_k \in T(\alpha_k)$ for each $k \in \mathbb{N}$. Since T is (usco), the set $\{f_k : k \in \mathbb{N}\}$ is countably compact; hence it is topologically bounded.

(iii) $\Leftrightarrow$ (i): To prove this equivalence replace in (9.8) X by $C_p(X)$. If $C_p(X)$ is strongly web-bounding, then

$$C_p(C_p(X)) \subset Z \subset \mathbb{R}^{C_p(X)},$$

and Z is Lindelöf Σ. If we need to assume that $C_p(X)$ is *only* web-bounded, one should have *more space* for Z to be a Lindelöf Σ-space. Indeed, if $C_p(X)$ is web-bounded, we deduce (analogously) that there exists a Lindelöf Σ-space Z such that

$$L_p(X) \subset Z \subset \mathbb{R}^{C_p(X)}.$$

Since $X \subset L_p(X)$, then $X \subset Z \subset \mathbb{R}^{C_p(X)}$. Now the classical procedure, see the proof of [32, Theorem IV.9.5], applies to show that υX is a Lindelöf Σ-space. Indeed, if Y is the closure of X in Z, the space Y is a Lindelöf Σ-space. Since every real-valued function on X can be continuously extended to $\mathbb{R}^{C_p(X)}$, then $\upsilon X = \upsilon Y = Y$ is a Lindelöf Σ-space. Conversely, if υX is a Lindelöf Σ-space, by Proposition 9.2.11 there exists a Lindelöf Σ-space Z such that

$$C_p(X) \subset Z \subset \mathbb{R}^X.$$

Then clearly $C_p(X)$ is web-bounded.

To prove (iii) $\Leftrightarrow$ (v) in (A) it is enough to apply Claim 1.

Claim 1. For a lcs E the following conditions are equivalent:

(a) E is web-bounded.
(b) The space $(E, \sigma(E, E'))$ is embedded in a locally convex Lindelöf Σ-space $(W, \sigma(W, E'))$, where $E \subset W \subset (E')^*$.
(c) $(E', \sigma(E', E))$ is web-bounded.
(d) The space $(E', \sigma(E', E))$ is embedded in a locally convex Lindelöf Σ-space $(Z, \sigma(Z, E))$, where $E' \subset Z \subset E^*$.

Indeed, $(a) \Rightarrow (d)$: Assume E is web-bounded, and that $\{A_\alpha : \alpha \in \Sigma\}$ is a covering of E such that, if $\alpha = (n_k) \in \Sigma$ and $x_k \in C_{n_1 n_2 \cdots n_k}$, then $(x_k)_k$ is bounded. Clearly, for each $\alpha \in \Sigma$ and each $x' \in E'$ there exists $k \in \mathbb{N}$ such that $x'(C_{n_1 n_2 \cdots n_k}) \subset [-k, k]$.

Set

$$Z := \{x' \in E^* : \forall \alpha = (n_i) \in \Sigma, \exists k \in \mathbb{N}, x'(C_{n_1 n_2 \cdots n_k}) \subset [-k, k]\}.$$

Since $(C_{n_1,n_2,\dots,n_k})_k$ is decreasing, Z is a vector subspace of E^* and

$$E' \subset Z \subset E^* \subset \mathbb{R}^E.$$

Similarly, as we proved in (A), we deduce that $(Z, \sigma(Z, E))$ is a Lindelöf Σ-space.

$(d) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (b)$: By the hypothesis $(E', \sigma(E', E))$ is web-bounded, and if we apply to this space the implication $(a) \Rightarrow (d)$ we note that the weak dual $(E, \sigma(E, E'))$ is embedded in a locally convex Lindelöf Σ-space $(W, \sigma(W, E'))$.

$(b) \Rightarrow (a)$ is trivial. Claim 1 is proved.

We continue the proof of Theorem 9.4.19.

(iii) $\Leftrightarrow$ (iv): Since $L_p(X)' = C_p(X)$ we apply claim 1 to prove the equivalence.

(B) Assume $C_p(X)$ is a web-bounded with realcompact X. By (A) the space $X = \upsilon X$ is a Lindelöf Σ-space. Then by Proposition 6.5.2 the space $L_p(X)$ is a Lindelöf Σ-space. This proves (i) $\Rightarrow$ (ii).

For the converse, if $L_p(X)$ is a Lindelöf Σ-space, the space $X \subset L_p(X)$ (as a closed subspace) is a Lindelöf Σ-space. Finally, by (A) the space $C_p(X)$ is web-bounded. □

We will need also the following general fact.

Lemma 9.4.20 *If E is a lcs such that $(E, \sigma(E, E'))$ has countable tightness, the weak* dual $(E', \sigma(E', E))$ is a realcompact space.*

Proof By Proposition 12.1.3 it is enough to show that every linear functional f on E that is $\sigma(E, E')$-continuous on each $\sigma(E, E')$-closed separable vector subspace is continuous on E. We prove that the kernel $K := \{x \in E : f(x) = 0\}$ of f is a closed subspace in E; this yields the continuity of f. Indeed, if $y \in \overline{K}$, there exists a countable set $D \subset K$ such that $y \in \overline{D}$ (where the closure is taken in $\sigma(E, E')$). By the assumption the restriction map $f|\overline{\text{span}(D)}$ is $\sigma(E, E')$-continuous. This shows that

$$f(y) \in \overline{f(\text{span}(D))} \subset \overline{f(K)} = \{0\}.$$

Hence $y \in K$ and $f \in E'$. □

Note the following proposition; see also [593, Proposition 2.11].

Proposition 9.4.21 *Let X be a Lindelöf P-space. Then X is countable if and only if $L_p(X)$ has countable tightness if and only if $C_p(X)$ is realcompact.*

Proof First, assume that X is countable. Then $C_p(X)$ is a separable metrizable lcs. Hence $L_p(X) \subset C_p(C_p(X))$ has countable tightness again by Proposition 9.4.1.

Now assume that the *weak** dual $L_p(X)$ of $C_p(X)$ has countable tightness. By Lemma 9.4.20 the space $C_p(X)$ is a realcompact space. Since X is a P-space, the topological group $C_p(X, [-1, 1])$ is a countably compact space; see, for example, [26]. Consequently, its closure K in $(C_p(X))$ is compact (as $C_p(X)$ is realcompact). Moreover K has countable tightness by Proposition 9.4.1. Since $C_p(X)$ embeds in

$C_p(X, [-1, 1])$ (take, e.g. $f \to f/(1+|f|)$, $f \in C(X)$), we apply Proposition 9.4.4 to deduce that $C_p(X)$ is metrizable. Hence X is countable. □

Since $C_p(X)$ is angelic if υX is a Lindelöf Σ-space (Theorem 4.3.3 and Lemma 9.2.10), Theorem 9.4.19 yields the following:

Corollary 9.4.22 *Every web-bounded space $C_p(X)$ is angelic.*

Since every topological space with a compact resolution is quasi-Suslin and X is closed in $L_p(X)$, an immediate consequence of Theorem 9.4.19, Proposition 6.5.2 combined with Proposition 9.4.17, implies the following dual version of Theorem 9.2.17.

Corollary 9.4.23 *The space $L_p(\upsilon X)$ is almost quasi-Suslin if and only if it is a Lindelöf Σ-space. The space $L_p(\upsilon X)$ is K-analytic if and only if it has a compact resolution.*

Now we prove the following:

Theorem 9.4.24 *For a Baire lcs E the following statements are equivalent.*

(i) E *is metrizable.*
(ii) $(E', \sigma(E', E))$ *is an almost quasi-Suslin space.*
(iii) E *is web-bounded.*

Proof (i) $\Rightarrow$ (ii): If $(U_n)_n$ is a decreasing basis of absolutely convex neighbourhoods of zero in E, its polars $K_n := (U_n)^\circ$ are $\sigma(E', E)$-compact sets in $(E', \sigma(E', E))$ and $E' = \bigcup_n K_n$. Therefore $(E', \sigma(E', E))$ is σ-compact.

(ii) $\Rightarrow$ (iii): Since the realcompactification $\upsilon(E'_\sigma)$ is a Lindelöf Σ-space, by Theorem 9.4.19 the space $C_p(E'_\sigma)$ is web-bounded, where $E'_\sigma := (E', \sigma(E', E))$. By $(E, \sigma(E, E')) \subset C_p(E'_\sigma)$ the space E is a web-bounded lcs.

(iii) $\Rightarrow$ (i): By the assumption there exists a set $\Sigma \subset \mathbb{N}^{\mathbb{N}}$, and a family $\{A_\alpha : \alpha \in \Sigma\}$ covering E such that for each $\alpha = (n_k) \in \Sigma$ and each

$$x_k \in C_{n_1, n_2, \ldots, n_k}$$

the sequence $(x_k)_k$ is bounded in E. Therefore, for each neighbourhood of zero W and each $\alpha = (n_k) \in \Sigma$ there exists $r \in \mathbb{N}$ such that $C_r \subset rW$, where $C_r := C_{n_1, n_2, \ldots n_r}$. Since

$$E = \bigcup_{n_1} C_{n_1}, \quad C_{n_1} = \bigcup_{n_2} C_{n_1, n_2},$$

etc., and since E is a Baire space, there exists $\alpha = (n_k) \in \Sigma$ such that $\overline{C_k} - \overline{C_k}$ is a neighbourhood of zero for each $k \in \mathbb{N}$. Let U and V be closed absolutely convex neighbourhoods of zero such that $V - V \subset U$. There exists $k \in \mathbb{N}$ such that

$$\overline{C_k} - \overline{C_k} \subset kV - kV \subset kU.$$

Hence the sets

$$W_k := k^{-1}(\overline{C_k} - \overline{C_k})$$

compose a countable basis of neighbourhoods of zero in E, so E is metrizable. □

Corollary 9.4.25 *Let $(E_i)_{i\in I}$ be a family of non-zero web-bounded lcs. Then $E := \prod_{i\in I} E_i$ is web-bounded if and only if $|I| \leq \aleph_0$.*

Proof Assume I is uncountable and E is web-bounded. Then E contains a subspace of the form $\mathbb{R}^A$ for some uncountable set A. Endow A with the discrete topology. Since $\mathbb{R}^A = C_p(A)$, the Baire space $\mathbb{R}^A$ is web-bounded. By Theorem 9.4.24 $\mathbb{R}^A$ is metrizable, so A is countable, a contradiction. □

Corollary 9.4.26 *Let $E = \prod_{t\in T} E_t$ be the product of Fréchet spaces. Then T is countable if and only if the space $(E', \sigma(E', E))$ is K-analytic if and only if $\upsilon(E', \sigma(E', E))$ is a Lindelöf Σ-space if and only if $(E', \sigma(E', E))$ is almost quasi-Suslin.*

Proof Assume T is uncountable and E'_σ is almost quasi-Suslin. Then $\upsilon E'_\sigma$ is a Lindelöf Σ-space by Proposition 9.4.17. By Claim 1 in Theorem 9.4.19 we deduce that E is web-bounded, and then we apply Theorem 9.4.24 to reach a contradiction. □

In order to prove Proposition 9.4.32 we need the following lemma due to Tkachuk [591].

Lemma 9.4.27 *Assume that $C_p(X) = \bigcup_n C_n$. Then there exists $f \in C_p(X)$, $\epsilon > 0$ and $n \in \mathbb{N}$ such that $(C_n + f) \cap C(X, (-\epsilon, \epsilon))$ is dense in $C_u(X, (-\epsilon, \epsilon))$ endowed with the uniform topology.*

Proof For each $n \in \mathbb{N}$ set

$$B_n := C_n \cap C_p^b(X),$$

where $C_p^b(X)$ is a subspace of $C_p(X)$ of continuous and bounded functions on X. Since $C_u^b(X) = \bigcup_n B_n$, and $C_u^b(X)$ is a Fréchet space, by the Baire category theorem there exist $n \in \mathbb{N}$, $g \in C^b(X)$, and $\epsilon > 0$ such that B_n is dense in

$$K_\epsilon(g) := \{h \in C^b(X) : \sup_{x\in X} |h(x) - g(x)| < \epsilon\}.$$

Then $B_n + f$ is dense in $C_u(X, (-\epsilon, \epsilon))$, where $f := -g$. Consequently,

$$(B_n + f) \cap C(X, (-\epsilon, \epsilon)) \subset (C_n + f) \cap C(X, (-\epsilon, \epsilon))$$

is dense in $C_u(X, (-\epsilon, \epsilon))$. This completes the proof. □

Lemma 9.4.27 yields the following two interesting corollaries, [591].

Corollary 9.4.28 *If $C_p(X)$ is covered by a sequence $(C_n)_n$ of closed subsets, there exists $m \in \mathbb{N}$ such that C_m contains a subset homeomorphic to $C_p(X)$.*

Corollary 9.4.29 *If $C_p(X)$ is covered by a sequence of metrizable closed subsets, the space $C_p(X)$ is metrizable.*

Motivated by Velichko's Theorem 9.4.8 one may ask about conditions for $C_p(X)$ to have a fundamental sequence of bounded sets. Since $C_p(X)$ is always a quasibarrelled space [322], this question is in fact to determine when $C_p(X)$ is a (DF)-space. Corollary 9.4.28 applies to deduce the following:

Proposition 9.4.30 *If $C_p(X)$ admits a fundamental sequence of bounded sets, the space X is finite.*

A family $\mathcal{U}$ is called an ω-cover of X if for any finite subset A of X there exists $U \in \mathcal{U}$ such that $A \subset U$. We will use also the following characterization of countable tightness of $C_p(X)$; see [273, 435].

Proposition 9.4.31 *$C_p(X)$ has countable tightness if and only if every open ω-cover of X contains a countable ω-subcover.*

If a topological space X has the property as in Proposition 9.4.31, we say that X is an ω-space. The following result is due to Tkachuk [591].

Proposition 9.4.32 (Tkachuk) *If $C_p(X)$ is covered by a sequence $(C_n)_n$ of subsets of countable tightness, the space $C_p(X)$ has countable tightness.*

Proof By Lemma 9.4.27 there exist $n \in \mathbb{N}$, $\epsilon > 0$, and $f \in C_p(X)$ such that

$$D := (f + C_n) \cap C(X, (-\epsilon, \epsilon))$$

is dense in $C_u(X, (-\epsilon, \epsilon))$. We show that every open ω-cover in X contains a countable ω-subcover. Let $\mathcal{U}$ be an ω-cover of non-empty open sets in X. Set

$$P := \{f \in D : \exists U \in \mathcal{U},\ f^{-1}(10^{-1}\epsilon, \epsilon) \subset U\}.$$

Note that $D \subset \overline{P}$, the closure in $C_p(X)$. Indeed, choose arbitrary $l \in D$, $\delta > 0$, and a finite set $x_1, x_2, \ldots x_n$ in X. The set

$$V(l) := \bigcap_{i=1}^{n} \{f \in C_p(X) : |l(x_i) - f(x_i)| < \delta\}$$

is a neighbourhood of l in $C_p(X)$. We need to show that there exists $h \in P$ contained in $V(l)$. There exists $U \in \mathcal{U}$ such that

$$\{x_1, x_2, \ldots, x_n\} \subset U.$$

For each $1 \leq i \leq n$ set $r_i := l(x_i)$. There exists a function $g \in C_p(X, (-\epsilon, \epsilon))$ such that $g(x_i) = r_i$ for each $1 \leq i \leq n$ and $g(X \setminus U) = \{0\}$. Since D is dense in $C_u(X, (-\epsilon, \epsilon))$, there exists $h \in D$ such that

$$\sup_{x \in X} |h(x) - g(x)| < \min\{\delta, 10^{-1}\epsilon\}.$$

Then clearly $h \in P$. On the other hand, $h \in V(l)$. The claim is proved.

Next, choose a function $f_0 \in D$ such that $f_0(x) > (10)^{-1}9\epsilon$ for each $x \in X$. Then $f_0 \in \overline{P}$. By the assumption there exists a countable subset $K \subset P$ such that $f_0 \in \overline{K}$. Since $K \subset P$, for each $s \in K$ we choose a set $U_s \in \mathcal{U}$ such that

$$s(10^{-1}\epsilon, \epsilon) \subset U_s.$$

Set $\mathcal{V} := \{U_s : s \in K\}$. Clearly, the family $\mathcal{V}$ is countable. The proof will be completed if we show that $\mathcal{V}$ is a ω-cover. Let $A := \{y_1, y_2, \ldots, y_n\} \subset X$. There exists $f_1 \subset K$ such that

$$|f_1(y_i) - f_0(y_i)| < 10^{-1}\epsilon$$

for each $1 \leq i \leq n$. Then

$$f_1(y_i) > (10)^{-1}8\epsilon > (10)^{-1}\epsilon$$

for each $1 \leq i \leq n$. This means that $A \subset U_{f_1}$. We proved that $\mathcal{V}$ is a ω-cover. □

From Theorem 9.4.19 (B) and Proposition 9.4.1 we know that, if $C_p(X)$ is web-bounded with countable tightness, $L_p(X)$ is a Lindelöf Σ-space. On the other hand, we have the following:

Example 9.4.33 Assume CH. There exists a compact space X of cardinality $\aleph_1$ such that $C_p(X)$ is Lindelöf and not a Lindelöf Σ-space, and $L_p(X)$ is web-bounded with countable tightness.

Proof Under CH Kunen constructed a compact scattered hereditarily separable space K of cardinality $\aleph_1$ such that the Banach space $C_c(K)$ is weakly (hereditarily) Lindelöf; see [502, 523]. Since K is a zero-dimensional space, by [32, Theorem IV.8.6] any finite product $C_p(K)^n$ is a Lindelöf space. Therefore, by Proposition 9.4.1, the space $L_p(K) \subset C_p(C_p(K))$ has countable tightness. We prove that $\upsilon C_p(K) = C_p(K)$ is not a Lindelöf Σ-space. As K is separable, it is enough to prove that K does not have a countable network (or equivalently K is not cosmic) because of Proposition 9.2.6. Assume K is a cosmic space. Then also $C_p(K)$ is cosmic by Proposition 9.2.5. Hence $C_p(K)$ is a separable space. Hence K admits a weaker metric topology. On the other hand, it is known that a metric compact scattered space is countable, a contradiction. □

Recall again that a compact space X is *scattered* if every closed subset $L \subset X$ has an isolated point in L. A point p is isolated in X if there exists a neighbourhood U of p in X such that $U \cap X = \{p\}$.

We refer the reader to [403], where examples of Corson compact spaces X such that $C_p(X)$ is not a Lindelöf Σ-space are presented; see also [7]. Example 9.4.34, see [31], provides additional restrictions on possible extensions of Theorem 9.4.19 (B).

Example 9.4.34 There exists a space Z that is not Lindelöf such that $C_p(Z)$ is K-analytic, $L_p(Z)$ has countable tightness, and $L_p(Z)$ is neither separable nor Lindelöf.

Proof Let Y be a Talagrand compact space such that $y \in Y$, $\beta Z = Y$, where $Z = Y \setminus \{y\}$, and Z is pseudocompact and not compact; see [31, Example 7.14] or [198, Part 8.4]. Hence $C_p(Z)$ is K-analytic and Z is not Lindelöf. Since Z is not Lindelöf, the space $C_p(Z)$ does not have countable tightness by Proposition 9.4.1. Also the space $L_p(Z)$ is not separable; otherwise there exists on $C_p(Z)$ a weaker metric topology, and then $C_p(Z)$, as K-analytic, must be analytic. Then, by Theorem 9.2.3 Z is σ-compact and hence Lindelöf, a contradiction. Clearly $L_p(Z)$ is not Lindelöf since it contains the closed subspace Z. On the other hand, $L_p(Z)$ has countable tightness, since $L_p(Z) \subset C_p(C_p(Z))$, and $C_p(C_p(Z))$ has countable tightness. □

9.5 Fundamental Bounded Resolutions for $C_p(X)$ and $C_c(X)$

This section is mostly based on results presented in [208].

First we recall the following easy:

Proposition 9.5.1 *For a Tychonoff space X the following assertions are equivalent:*

(i) $C_c(X)$ *is covered by a sequence of bounded sets.*
(ii) $C_c(X)$ *does not contain a copy of* $\mathbb{R}^{\mathbb{N}}$.

Indeed, if (i) holds, then (ii) holds also since $\mathbb{R}^{\mathbb{N}}$ is a Baire space not being normed. Conversely, if (ii) holds, then X is pseudocompact by Theorem 2.6.3, what implies that $C_c(X)$ admits a stronger Banach topology. This yields Claim (i). Note that the same fact holds if $C_c(X)$ is replaced by $C_p(X)$.

We need the following concept introduced by Banakh and Zdmoskyy in [67]. Recall that a family $\mathcal{N}$ of subsets of a topological space X is called a *cs*-network at a point* $x \in X$ if for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converging to x and for each neighbourhood O_x of x there is a set $N \in \mathcal{N}$ such that $x \in N \subseteq O_x$ and the set $\{n \in \mathbb{N} : x_n \in N\}$ is infinite; $\mathcal{D}$ is a *cs*-network* in X if $\mathcal{N}$ is a *cs**-network at each point $x \in X$.

We shall say that a resolution of bounded sets $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in a lcs E is *fundamental* if every bounded set in E is contained in some A_α.

Tkachuk proved Theorem 9.4.16 stating that the space $C_p(X)$ has a fundamental compact resolution if and only if X is countable and discrete. Hence, if for an infinite compact space K, the space $C_p(K)$ is K-analytic, then $C_p(K)$ has a compact resolution but it does not have a fundamental compact resolution.

Below we prove a similar result for $C_p(X)$ having a fundamental bounded resolution.

Let X be a Tychonoff space. Denote by $\mathfrak{F}(X)$ and $\mathfrak{K}(X)$ the families of all finite subsets and all compact subsets of X, respectively. For $\tau = \tau_p$ or $\tau = \tau_k$, we denote by $C_\tau(X)$ the space $C_p(X)$ or the space $C_c(X)$ and set $\tau(X) = \mathfrak{F}(X)$ or $\tau(X) = \mathfrak{K}(X)$, respectively. We need the following notion. For every $\alpha \in \mathbb{N}^{\mathbb{N}}$ and each $k \in \mathbb{N}$, set

$$I_k(\alpha) := \{\beta \in \mathbb{N}^{\mathbb{N}} : \ \beta(1) = \alpha(1), \ldots, \beta(k) = \alpha(k)\}.$$

Let $\mathcal{B}$ and $\mathcal{C}$ be two families of subsets of a set Ω. We shall say that $\mathcal{B}$ *swallows* $\mathcal{C}$ if for every $C \in \mathcal{C}$ there is a $B \in \mathcal{B}$ such that $C \subseteq B$.

Let X be a Tychonoff space and assume that $\tau \in \{\tau_p, \tau_k\}$. A family

$$\left\{U_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\right\}$$

of closed subsets of X is called τ*-framing* if for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, the sequence $\left\{U_{\alpha,n} : n \in \mathbb{N}\right\}$ is increasing and swallows $\tau(X)$, and for every $n \in \mathbb{N}$, $U_{\beta,n} \subseteq U_{\alpha,n}$ whenever $\alpha \leq \beta$.

We start with the following:

Proposition 9.5.2 *Let X be a Tychonoff space and let $\tau \in \{\tau_p, \tau_k\}$. If the space $C_\tau(X)$ has a fundamental bounded resolution, then $C_\tau(X)$ has a countable cs*-network at zero.*

Proof Let $\left\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ be a fundamental bounded resolution in $C_\tau(X)$. We show our result in several partial steps.

Claim 1. A subset Q of $C_\tau(X)$ is bounded if and only if there exists an increasing sequence $\{V_n : n \in \mathbb{N}\}$ consisting of closed subsets of X and swallowing $\tau(X)$ such that

$$\sup_{f \in Q} |f(x)| \leq n \text{ for all } x \in V_n.$$

Indeed, choose a bounded subset Q of $C_\tau(X)$. For every $n \in \mathbb{N}$, set

$$V_n = \left\{x \in X : \sup_{f \in Q} |f(x)| \leq n\right\}.$$

Clearly, all V_n are closed, $V_n \subseteq V_{n+1}$ for each $n \in \mathbb{N}$, and $\sup_{f \in Q} |f(x)| \leq n$ for all $x \in V_n$. If $K \in \tau(X)$, the τ-boundedness of Q implies that there is an $m \in \mathbb{N}$ such that $Q \subseteq [K; m]$, so that $K \subseteq V_m$. This shows that $\{V_n : n \in \mathbb{N}\}$ swallows $\tau(X)$.

The converse implication is obvious.

Claim 2. There exists a τ-framing family $\{U_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ in X enjoying the following property: If $\{V_n : n \in \mathbb{N}\}$ is an increasing sequence consisting of closed subsets of X and swallowing $\tau(X)$, then there exists a $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $U_{\gamma,n} \subseteq V_n$ for all $n \in \mathbb{N}$.

Indeed, for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ and every $n \in \mathbb{N}$ we set

$$U_{\alpha,n} = \{x \in X : \sup_{f \in B_\alpha} |f(x)| \leq n\}.$$

For each $\alpha \in \mathbb{N}^{\mathbb{N}}$, the increasing sequence $\{U_{\alpha,n} : n \in \mathbb{N}\}$ consists of closed subsets of X and $U_{\beta,n} \subseteq U_{\alpha,n}$ if $\alpha \leq \beta$ and $\sup_{f \in B_\alpha} |f(x)| \leq n$ for each $x \in U_{\alpha,n}$ and $n \in \mathbb{N}$.

By Claim 1 the sequence $\{U_{\alpha,n} : n \in \mathbb{N}\}$ swallows $\tau(X)$. Hence $\mathcal{U} := \{V_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ is framing.

We claim that $\mathcal{U}$ satisfies the required property. Fix an increasing sequence $\{V_n : n \in \mathbb{N}\}$ of closed subsets of X and swallowing $\tau(X)$. Set

$$P := \{f \in C(X) : \sup_{x \in V_n} |f(x)| \leq n \ \forall n \in \mathbb{N}\}.$$

Then P is a bounded subset of $C_\tau(X)$ by Claim 1. Since $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution for $C_\tau(X)$, there exists $\delta \in \mathbb{N}^{\mathbb{N}}$ such that $P \subseteq B_\delta$.

We show that $U_{\delta,n} \subseteq V_n$ for every $n \in \mathbb{N}$. Take any $x \in U_{\delta,n}$. Then

$$\sup_{f \in P} |f(x)| \leq \sup_{f \in B_\delta} |f(x)| \leq n.$$

Now if $x \notin \overline{V_n} = V_n$, there is $h \in C(X)$ with $0 \leq h \leq n+1$ such that $h(x) = n+1$ and $h(y) = 0$ for every $y \in V_n$. By the construction of h, and since $\{V_n\}_n$ is increasing, we note that $|h(y)| \leq n$ for every $n \in \mathbb{N}$ and $y \in V_n$. Therefore $h \in P \subseteq B_\delta$. So we conclude that $x \in U_{\delta,n}$ and $|h(x)| = n+1$ with $h \in B_\delta$. This provides a contradiction. Thus $U_{\delta,n} \subseteq V_n$ for every $n \in \mathbb{N}$.

Claim 3. For every $n \in \mathbb{N}$ and each $\alpha \in \mathbb{N}^{\mathbb{N}}$, set $M_n(\alpha) := \bigcup_{\beta \in I_n(\alpha)} B_\beta$ and

$$K_n(\alpha) := \bigcap_{\beta \in I_n(\alpha)} U_{\beta,n} = \left\{x \in X : \sup_{f \in M_n(\alpha)} |f(x)| \leq n\right\}.$$

Then all $K_n(\alpha)$ are closed satisfying conditions below:

(i) $K_n(\alpha) \subseteq K_{n+1}(\alpha)$ for every $n \in \mathbb{N}$ and each $\alpha \in \mathbb{N}^{\mathbb{N}}$.
(ii) $K_n(\alpha) \supseteq K_n(\beta)$ for every $n \in \mathbb{N}$ whenever $\alpha \leq \beta$.
(iii) For each $\alpha \in \mathbb{N}^{\mathbb{N}}$, the sequence $\{K_n(\alpha)\}_{n \in \mathbb{N}}$ swallows $\tau(X)$.
(iv) For every increasing sequence $\{V_n : n \in \mathbb{N}\}$ of closed subsets of X swallowing $\tau(X)$ there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $K_n(\gamma) \subseteq V_n$ for all $n \in \mathbb{N}$.

Moreover, $\mathcal{K} := \{K_n(\alpha) : n \in \mathbb{N}, \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is countable.

The items (i) and (ii) are clear.

Now we prove (iii). Assume there exist $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $K \in \tau(X)$ such that $K \nsubseteq K_n(\alpha)$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, choose $\beta_n \in I_n(\alpha)$ and $x_n \in K$ such that $x_n \notin U_{\beta_n,n}$. Set

$$\gamma := \sup\{\beta_n : n \in \mathbb{N}\}.$$

Then, for every $n \in \mathbb{N}$, $\beta_n \leq \gamma$ and hence $x_n \notin U_{\gamma,n}$ since

$$U_{\gamma,n} \subseteq U_{\beta_n,n}$$

by the definition of a τ-framing family. Therefore $K \nsubseteq U_{\gamma,n}$ for every $n \in \mathbb{N}$, a contradiction. Thus (iii) holds.

Now we prove (iv). By Step 2, there is a $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $U_{\gamma,n} \subseteq V_n$ for all $n \in \mathbb{N}$. Then $K_n(\gamma) \subseteq V_n$ since $K_n(\gamma) \subseteq U_{\gamma,n}$ for all $n \in \mathbb{N}$. Finally, the family $\mathcal{K}$ is countable since, by construction, the set $K_n(\alpha)$ depends only on $\alpha(1), \ldots, \alpha(n)$.

Claim 4. For every $m, n \in \mathbb{N}$ and each $\alpha \in \mathbb{N}^{\mathbb{N}}$, set

$$N_{mn}(\alpha) := \left\{ f \in C(X) : |f(x)| \leq \frac{1}{m} \ \ \forall x \in K_n(\alpha) \right\}$$

(if $K_n(\alpha)$ is empty we set $N_{mn}(\alpha) := \{0\}$). We claim that the family

$$\mathcal{D} := \left\{ N_{mn}(\alpha) : \ m, n \in \mathbb{N} \text{ and } \alpha \in \mathbb{N}^{\mathbb{N}} \right\}$$

is a countable cs-network at $0 \in C_\tau(X)$.*

Indeed, the family $\mathcal{D}$ is countable as the family $\mathfrak{K}$ is countable. To show that $\mathcal{D}$ is a cs^*-network at $0 \in C_\tau(X)$, let $S = \{g_n : n \in \mathbb{N}\}$ converge to zero in $C_\tau(X)$. Let U be a neighbourhood of zero in $C_\tau(X)$ of the form

$$U = [F, \epsilon] := \{f \in C(X) : |f(x)| < \epsilon \ \ \forall x \in F\},$$

where $F \in \tau(X)$ and $\epsilon > 0$. Fix arbitrarily $m \in \mathbb{N}$ such that $m > \frac{1}{\epsilon}$. For every $n \in \mathbb{N}$, set

$$T_n := \bigcap_{i \geq n} R_i, \quad \text{where} \quad R_i := \left\{ x \in X : \ |g_i(x)| \leq \frac{1}{m} \right\}.$$

Clearly $\{T_n\}_{n\in\mathbb{N}}$ is an increasing sequence of closed subsets of X. Moreover, since $g_n \to 0$ in $C_\tau(X)$ we obtain that the sequence $\{T_n\}_{n\in\mathbb{N}}$ swallows $\tau(X)$. Therefore, by (iv) of Step 3, there is a $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that

$$K_n(\gamma) \subseteq T_n \ \text{ for every } n \in \mathbb{N}. \tag{9.9}$$

Now, by (iii) of Step 3, choose an $l \in \mathbb{N}$ such that $F \subseteq K_l(\gamma)$. Then (9.9) implies

$$\{g_i\}_{i \geq l} \subseteq N_{ml}(\gamma) \subseteq U = [F, \epsilon].$$

Thus $\mathcal{D}$ is a countable cs^*-network at zero of $C_\tau(X)$. □

Now we prove the main theorem of this section; see [208].

Theorem 9.5.3 (Ferrando–Gabriyelyan–Kąkol) *Let X be a Tychonoff space.*

(i) *The space $C_p(X)$ is metrizable if and only if $C_p(X)$ admits a fundamental bounded resolution (so exactly when X is countable).*
(ii) *The space $C_c(X)$ is metrizable if and only if $C_c(X)$ is Fréchet–Urysohn and admits a fundamental bounded resolution.*

Proof (i) If X is countable, the space $C_p(X)$ has a fundamental bounded resolution since $C_p(X)$ is metrizable. If $C_p(X)$ has a fundamental bounded resolution, then X is countable by Proposition 9.5.2.
(ii) If $C_c(X)$ is metrizable, it admits a fundamental bounded resolution. Conversely, note that every Fréchet–Urysohn group with countable cs^*-character is metrizable by [67] and Proposition 9.5.2 applies. □

Note that the condition on $C_c(X)$ of being Fréchet–Urysohn in (ii) of Theorem 9.5.3 is essential. Indeed, $C_c(\mathbb{Q})$ has a fundamental bounded resolution by (iv) of Corollary 9.5.4 (from [208]), but $C_c(\mathbb{Q})$ is not metrizable. On the other hand, note also that $C_c(\mathbb{Q})$ is not even covered by a sequence of bounded sets. Indeed, for the contrary assume that $C_c(\mathbb{Q})$ is covered by a sequence of bounded (absolutely convex and closed) sets. Then $C_c(\mathbb{Q})$ does not contain a copy of the space $\mathbb{R}^{\mathbb{N}}$; otherwise $\mathbb{R}^{\mathbb{N}}$ would be a Banach space (by application of the Baire theorem). This contradiction implies that the space of rationals $\mathbb{Q}$ is pseudocompact, what is not true.

Corollary 9.5.4 (Ferrando–Gabriyelyan–Kąkol) *Let X be such that $C_c(X)$ is locally complete (for instance, X is a k_r-space). Then the following assertions are equivalent:*

(i) *$C_c(X)$ has a bounded resolution.*
(ii) *$C_c(X)$ has a fundamental bounded resolution.*

If in addition X is metrizable, then (i)–(iv) are equivalent to

(iii) *$C_p(X)$ has a bounded resolution.*
(iv) *X is σ-compact.*

We refer also the readers to articles of Ferrando and López-Pellicer [229] and Ferrando [205] for additional results concerning spaces $C_c(X)$ with fundamental bounded resolutions.

In [205, Proposition 3] Ferrando proved if X is metrizable, then $C_c(X)$ has a fundamental bounded resolution if and only if X is σ-compact.

Proposition 9.5.5 (Ferrando) *Let X be a metrizable space. Then $C_c(X)$ has a fundamental bounded resolution if and only if X is σ-compact.*

Proof Assume X is σ-compact. Then by [263, Corollary 2.10] the space $C_c(X)$ has a bounded resolution. Since X is a k_r-space, the space $C_c(X)$ is complete, and then we can apply Valdivia's Theorem 3.3.12 to derive that $C_c(X)$ has a fundamental bounded resolution. Conversely, if $C_c(X)$ has a fundamental bounded resolution, the space $C_p(X)$ has a bounded resolution, and we apply Corollary 9.2.14 to deduce that X is σ-compact. □

Let $\mathcal{N}$ be a uniformity for a (non-empty) set X and denote by $\tau_{\mathcal{N}}$ the uniform topology defined by $\mathcal{N}$. A base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of the uniformity $\mathcal{N}$ is called an $\mathfrak{G}$-base if $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$. We may assume that each U_α is a symmetric vicinity.

We show another result due to Ferrando from [205, Theorem 5].

Theorem 9.5.6 (Ferrando) *The space $C_c(X)$ has a fundamental bounded resolution if and only if $(X, \mathcal{M})$, where $\mathcal{M}$ is the uniformity for X generated by the pseudometrics*

$$d_A(x, y) = \sup_{f \in A} |f(x) - f(y)|$$

for each bounded set A of $C_c(X)$, has a $\mathfrak{G}$-base.

Proof Let E be the topological dual of $C_c(X)$. Denote by $\mathcal{B}$ the family of all bounded sets of $C_c(X)$ and by $\beta(E, C(X))$ the strong topology on E. As usual identify X with its canonical homeomorphic embedding in $L_p(X)$. Note that $X \subseteq L(X) \subseteq E$. The strong topology $\beta(E, C(X))$ generates a unique admissible translation-invariant uniformity $\mathcal{N}$ on E, so that $\tau_{\mathcal{N}} = \beta(E, C(X))$. By considering also $f \in C(X)$ as a linear functional on E note that for each $N \in \mathcal{N}$ there is $A \in \mathcal{B}$ such that

$$\left\{(u, v) \in E \times E : \sup_{f \in A} |\langle f, u - v\rangle| \leq 1\right\} \subseteq N.$$

This implies $M \subseteq X \times X$ belongs to the relative uniformity $\mathcal{M}$ of $\mathcal{N}$ to $X \times X$ if and only if there exists $A \in \mathcal{B}$ such that

$$\left\{(x, y) \in X \times X : \sup_{f \in A} |f(x) - f(y)| \leq 1\right\} \subseteq M.$$

Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a fundamental bounded resolution for $C_c(X)$. Set

$$U_\alpha = \{(x, y) \in X \times X : \sup_{f \in A_\alpha} |f(x) - f(y)| \leq 1\}.$$

Then $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base of $\mathcal{M}$. Indeed, if $\alpha \leq \beta$, then $U_\beta \subseteq U_\alpha$, and if $M \in \mathcal{M}$ there is $A \in \mathcal{B}$ such that $(x, y) \in M$ whenever $\sup_{f \in A} |f(x) - f(y)| \leq 1$, so that if $\gamma \in \mathbb{N}^{\mathbb{N}}$ is such that $A \subseteq A_\gamma$, clearly $U_\gamma \subseteq M$.

Conversely, assume that the uniform structure for X is generated by the family of pseudometrics $\{d_A : A \in \mathcal{B}\}$, where

$$d_A(x, y) = \sup_{f \in A} |f(x) - f(y)|$$

admits a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Then for each $A \in \mathcal{B}$ there is $\delta \in \mathbb{N}^{\mathbb{N}}$ such that $\sup_{f \in A} |f(x) - f(y)| \leq 1$ for every $(x, y) \in U_\delta$. Set

$$A_\alpha = \{f \in C(X) : \sup_{(x,y) \in U_\alpha} |f(x) - f(y)| \leq 1\},$$

for each $\alpha \in \mathbb{N}^{\mathbb{N}}$. Then $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$ and $A \subseteq A_\delta$. Consequently $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution for $C_c(X)$. This completes the proof. □

9.6 Some Examples of K-Analytic Spaces $C_p(X)$ and $C_p(X, E)$

In this section we provide more examples of K-analytic spaces $C_p(X)$, $\mathcal{L}_p(X, E)$ and $C_p(X, E)$. Most of results of this part are due to Canela [115]. For two lcs X and E by $\mathcal{L}_p(X, E)$ we denote the subspace of $C_p(X, E)$ of linear maps from X into E. The topology of $\mathcal{L}_p(X, E)$ is the topology of the pointwise convergence, also called a simple topology; this space is sometimes denoted by $\mathcal{L}_s(X, E)$. As usual, $L_p(X)$ means the *weak** dual of $C_p(X)$. We start with the following simple observation.

Proposition 9.6.1 *Let X be a separable metrizable lcs. Let E be a separable Fréchet space. Then $\mathcal{L}_p(X, E)$ is analytic.*

Proof Firs assume that E is a separable Banach space. Let $(V_n)_n$ be a countable basis of neighbourhoods of zero in X. Set $V(V_n, B) := \{f \in \mathcal{L}(X, E) : f(V_n) \subset B\}$, where B denotes the unit ball in E, and the sets $V(V_n, B)$ are endowed with the topology from $\mathcal{L}_p(X, E)$. If $\{x_n : n \in \mathbb{N}\}$ is a countable dense subset of X, the map $T : V(V_n, B) \to E^{\mathbb{N}}$, defined by $f \mapsto (f(x_n)_n)$, is a homeomorphism onto a closed subspace of $E^{\mathbb{N}}$. Therefore $V(V_n, B)$ is analytic. Hence $\mathcal{L}_p(X, E) = \bigcup_n V(V_n, B)$ is analytic. Now assume that E is a separable Fréchet space. Then, since E is topologically isomorphic to a closed subset of a countable product $\prod_n E_n$ of separable Banach spaces, $\mathcal{L}_p(X, E)$ is topologically isomorphic to a closed subspace of the analytic space $\prod_n \mathcal{L}_p(X, E_n)$. Hence $\mathcal{L}_p(X, E)$ is analytic. □

Similarly we prove the following:

Proposition 9.6.2 *Let X be a separable normed space. Let E be a complete K-analytic (analytic) lcs with a fundamental sequence of bounded sets. Then $\mathcal{L}_p(X, E)$ is K-analytic (analytic).*

Example 9.6.3 Let $\ell^2(I)_\sigma$ and $c_0(I)_\sigma$ be the Banach spaces $\ell^2(I)$ and $c_0(I)$ endowed with the weak topologies, respectively. If I is uncountable, the space $C_p(\ell^2(I)_\sigma, c_0(I)_\sigma)$ is not K-analytic and $c_0(I)_\sigma$ is K-analytic.

Proof Note that the space $\mathcal{L}_p(\ell^2(I)_\sigma, c_0(I)_\sigma)$ equals $\mathcal{L}_p(\ell^2(I), c_0(I)_\sigma)$. Since the space $\mathcal{L}_p((\ell^2(I)_\sigma, c_0(I)_\sigma)$ is a closed subspace of $C_p(\ell^2(I)_\sigma, c_0(I)_\sigma)$, it is enough to show that $\mathcal{L}_p(\ell^2(I), c_0(I)_\sigma)$ is not K-analytic; see [576]. Clearly $c_0(\mathbb{N} \times I)$ is isomorphic to $c_0(I)$.

Let $(e_i)_{i\in I}$ be a canonical basis in $\ell^2(I)$. We define a homeomorphism T from $\mathbb{N}^I$ into a closed subspace of $\mathcal{L}_p(\ell^2(I), c_0(I)_\sigma)$. For $s = (s_i) \in \mathbb{N}^I$ define $T_s(e_i) := f_i$, where

$$f_i(n, j) = \begin{cases} 1 & j = i, \ \ n \le s_i, \\ 0 & \text{otherwise} \end{cases}$$

Then the map $T : s \mapsto T_s$ is a homeomorphism. Assume that the space $\mathcal{L}_p(\ell^2(I)_\sigma, c_0(I)_\sigma)$ is K-analytic. Then $\mathbb{N}^I$ is also K-analytic. Since $\mathbb{N}^I$ is not Lindelöf, we reach a contradiction. The space $c_0(I)$ is a (WCG) Banach space; hence $c_0(I)_\sigma$ is K-analytic; see Theorem 12.4.6. □

Example 9.6.4 If X is Eberlein compact and E is a nuclear Fréchet space, $C_p(X, E_\sigma)$ is K-analytic.

Proof Since X is Eberlein compact, X is homeomorphic with a weakly compact subset of a Banach space. Since the weak topology of a metrizable lcs is angelic, X is angelic. Hence X is Fréchet–Urysohn. On the other hand, since E is a nuclear Fréchet space, E is a closed subspace of a countable product of the space ℓ^1 (so E has the Schur property, i.e. every weakly convergent sequence in E converges in the original topology of E; see also [543, Corollary 2, p.101]). As every sequentially continuous map on a Fréchet–Urysohn space is continuous, we deduce that $C(X, E_\sigma) = C(X, E)$. The space $C_p(X, \ell^1)$ is K-analytic by Example 11.2.5. Since $C_p(X, E)$ is closed in a countable product of the space $C_p(X, \ell^1)$, and the topology of $C_p(X, E_\sigma)$ is equal to the original one of $C_p(X, E)$, the conclusion follows. □

9.7 K-Analytic Spaces $C_p(X)$ over a Locally Compact Group X

It is still unknown, see [31, Problem 44, p.29], when exactly for a given X the space $C_p(X)$ is a Lindelöf space. It is known [195, 3.8.D] that for any second countable X the space $C_p(X)$ is Lindelöf. The same holds for (not necessarily second countable) Corson compact spaces X [32, Theorem IV.2.22]. We refer the reader to [29, 31, 112, 123, 178] for more known results concerning this problem.

Theorem 9.7.1, published in [338], collects several equivalent conditions for $C_p(X)$ to be a Lindelöf space for a locally compact group X.

Theorem 9.7.1 *For a locally compact topological group X the following assertions are equivalent:*

(1) $C_p(X)$ *is analytic.*
(2) $C_p(X)$ *is K-analytic.*
(3) $C_p(X)$ *is Lindelöf.*
(4) X *is metrizable and σ-compact.*
(5) X *is analytic.*
(6) $C_p(X)$ *has a bounded resolution and X is metrizable.*
(7) $C_c(X)$ *is a separable Fréchet space.*
(8) $C_c(X)$ *has a compact resolution.*
(9) $C_c(X)$ *has a bounded resolution and X is metrizable.*

We need the following two propositions; for the first one see [152]; see also [141, Theorem 1 and Remark (ii)].

Proposition 9.7.2 *Let X be a locally compact topological group. Then there exist a compact subgroup G of X, $n \in \mathbb{N} \cup \{0\}$, and a discrete subset $D \subset X$ such that X is homeomorphic to the product $\mathbb{R}^n \times D \times G$.*

Proposition 9.7.3 *For a locally compact topological group X the following assertions are equivalent:*

(i) X *is angelic.*
(ii) *Every compact subgroup of X has countable tightness.*
(iii) X *is metrizable.*
(iv) X *has countable tightness.*

Proof The only non-trivial implication is (ii) $\Rightarrow$ (iii): Assume first that X is compact. By Kuz'minov's theorem [396, Theorem] every compact Hausdorff group X is dyadic, i.e. X is a continuous image of $\{0, 1\}^\alpha$, where α is some cardinal number. Since every dyadic Hausdorff space with countable tightness is metrizable [195, 3.12.12(h)], [25, Theorem 3.1.1], the conclusion holds. Now assume that X is a locally compact group. The previous case combined with Proposition 9.7.2 completes the proof. □

Now we are ready to prove our Theorem 9.7.1.

Proof Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4): Assume that $C_p(X)$ is Lindelöf. By Asanov's Theorem 9.4.2 every finite product X^n of X has countable tightness. By Proposition 9.7.3 the space X is metrizable. This proves that X is a metrizable space and separable by Proposition 9.7.2. Indeed, note that $C_p(X)$ is continuously mapped onto $C_p(D)$, so $C_p(D) = \mathbb{R}^D$ is Lindelöf. Hence D is countable.

(4) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (4): Since X is an analytic Baire topological group, Theorem 7.1.7 applies to get that X is metrizable and separable.

(4) $\Rightarrow$ (6): Since X is locally compact and σ-compact, it is hemicompact, and then $C_c(X)$ is a metrizable lcs. Hence $C_c(X)$ has a bounded resolution, and consequently $C_p(X)$ has also a bounded resolution.

(6) $\Rightarrow$ (4): Since X a is metrizable locally compact topological group, X is paracompact, X is σ-compact by Corollary 7.1.6.

(4) $\Rightarrow$ (7): If X is locally compact, metrizable, and σ-compact, the space $C_c(X)$ is a separable Fréchet space.

(7) $\Rightarrow$ (8) is obvious.

(8) $\Rightarrow$ (9): Since $C_c(X)$ is trans-separable by Proposition 6.4.4, every compact subset of X is metrizable; see again Proposition 6.4.7. Now Proposition 9.7.2 applies to get that X is metrizable.

(7) $\Rightarrow$ (1) is clear.

We need to prove (9) $\Rightarrow$ (4): Since X is paracompact and locally compact, the space $C_c(X)$ is Baire by Proposition 2.1.3. Now Proposition 7.1.3 shows that $C_c(X)$ is metrizable. Hence X is hemicompact, so (4) holds, too, and the proof is complete. □

We note a few remarks related with Theorem 9.7.1.

(1) Arkhangel'skii asked if every compact homogeneous Hausdorff topological space with countable tightness is first countable. Dow in [174, Theorem 6.3] answered this question positively under PFA. Although Eberlein compact spaces provide a large class of spaces with countable tightness, and it is known that homogeneous Eberlein compact spaces are first countable [32, III.3.10], non-metrizable homogeneous Eberlein compact spaces exist [629]. On the other hand, Gruenhage showed [291] that every Gul'ko compact space contains a dense G_δ-subset that is metrizable; hence each compact group which is Gul'ko compact is metrizable. Nevertheless, there are Corson compact spaces without any dense metrizable subspace [600]. Since every Corson compact space has countable tightness [358], Proposition 9.7.3 yields: In the class of topological groups the Eberlein, Talagrand, Gul'ko, and Corson compactness are equivalent, and each such a compact group is metrizable.

(2) A locally compact topological group X is metrizable and σ-compact if and only if $C_c(X)$ is weakly Lindelöf.

(3) From Theorem 9.7.1 it follows that the analyticity, K-analyticity, and the Lindelöf property for $C_c(X)$ over a locally compact topological group X are equivalent conditions. Moreover, for a locally compact topological group X the space $C_p(X)$ is analytic (K-analytic) if and only if $C_c(X)$ is weakly analytic (weakly K-analytic).

(6) Note that ((i)+(ii)+(iv)) $\Rightarrow$ (iii) in Proposition 9.7.3 fails for topological groups that are not locally compact. Indeed, let X be an infinite-dimensional reflexive separable (real) Banach space endowed with the weak topology. Then X is a σ-compact angelic lcs (Corollary 4.3.7) whose compact subsets are metrizable

(Corollary 4.4.4), X has countable tightness (Theorem 12.2.1), and X is not metrizable.

(7) Note also that the group structure in (iv) $\Rightarrow$ (i) in Proposition 9.7.3 is essential: The one-point compactification of the space Ψ of Isbell is an example of a compact Hausdorff space with countable tightness not angelic; see [240, p. 54–55].

It is known that there are non-metrizable absolutely convex weakly compact sets in Banach spaces over the field of real or complex numbers. This situation is different if the valued field $\mathbb{K}$ is non-archimedean and locally compact. Recall that a non-trivially valued field $\mathbb{K} := (\mathbb{K}, |\ |)$ is *non-archimedean* if $|t+s| \leq max\ \{|t|, |s|\}$ for all $t, s \in \mathbb{K}$; see [631]. A subset B of a vector space E over a non-archimedean non-trivially valued field $\mathbb{K}$ is called *absolutely* $\mathbb{K}$*-convex*, if from $x, y \in B$, $t, s \in \mathbb{K}$, and $|t| \leq 1$, $|s| \leq 1$ it follows that $tx + sy \in B$. If E is a tvs over a non-archimedean non-trivially valued complete field $\mathbb{K}$, and E contains a non-zero compact absolutely $\mathbb{K}$-convex set, $\mathbb{K}$ must be locally compact; see [154].

We note the following proposition.

Proposition 9.7.4 *Let E be a tvs over a locally compact non-trivially valued field $\mathbb{K}$. Then:*

(i) If $\mathbb{K}$ is archimedean, every locally compact subgroup X of E is metrizable.

(ii) If $\mathbb{K}$ is non-archimedean and E is metrizable, every absolutely $\mathbb{K}$-convex locally compact subset X of $\big(E, \sigma(E, E')\big)$ is metrizable in $\sigma(E, E')$.

Proof (i) Since $\mathbb{K}$ is archimedean, by Ostrowski's theorem, [631, Theorem 1.2], $\mathbb{K}$ is either the field of the real or complex numbers. By the assumption X is locally compact, so it is homeomorphic to the product $\mathbb{R}^n \times D \times G$, where D and G are as in Proposition 9.7.2. As any compact subgroup in a (real or complex) topological vector space is trivial, the conclusion follows.

(ii) Since $(\mathbb{K}, +)$ is a locally compact Abelian group, $\mathbb{K}^\wedge$ separates points of $\mathbb{K}$. Fix a non-constant $\chi \in \mathbb{K}^\wedge$. Then $E^\wedge = \{\chi \circ x' : x' \in E'\}$, by [637, Theorem 2]. As $E' =: \mathrm{Hom}(E, \mathbb{T})$, where $\mathbb{T}$ denotes the torus in the complex plane, separates points of E, see [545], we deduce that the Bohr topology of the group $(E, +)$ is Hausdorff. The equality $E^\wedge = \{\exp(if) : f \in E'\}$ ensures that $\sigma(E, E^\wedge) \leq \sigma(E, E')$. E is metrizable, so by Proposition 4.3.8 we deduce that $\big(E, \sigma(E, E^\wedge)\big)$ is angelic. Next, we apply Corollary 4.1.3 to see that $(E, \sigma(E, E'))$ is angelic. On the other hand, since $\mathbb{K}$ is non-archimedean and X is an absolutely $\mathbb{K}$-convex subset of E, X is an additive subgroup of E. Hence X is an angelic locally compact group, and by Proposition 9.7.3 we note that X is metrizable. □

Example 9.7.5 Proposition 9.7.4 (i) fails for a non-archimedean $\mathbb{K}$.

Proof Set $\mathbb{K} := \mathbb{Q}_2$ and $B := \{\alpha \in \mathbb{Q}_2 : |\alpha|_2 \leq 1\}$. Then $B^{\mathfrak{c}}$ is a non-metrizable compact additive subgroup of the topological vector space $\mathbb{K}^{\mathfrak{c}}$. Using the same example we deduce that the metrizability assumption of E is essential in Proposition 9.7.4 (ii). □

Since the weak topology $\sigma(E, E')$ of a metrizable lcs E is angelic and has countable tightness, Proposition 9.7.4 may suggest the following problem.

Problem 9.7.6 *Let E be a real lcs. For which compact subsets X of E does there exist a compact topological group which is homeomorphic to X?*

Note that every such X must be homogeneous. Also X cannot be convex, since the Schauder fixed point theorem fails for compact topological groups (translations do not have a fix point).

9.8 K-Analytic Group $X_p^\wedge$ of Homomorphisms

This section deals with a variant of Theorem 9.7.1 for the group of homomorphisms $X_c^\wedge$. For Abelian topological groups X and Y by $\mathrm{Hom}_p(X, Y)$ and $\mathrm{Hom}_c(X, Y)$ we denote the set $\mathrm{Hom}(X, Y)$ of all continuous homomorphisms from X into Y endowed with the pointwise and compact-open topology, respectively. Set

$$X_p^\wedge =: \mathrm{Hom}_p(X, \mathbb{T}), \quad X_c^\wedge =: \mathrm{Hom}_c(X, \mathbb{T}),$$

where $\mathbb{T}$ denotes the unit circle of the complex plane. For every $x \in X$ the function $x^\wedge : X^\wedge \to \mathbb{T}$, defined by $x^\wedge(f) := f(x)$ for $f \in X^\wedge$, is a continuous homomorphism on $X_c^\wedge$, and $\{x^\wedge : x \in X\} \subset (X_c^\wedge)^\wedge$, and by the Pontryagin–van Kampen theorem, see [310, Theorem 24.8], the map $x \mapsto x^\wedge$ is a topological isomorphism between a locally compact Abelian group X and $(X_c^\wedge)_c^\wedge$. If X is an Abelian locally compact group, $X_c^\wedge$ is also locally compact and Abelian, and by the Peter–Weyl–van Kampen theorem [456, Theorem 21] the space $X_c^\wedge$ is *dual-separating*, i.e. for different $x, y \in X$, there exists $f \in X^\wedge$ such that $f(x) \neq f(y)$.

For an Abelian group X the set of all homomorphisms from X into $\mathbb{T}$ endowed with the pointwise convergence topology is a compact Abelian group, as a closed subgroup of the product $\mathbb{T}^X$; see [312, Proposition 1.16]. For a metrizable Abelian topological group X the group $X_c^\wedge$ is always an Abelian Hausdorff complete and hemicompact group. Moreover, it is a k-space; see [42, Corollary 4.7] and [134].

Proposition 9.8.1 *If X be a separable and metrizable Abelian topological group, $X_c^\wedge$ is locally compact if and only if $X_c^\wedge$ is metrizable.*

Proof If $X_c^\wedge$ is metrizable, the evaluation map $e : X \times X_c^\wedge \to \mathbb{T}$ is continuous. Then, by [432, Proposition 1.2], the group $X_c^\wedge$ is locally compact. To prove the converse, note that every compact subset of $X_c^\wedge$ is metrizable (since X is separable), and Proposition 9.7.3 applies. □

We need also a couple of additional results.

Proposition 9.8.2 *A locally compact Lindelöf topological group X is hemicompact.*

Proof Take an open neighbourhood U of the unit whose closure $\overline{U}$ is compact. Since $X = \bigcup_{x\in X} xU$, and X is a Lindelöf space, there exists a sequence $(x_n)_n$ such that $X = \bigcup_n x_n\overline{U}$. Set $K_n := \bigcup_{i=1}^n x_i\overline{U}$. Then $(K_n)_n$ is a fundamental sequence of compact sets in X. □

Proposition 9.8.3 *If X is a metrizable locally compact Abelian group, $X_c^\wedge$ is a hemicompact k-space.*

Proof Let $(U_n)_n$ be a decreasing basis of neighbourhoods of the unit in X. Then

$$U_n^\triangleright := \{\varphi \in X_c^\wedge : \varphi(U_n) \subset \mathbb{T}_+\}$$

is compact in the compact-open topology for each $n \in \mathbb{N}$, where $\mathbb{T}_+ := \{z \in \mathbb{T} : \operatorname{Re} z \geq 0\}$. Moreover, $X^\wedge = \bigcup_n U_n^\triangleright$. If K is a compact set in the space $X_c^\wedge$, $K^\triangleright \subset (X_c^\wedge)^\wedge$ is a neighbourhood of the unit. By the Pontryagin–van Kampen theorem [310], it can be identified with

$$K^\triangleleft := \{x \in X : \operatorname{Re}\varphi(x) \geqslant 0, \varphi \in K\}.$$

Since the last set is a neighbourhood of the unit, $U_m \subset K^\triangleleft$ for some $m \in \mathbb{N}$. Hence

$$K \subset K^{\triangleright\triangleleft} \subset U_m^\triangleright.$$

For the proof that $X_c^\wedge$ is a k-space we refer to [42, Corollary 4.7]. □

Reference [42, Proposition 2.8] implies the following:

Proposition 9.8.4 *If a topological Abelian group X is hemicompact, $X_c^\wedge$ is metrizable.*

We need also the following simple:

Proposition 9.8.5 *If $C_p(X, \mathbb{R})$ has countable tightness, $C_p(X, Y)$ has countable tightness for any metric space (Y, d).*

Proof Let $A \subset C_p(X, Y)$. Assume $f \in \overline{A}$. Define a continuous map $T : C_p(X, Y) \to C_p(X, \mathbb{R})$ by $T(g)(x) := d(g(x), f(x))$, where $g \in C_p(X, Y)$, $x \in X$. Note that $0 = T(f) \in T(\overline{A}) \subset \overline{T(A)}$. By the assumption there exists in A a countable subset B such that $T(f) \in \overline{T(B)}$; hence $f \in \overline{B}$. □

Now we are ready to prove the following main result of [342].

Theorem 9.8.6 *Let X be a locally compact Abelian group. The following assertions are equivalent:*

(1) X is metrizable.
(2) $X_p^\wedge$ is σ-compact.
(3) $X_p^\wedge$ is K-analytic.
(4) $(X, \sigma(X, X^\wedge))$ has countable tightness.

Moreover, if X is Lindelöf, any condition above is equivalent to:

(5) $X_c^\wedge$ is metric, complete, and separable.

Proof (1) $\Rightarrow$ (2): By Proposition 9.8.3 the group $X_c^\wedge$ is hemicompact. So $X_p^\wedge$ is σ-compact.

(2) $\Rightarrow$ (3): For an increasing sequence $(B_n)_n$ of compact sets covering $X_p^\wedge$ set $T(\alpha) := B_{n_1}$ for $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$. Clearly T is a (usco) compact-valued map with values covering $X_p^\wedge$.

(3) $\Rightarrow$ (4): Since $X_p^\wedge$ is K-analytic, any finite product $(X_p^\wedge)^n$ is Lindelöf. By Proposition 9.4.1 the space $C_p(X_p^\wedge, \mathbb{R})$ has countable tightness. Now Proposition 9.8.5 applies to say that $C_p(X_p^\wedge, \mathbb{C})$ has countable tightness. Hence $(X, \sigma(X, X^\wedge))$ (as topologically included in $C_p(X_p^\wedge, \mathbb{C})$) has countable tightness.

(4) $\Rightarrow$ (1): Since X is a locally compact group, there exist a compact subgroup G of X, $n \in \mathbb{N} \cup \{0\}$, and a discrete subset $D \subset X$ such that X is homeomorphic to the product $\mathbb{R}^n \times D \times G$; see Proposition 9.7.2. Therefore the induced topology $\sigma(X, X^\wedge)|G$ coincides with the original one of G. Hence G has countable tightness. A compact group with countable tightness is metrizable by Proposition 9.7.3; hence X is metrizable.

The remaining part follows from Propositions 9.8.2, 9.8.3, and 9.8.4. □

A topological space is *sequential* if every sequentially closed subset of X is closed. For a metrizable topological group X the dual group $X_c^\wedge$ is Fréchet–Urysohn if and only if $X_c^\wedge$ is locally compact and metrizable; see, for example, [141, Theorem 2.1]. This provides a large class of complete angelic and hemicompact sequential groups that are not Fréchet–Urysohn [141, Theorem 2.3].

The Glicksberg theorem states that for a locally compact topological group X the compact sets in X and $(X, \sigma(X, X^\wedge))$ coincide; see [42].

Corollary 9.8.7 *If X is a metrizable locally compact non-compact Abelian group, the group $(X, \sigma(X, X^\wedge))$ has countable tightness and cannot be sequential, so neither Fréchet–Urysohn.*

Proof By the Glicksberg theorem the space $(X, \sigma(X, X^\wedge))$ has the same compact subsets as X. Since X is metrizable, X is a k-space and X does not admit another k-space topology with the same compact sets. Therefore $(X, \sigma(X, X^\wedge))$ is not a k-space, and in particular cannot be sequential, neither Fréchet–Urysohn. □

Chapter 10
Precompact Sets in (LM)-Spaces and Dual Metric Spaces

Abstract This chapter presents unified and direct proofs of Pfister, Cascales, and Orihuela and Valdivia's theorems about metrizability of precompact sets in (LF)-spaces, (DF)-spaces, and dual metric spaces, respectively. The proofs do not require the typical machinery of quasi-Suslin spaces, upper semicontinuous compact-valued maps, and so on.

10.1 The Case of (LM)-Spaces, Elementary Approach

In [239] Floret (motivated by results of Grothendieck, Fremlin, De Wilde, and Pryce) proved an extended version of the Eberlian–Šmulian theorem with many applications. Nevertheless, Floret's result said nothing about the metrizability of compact sets. Cascales and Orihuela [126] (answering a question of Floret [240]) showed that the weight of any precompact set in an (LM)-space is countable. Pfister and Valdivia proved the same result for (DF)-spaces and dual metric spaces, respectively, [492], [610, Note 4]. In [346] Kąkol and Saxon presented alternative proofs for (LM)-spaces and dual metric spaces.

In this chapter we present elementary proofs (mostly due to Kąkol and Saxon [346]) of results mentioned above, which do not require typical machinery involving (usco) maps, etc., showing that precompact sets in (LM)-spaces, dual metric spaces, and (DF)-spaces are metrizable.

If $A \subset E$ is a subset of a lcs E by $\mathrm{ac}(A)$ and $\overline{\mathrm{ac}}(A)$ we mean the absolutely convex and the closed absolutely convex envelope of the set A, respectively.

Recall again, for the convenience, that (DF)-spaces are $\aleph_0$-quasibarrelled spaces having a fundamental sequence of bounded sets (see the text below Proposition 2.6.9).

A lcs E is ℓ^∞-quasibarrelled if every $\beta(E', E)$-bounded sequence in E' is equicontinuous. A lcs E is said to be *dual metric* if E has a fundamental sequence of bounded sets and every $\beta(E', E)$-bounded sequence in E' is equicontinuous. Every semi-Montel dual metric space is a Montel (DF)-space [346]. A lcs E is a Montel (DF)-space if and only if E is a Montel (LB)-space if and only if E is the strong dual of a Fréchet–Montel space [346].

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We need the following:

Lemma 10.1.1 *Let P be a precompact set for the inductive limit space E with a defining sequence $(E_n)_n$ of lcs. Assume that U_n is an absolutely convex neighbourhood of zero in E_n with $U_n \subset U_{n+1}$ for each $n \in \mathbb{N}$. Then there exist $m \in \mathbb{N}$, a finite subset $F \subset P$ such that $P \subset F + \overline{U_m}$, where the closure is taken in E.*

Proof Assume the conclusion fails. Then there exists a sequence $(x_n)_n$ in P such that

$$x_{n+1} \notin \{x_1, x_2, \ldots, x_n\} + \overline{U_n}$$

for each $n \in \mathbb{N}$. Since the above right-hand set is closed in E, for each $n \in \mathbb{N}$ there exists a decreasing sequence $(V_n)_n$ of closed absolutely convex neighbourhoods of zero V_n in E such that

$$x_{n+1} \notin \{x_1, x_2, \ldots, x_n\} + U_n + V_n.$$

Then $U_k \cap V_k \subset U_n + V_n$ for all $k, n \in \mathbb{N}$. This implies that

$$V := \operatorname{ac}\left(\bigcup_k U_k \cap V_k\right) \subset U_n + V_n$$

for all $n \in \mathbb{N}$. Hence

$$x_{n+1} \notin \{x_1, x_2, \ldots, x_n\} + V$$

for all $n \in \mathbb{N}$. Clearly V is a neighbourhood of zero in E. We proved that for each finite subset F of $\{x_n : n \in \mathbb{N}\}$ the set $\{x_n : n \in \mathbb{N}\}$ is not included in $F + V$, showing that P is not precompact, a contradiction. □

Lemma 10.1.1 applies to get the following:

Theorem 10.1.2 *Let P be a precompact set for the inductive limit space E with a defining sequence $(E_n)_n$ of lcs. Let $\mathcal{W}_n$ be a basis of absolutely convex neighbourhoods of zero in E_n for each $n \in \mathbb{N}$. Then the induced topology from E onto P has a basis $\mathcal{W}$ of neighbourhoods such that $|\mathcal{W}| \leq \sup_n |\mathcal{W}_n|$.*

Proof Assume that each $\mathcal{W}_n$ is infinite. For each $n \in \mathbb{N}$ select $W_n \in \mathcal{W}_n$ and set $u := (W_n)_n$. Let $\mathcal{U}$ be the set of all such u. For $u \in \mathcal{U}$ set

$$U_n := 3^{-1} \operatorname{ac}\left(\bigcup_{i \leq n} W_i\right).$$

By Lemma 10.1.1 there exist a finite subset $F_u \subset P$ and $n_u \in \mathbb{N}$ such that $P \subset F_u + 3^{-1} T_u$, where $T_u := \overline{U_{n_u}}$. Set

$$\mathcal{W} := \{P \cap (x + T_u) : u \in \mathcal{U}, x \in F_u\}.$$

Since each set F_u is finite and T_u is described by a finite subset of the union $\bigcup_n \mathcal{W}_n$, we have

$$|\mathcal{W}| \leq \sup_n |\mathcal{W}_n|.$$

We need to show that $\mathcal{W}$ is a basis of neighbourhoods for P. Fix $y \in P$. Let U be a closed absolutely convex neighbourhood of zero in E. Since E is the inductive limit space, there exists $u = (W_n)_n \in \mathcal{U}$ such that $W_n \subset 3(4)^{-1}U$ for each $n \in \mathbb{N}$. Hence $T_u \subset 4^{-1}U$. Take $x \in F_u$ such that $y \in x + 3^{-1}T_u$. We need to prove that $(x + T_u) \cap P$ is a relative neighbourhood of $y \in P$ and $(x + T_u) \cap P \subset y + U$. Set

$$C := \bigcup \{x' + 3^{-1}T_u : x' \in F_u,\ \ y \notin x' + 3^{-1}T_u\}.$$

Hence C (since F_u is finite) is closed and $y \notin C$. Consequently, $y \in P \setminus C$ is a relative open set, and

$$P \setminus C \subset y + 2(3)^{-1}T_u \subset x + T_u.$$

Finally,

$$x + T_u = (x + 3^{-1}T_u) + 2(3)^{-1}T_u \subset (y + 2(3)^{-1}T_u) + 2(3)^{-1}T_u =$$

$$y + 4(3)^{-1}T_u \subset y + U.$$

□

Since every second countable regular space is metrizable [195], every second countable subset of a Hausdorff lcs is metrizable. Therefore we note the following result due to Cascales and Orihuela [126].

Theorem 10.1.3 (Cascales–Orihuela) *Every precompact set in an (LM)-space E is metrizable. Hence E is angelic.*

10.2 The Case of Dual Metric Spaces, Elementary Approach

Recall that a sequence $(A_n)_n$ of absolutely convex sets in a lcs E ic called *bornivorous* if for every bounded set B in E there exists $m \in \mathbb{N}$ such that A_m absorbs B, i.e. there exists $t > 0$ such that $B \subset tA_m$. Let us start with the following:

Lemma 10.2.1 *Let P be a precompact set. Let $(A_n)_n$ be an increasing bornivorous sequence of absolutely convex sets in a ℓ^∞-quasibarrelled space E. Then there exist $m \in \mathbb{N}$, a finite subset $F \subset P$ such that $P \subset F + \overline{A_m}$.*

Proof For elements $x_1, \ldots, x_k$ in P, $k \in \mathbb{N}$, let T_k be the absolutely convex hull of $\{x_1, \ldots, x_k\}$. Set $C_k := 2T_k + 2^{-2}\overline{A_k}$. Assume that there exist $x_1, x_2, \ldots, x_k \in P$ such that $P \subset C_k$. Since $2T_k$ is a compact subset of a finite-dimensional vector space, there exist $m \in \mathbb{N}$, $m \geq k$, and a finite subset W of $2T_k$ such that $C_k \subset W + 2^{-1}\overline{A_m}$. Consequently, P is covered by finitely many of non-void sets of the form $(y + 2^{-1}\overline{A_m}) \cap P$. If $z \in (y + 2^{-1}\overline{A_m}) \cap P$, we note that P is also covered by finitely many sets $z + \overline{A_m}$ with $z \in P$. This proves that there exists a finite set $F \subset P$ such that $P \subset F + \overline{A_m}$.

Now assume that $P \nsubseteq C_k$ for each C_k of the above form. Inductively we select a sequence $(x_k)_k$ in P such that $x_{k+1} \notin C_k$, $k \in \mathbb{N}$. Since every C_k is absolutely convex and closed, for every $k \in \mathbb{N}$ there exists a continuous linear functional f_k over E such that $f_k(x_{k+1}) > 1$ and $|f_k(y)| \leq 1$ for all

$$y \in 2T_k + 2^{-2}A_k.$$

Since $(A_k)_k$ is bornivorous, the sequence $(f_k)_k$ is strongly bounded, i.e. bounded in $\beta(E', E)$. The space E is ℓ^∞-quasibarrelled, so $(f_k)_k$ is equicontinuous. Fix $0 < t < 2^{-1}$. Then

$$U := \bigcap_k f_k^{-1}(-t, t)$$

is a neighbourhood of zero in E and $x_{k+1} - x_j \notin U$ for all $k \in \mathbb{N}$, $j = 1, 2, \ldots, k$. Hence P is not precompact, a contradiction. □

We are ready to prove the following result due to Valdivia [610, Note 4].

Theorem 10.2.2 (Valdivia) *Every precompact set in a dual metric space E is metrizable. Hence E is angelic.*

Proof Let $(B_n)_n$ be a fundamental sequence of bounded absolutely convex sets in E. Let $\mathcal{U}$ be the family of all sequences $u := (U_n)_n$ of the sets of the form $U_n := k^{-1}B_n$, $k, n \in \mathbb{N}$. For $u = (U_n)_n$ set

$$A_n := 3^{-1}\,\mathrm{ac}(\bigcup_{j \leq n} U_j).$$

Since the sequence $(A_n)_n$ of absolutely convex sets is increasing and bornivorous, we apply Lemma 10.2.1 to get a finite subset $F_u \subset P$ and $n(u) \in \mathbb{N}$ such that

$$P \subset F_u + 3^{-1}\overline{\mathrm{ac}}(\bigcup_{j \leq n(u)} U_j).$$

Set

$$T_u := 3^{-1}\overline{\mathrm{ac}}(\bigcup_{j \leq n(u)} U_j).$$

As the sets B_n are bounded, for each neighbourhood of zero U in E there exists $u \in \mathcal{U}$ such that $T_u \subset 4^{-1}U$. Then we follow similarly as in the proof of Theorem 10.1.2 to get the metrizability of the set P. □

Last theorem yields the following result due to Pfister [492].

Theorem 10.2.3 (Pfister) *Every precompact set in a (DF)-space E is metrizable. Hence E is angelic.*

Chapter 11
Metrizability of Compact Sets in the Class $\mathfrak{G}$

Abstract This chapter introduces (after Cascales and Orihuela) a large class of locally convex spaces under the name of the class $\mathfrak{G}$. The class $\mathfrak{G}$ contains among others all (LM)-spaces (hence (LF)-spaces), and dual metric spaces (hence (DF)-spaces), spaces of distributions $D'(\Omega)$, and spaces $A(\Omega)$ of real analytic functions on open $\Omega \subset \mathbb{R}^n$. We show (following Cascales and Orihuela) that every precompact set in a lcs in the class $\mathfrak{G}$ is metrizable. This general result covers many already known theorems for (DF)-spaces, (LF)-spaces, and dual metric spaces.

11.1 The Class $\mathfrak{G}$, Examples

Following Cascales and Orihuela [127] a lcs E is said to be in the *class* $\mathfrak{G}$ if there is a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of E' (called a $\mathfrak{G}$-representation of E) such that:

(a) $E' = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.
(b) $A_\alpha \subset A_\beta$ if $\alpha \leq \beta$.
(c) In each A_α all sequences are equicontinuous.

Condition (c) implies that every set A_α is $\sigma(E', E)$-relatively countably compact. Therefore, if E is in the class $\mathfrak{G}$, the space $(E', \sigma(E', E))$ has a relatively countably compact resolution.

The class $\mathfrak{G}$ is reach; $\mathfrak{G}$ contains (LM)-spaces, the dual metric spaces (hence (DF)-spaces), the space of distributions $D'(\Omega)$, and the space $A(\Omega)$ of the real analytic functions for open $\Omega \subset \mathbb{R}^{\mathbb{N}}$, etc.; see [119], [220]. The next proposition shows that the class $\mathfrak{G}$ is stable by taking subspaces, separated quotients, completions, countable direct sums, and countable products [127].

Proposition 11.1.1 *(i) Let $(E_n)_n$ be a sequence of lcs in the class $\mathfrak{G}$. Then the topological direct sum $E := \bigoplus_n E_n$ belongs to the class $\mathfrak{G}$.*
(ii) Let $(E_n)_n$ be a sequence of lcs in the class $\mathfrak{G}$. Then the topological product $E := \prod_n E_n$ belongs to the class $\mathfrak{G}$.

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(iii) If E is a lcs in the class $\mathfrak{G}$ and F is a closed subspace, the quotient space E/F belongs to the class $\mathfrak{G}$.
(iv) Every subspace F of a lcs E in the class $\mathfrak{G}$ belongs to the class $\mathfrak{G}$.
(v) The completion F of a lcs E in the class $\mathfrak{G}$ belongs to the class $\mathfrak{G}$.

Proof (i) Let $\{A^n_{\alpha^n} : \alpha^n = (a^n_1, a^n_2, \dots,) \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of E_n for each $n \in \mathbb{N}$. Set $A_\alpha := \prod_j A^j_{\alpha^j}$, with $\alpha = (a^1_1, a^1_2, a^2_1, a^1_3, a^2_2, a^3_1, a^1_4, a^2_3, \dots,)$. Then the family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation of E of sets in E' (that is isomorphic to the product $\prod_n E'_n$).

(ii) Let $\{A^n_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of E_n for each $n \in \mathbb{N}$. Set

$$A_\alpha := A^1_\alpha \oplus A^2_\alpha \cdots \oplus A^{a_1}_\alpha$$

for $\alpha = (a_k) \in \mathbb{N}^{\mathbb{N}}$. Then $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation for E in E' (that is isomorphic to the direct sum $\bigoplus_n E'_n$).

(iii) Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of E. Then

$$\{A_\alpha \cap F^\perp : \alpha \in \mathbb{N}^{\mathbb{N}}\}$$

is a $\mathfrak{G}$-representation of E/F in $(E/F)'$, where $F^\perp$ is the orthogonal subspace to F in E'.

(iv) If $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation of E in E',

$$\{A_\alpha | F : \alpha \in \mathbb{N}^{\mathbb{N}}\}$$

is a $\mathfrak{G}$-representation of F in F'.

(v) Since E and F have the same equicontinuous sets, the conclusion follows. □

We provide short arguments showing that (DF)-spaces and (LM)-spaces E admit a $\mathfrak{G}$-representation.

Let E be a (DF)-space. Let $(B_n)_n$ be a fundamental sequence of absolutely convex bounded subsets of E. For every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ set $A_\alpha := \bigcap_k n_k(B_k)^\circ$. Clearly the conditions (a), (b), and (c) are satisfied. Note also that for a bounded subset B in $(E', \beta(E', E))$ there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that A_α absorbs B; such a $\mathfrak{G}$-representation will be called *bornivorous*.

Let E be an (LM)-space. Let $(E_j)_j$ be a defining sequence for E of metrizable lcs. For every $j \in \mathbb{N}$ let $(U^j_n)_n$ be a decreasing basis of absolutely convex neighbourhoods of zero in E_j such that

$$U^j_{n+1} + U^j_{n+1} \subset U^j_n$$

for all $j, n \in \mathbb{N}$. For every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ set $A_\alpha := \bigcap_k (U^k_{n_k})^\circ$. Clearly the conditions (a), (b), (c) are satisfied. Since every bounded set B in $(E', \beta(E', E))$ is equicontinuous, the polar D of B is a neighbourhood of zero in E. Hence there

exists a sequence $\alpha = (n_k)$ in $\mathbb{N}^{\mathbb{N}}$ such that $U^k_{n_k} \subset D$ for any $k \in \mathbb{N}$. Consequently,

$$B \subset D^\circ \subset \bigcap_k (U^k_{n_k})^\circ.$$

Applying Proposition 3.2.2 we have that for an uncountable-dimensional vector space E neither $(E, \sigma(E, E^*))$ nor $(E^*, \sigma(E^*, E))$ is in the class $\mathfrak{G}$. This yields the following examples of spaces not in the class $\mathfrak{G}$.

Corollary 11.1.2 *Let $\{E_l : i \in I\}$ be an uncountable family of non-zero lcs. Then neither the topological direct sum $S := \bigoplus_l E_l$ nor the topological product $P := \prod_l E_l$ is in the class $\mathfrak{G}$.*

Proof Let F_l be a one-dimensional vector subspace of E_l for each $l \in I$. Then $M := \bigoplus_l F_l \subset S$ and $N := \prod_l F_i \subset P$. Note that the weak dual of M is $(M^*, \sigma(M^*, M))$, and the *weak** dual of N is $(M, \sigma(M, M^*))$. By the above remark, neither M nor N is in the class $\mathfrak{G}$. Therefore, neither S nor P is in the class $\mathfrak{G}$. □

We have also the following:

Proposition 11.1.3 *If E is a lcs such that E' is uncountable-dimensional, the space $(E, \sigma(E, E'))$ is not in the class $\mathfrak{G}$.*

Proof Assume $(E, \sigma(E, E'))$ is in the class $\mathfrak{G}$. Since the completion of a lcs in the class $\mathfrak{G}$ belongs to $\mathfrak{G}$, the completion $(E'^*, \sigma(E'^*, E'))$ is in the class $\mathfrak{G}$. This is impossible by the remark before Corollary 11.1.2. □

11.2 Cascales–Orihuela's Theorem and Applications

Spaces in the class $\mathfrak{G}$ enjoy another important general property.

Theorem 11.2.1 (Cascales–Orihuela) *Every precompact set in a lcs E in the class $\mathfrak{G}$ is metrizable.*

We already provided simple proofs for (LM)-spaces and (DF)-spaces. The following simple and short proof is due to Ferrando, Kąkol, and López-Pellicer; see [215].

Proof Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of E. For $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ set $C_{n_1,\ldots,n_k} := \bigcup\{A_\beta : \beta = (m_k) \in \mathbb{N}^{\mathbb{N}}, n_j = m_j, 1 \le j \le k\}$. By $D_{n_1,n_2,\ldots,n_k}$ we denote the polar of $C_{n_1,n_2,\ldots,n_k}$ for each $k \in \mathbb{N}$. Let P be a precompact set in E. Since the completion of a lcs in the class $\mathfrak{G}$ belongs to $\mathfrak{G}$, we may assume that P is compact. Note that for each $\epsilon > 0$ there is a countable subset H_ϵ in E' such that

$$E' = H_\epsilon + \epsilon(P)^\circ.$$

Indeed, otherwise (by the Zorn lemma) there exists an uncountable subset F in E', $\epsilon > 0$, such that the condition $f - g \in \epsilon(P)^\circ$ for $f, g \in F$ implies $f = g$. By an obvious induction procedure we select a sequence $(n_k)_k$ in $\mathbb{N}$, a sequence $(f_k)_k$ in E' of different elements with $f_k \in C_{n_1,n_2,\dots,n_k}$ such that the condition

$$f_n - f_m \in \epsilon(P)^\circ$$

implies $m = n$. Indeed, there exists $n_1 \in \mathbb{N}$ such that $F \cap C_{n_1}$ is uncountable. Choose $f_1 \in F \cap C_{n_1}$. Since

$$C_{n_1} = \bigcup \{C_{n_1,m_2} : m_2 \in \mathbb{N}\},$$

there exists $n_2 \in \mathbb{N}$ such that $(F \setminus \{f_1\}) \cap C_{n_1,n_2}$ is uncountable. Select

$$f_2 \in (F \setminus \{f_1\}) \cap C_{n_1,n_2}.$$

Using a simple induction we obtain both the sequences as desired. As $f_k \in C_{n_1,n_2,\dots,n_k}$ for all $k \in \mathbb{N}$, the sequence $(f_k)_k$ is equicontinuous. Indeed, for every $k \in \mathbb{N}$ there exists $\beta_k = (m_n^k)_n \in \mathbb{N}^{\mathbb{N}}$ such that $f_k \in A_{\beta_k}$, where $n_j = m_j^k$ for $j = 1, 2, \dots, k$. Define

$$a_n = \max\left\{m_n^k : k \in \mathbb{N}\right\}$$

and $\gamma = (a_n) \in \mathbb{N}^{\mathbb{N}}$. Note that $\gamma \geq \beta_k$ for every $k \in \mathbb{N}$. Therefore $A_{\beta_k} \subset A_\gamma$, and so $f_k \in A_\gamma$ for all $k \in \mathbb{N}$ (by the condition (b)). Also by (c) the sequence $(f_k)_k$ is equicontinuous. Applying the Ascoli theorem for the Banach space $C_c(P)$ we obtain two different natural numbers j, k such that

$$f_j - f_k \in \epsilon(P)^\circ,$$

which yields a contradiction. This proves the claim. Since

$$H := \{H_{n^{-1}} : n \in \mathbb{N}\}$$

is countable, the topology τ_H on E of the pointwise convergence on H restricted to P is Hausdorff and metrizable and coincides with the original topology of P. Hence P is metrizable. □

For a lcs E let $\tau_{pc}(E', E)$ be the topology on E' of the uniform convergence on precompact subsets of E. Note that, if E is in the class $\mathfrak{G}$, the space $(E', \sigma(E', E))$ admits a relatively countably compact resolution; this resolution is also $\tau_{pc}(E', E)$-precompact (since every $\sigma(E', E)$-relatively countably compact set is $\tau_{pc}(E', E)$-precompact). This may suggest the following question: Let E be a lcs whose dual E' endowed with the topology $\tau_{pc}(E', E)$ admits a precompact resolution. Does E

belong to the class $\mathfrak{G}$? The answer is "no". Indeed, let E be an infinite-dimensional separable reflexive Banach space. Since $(E', \beta(E', E))$ is a separable Banach space, it admits a compact resolution. Clearly $\beta(E', E)$ is the topology $\tau_{pc}(E', E)$ of the uniform convergence on $\sigma(E, E')$-precompact sets. $(E, \sigma(E, E'))$ does not belong to the class $\mathfrak{G}$; this is a consequence of Proposition 11.1.3.

Since precompact sets in a lcs in the class $\mathfrak{G}$ are metrizable, each lcs in $\mathfrak{G}$ is angelic. It turns out that the following stronger fact also holds; see [127, Theorem 11].

Proposition 11.2.2 *The weak topology $\sigma(E, E')$ of a lcs E in the class $\mathfrak{G}$ is angelic.*

Proof By the assumption the space $(E', \sigma(E', E))$ is web-compact. Applying Theorem 4.3.3 we derive that $C_p(E', \sigma(E', E))$ is angelic. $(E, \sigma(E, E'))$ is a topological subspace of $C_p(E', \sigma(E', E))$, and the conclusion follows. □

We proved that every lcs in the class $\mathfrak{G}$ is weakly angelic. This refers to many spaces, except the class of spaces $C_p(X)$. Indeed, Proposition 12.2.2 below states that $C_p(X)$, for uncountable spaces X, does not belong to the class $\mathfrak{G}$. Nevertheless, it is known that $C_p(X, E)$ is weakly angelic for any web-compact X and any lcs E in the class $\mathfrak{G}$; see [128, Theorem 8, Corollary 1.8]. Proposition 11.2.3 provides a direct proof of this fact.

Proposition 11.2.3 *If X is a web-compact space and E is a lcs in the class $\mathfrak{G}$, the space $C_p(X, E)$ is weakly angelic. If $E \in \mathfrak{G}$ is separable and $C_p(X)$ is angelic, the space $C_p(X, E_\sigma)$ is angelic, where $E_\sigma := (E, \sigma(E, E'))$.*

Proof Let $\{A_\alpha : \alpha \in \Sigma\}$ be a web-compact representation for X. Set $G := C_p(X, E)$. By Corollary 4.4.4 it is enough to show that $(G', \sigma(G', G))$ contains a dense web-compact subset. If $(g_t)_t \to g$ in $C_p(X, E)$, for each $s \in X$ and each $x' \in E'$, then

$$(x' g_t(s))_t \to x' g(s).$$

Therefore the map

$$\delta_{sx'} : C_p(X, E) \to \mathbb{R}$$

defined by

$$\delta_{sx'}(g) := x' g(s)$$

is continuous. The set

$$Z := \left\{ \delta_{sx'} : s \in \bigcup \{A_\alpha : \alpha \in \Sigma\}, x' \in E' \right\}$$

is a dense subset of $(G', \sigma(G', G))$. Indeed, if $f \in C_p(X, E)$, and

$$0 = \delta_{sx'}(f) = x' f(s)$$

for each $s \in \bigcup\{A_\alpha : \alpha \in \Sigma\}$, and $x' \in E'$, then $f(s) = 0$ for each $s \in \bigcup\{A_\alpha : \alpha \in \Sigma\}$. Then, by the continuity, $f(s) = 0$ for each $s \in X$. Hence $f = 0$.

Let $E' = \bigcup\{B_\beta : \beta \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of E. Then

$$Z = \bigcup\{D_{\alpha\beta} : (\alpha, \beta) \in \Sigma \times \mathbb{N}^{\mathbb{N}}\},$$

where

$$D_{\alpha\beta} = \{\delta_{sx'} : s \in A_\alpha, x' \in B_\beta\}.$$

To show that Z is a web-compact subset of $(G', \sigma(G', G))$ we need to prove that, if $((\alpha_n, \beta_n))_n \to (\alpha, \beta)$ in $\Sigma \times \mathbb{N}^{\mathbb{N}}$, and for each $n \in \mathbb{N}$

$$\delta_{s_n x'_n} \in D_{\alpha_n \beta_n},$$

the sequence $(\delta_{s_n x'_n})_n$ has an adherent point in $(G', \sigma(G', G))$. As

$$((\alpha_n, \beta_n))_n \to (\alpha, \beta),$$

we note that $\{s_n : n \in \mathbb{N}\}$ is a relatively countably compact subset of X, and $\{x'_n : n \in \mathbb{N}\}$ is an equicontinuous subset of B_γ, where γ is an element of $\mathbb{N}^{\mathbb{N}}$ that verifies $\beta_n \leq \gamma$ for each $n \in \mathbb{N}$. Then the sequence $(\delta_{s_n x'_n})_n$ has a subnet

$$(\delta_{s_{n(d)} x'_{n(d)}})_{d \in D}$$

such that $(s_{n(d)})_{d \in D} \to s \in X$ and

$$(x'_{n(d)})_{d \in D} \to x'$$

in $(E', \sigma(E', E))$. From the equicontinuity it follows that $(x'_{n(d)})_{d \in D} \to x'$ uniformly on the precompact subsets of E. The proof will be finished if we show that

$$(\delta_{s_{n(d)} x'_{n(d)}})_{d \in D} \to \delta_{sx'}$$

in $(G', \sigma(G', G))$. In other words, we have to prove that for each $f \in C(X, E)$ we have

$$\lim_{d \in D} x'_{n(d)}[f(s_{n(d)})] = x'[f(s)].$$

This equality follows from the following facts:

(*) $\lim_{d\in D} x'[f(s_{n(d)})] = x'[f(s)]$.
(**) As $\{f(s_n) : n \in \mathbb{N}\}$ is a relatively countably compact subset of E (therefore precompact),

$$\lim_{d\in D} x'_{n(d)}[f(s_{n(d)})] = \lim_{d\in D} x'[f(s_{n(d)})].$$

Now assume that E is separable. Then $(E', \sigma(E', E))$ is separable; see Corollary 12.3.3. If G is a countable and dense subset in $(E', \sigma(E', E))$, $\xi := \sigma(E, G)$ is a metrizable locally convex topology on E with $\xi \leq \sigma(E, E')$. The assumptions of Fremlin's Theorem 4.1.5 are satisfied: $C_p(X)$ is angelic and E_ξ is metrizable, so $C_p(X, E_\xi)$ is angelic, where $E_\xi := (E, \xi)$. Note that $C_p(X, E_\sigma) \subset C_p(X, E_\xi)$. Then we deduce that $C_p(X, E_\sigma)$ is angelic. □

Applying Proposition 11.2.3 we provide the following vector-valued version of Theorem 9.2.17.

Theorem 11.2.4 *Let E be a separable lcs in the class $\mathfrak{G}$. Let ξ be a regular topology on $C(X, E)$ stronger than the pointwise topology of $C(X, E)$. The following assertions are equivalent:*

(i) $(C(X, E), \xi)$ *is K-analytic.*
(ii) $(C(X, E), \xi)$ *admits a compact resolution.*
(iii) $(C(X, E), \xi)$ *admits a relatively countably compact resolution.*

Proof Since each K-analytic space admits a compact resolution, it is enough to show (iii) ⇒ (i): If $(C(X, E), \xi)$ admits a relatively countably compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, the family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bounded resolution on $C_p(X, E)$ in the topology τ_p. $C_p(X)$ is isomorphic to a subspace of $C_p(X, E)$. Hence $C_p(X)$ admits a bounded resolution. By Theorem 9.4.19(i) the space υX is web-compact, so $C_p(\upsilon X)$ is angelic by Proposition 4.3.4. Consequently, $C_p(X)$ is angelic by Lemma 9.2.10. Now we apply Proposition 11.2.3 to conclude that $C_p(X, E_\sigma)$ is angelic. As $C_p(X, E) \subset C_p(X, E_\sigma)$, the space $C_p(X, E)$ is angelic, and then by Theorem 4.1.1, the space $(C(X, E), \xi)$ is also angelic. Finally, since angelic spaces having a resolution consisting of relatively countably compact sets are K-analytic (Corollary 3.2.9), the space $(C(X, E), \xi)$ is K-analytic. □

We present a couple of examples (adopted from [115]) motivated by Theorem 11.2.4.

Example 11.2.5 If X is an Eberlein compact space and Y is a Polish space, $C_p(X, Y)$ is an angelic space having a compact resolution. Hence $C_p(X, Y)$ is K-analytic.

Proof Since every Polish space is a G_δ-subset of $\mathbb{R}^{\mathbb{N}}$, it is enough to show that $C_p(X, Y)$ has a compact resolution if Y is an open subset of $\mathbb{R}^{\mathbb{N}}$. Since

$$C_p(X, \mathbb{R}^{\mathbb{N}}) = (C_p(X))^{\mathbb{N}},$$

and X is Eberlein compact, $C_p(X, \mathbb{R}^{\mathbb{N}})$ has a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Let ϱ be a metric defining the topology of $\mathbb{R}^{\mathbb{N}}$. For an open subset Y of $\mathbb{R}^{\mathbb{N}}$ the family $\{K_\alpha \cap C_p(X, Y) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution on $C_p(X, Y)$. If $\alpha = (a_n)_n$, then

$$H_\alpha := \{f \in K_\alpha \cap C_p(X, Y) : \varrho(f(x), \mathbb{R}^{\mathbb{N}} \setminus Y) \geq a_1^{-1}, x \in X\}$$

is a closed subset of K_α, hence compact. If $f \in C_p(X, Y)$, then $f(X) \subset Y$ is compact, and there exists $p \in \mathbb{N}$ such that

$$\varrho(f(x), \mathbb{R}^{\mathbb{N}} \setminus Y) \geq p^{-1}$$

for each $x \in X$. There exists $\beta = (b_n)_n$ such that $f \in K_\beta$. Define $\alpha = (a_n)_n$ by $a_1 = b_1 + p$, $a_n = b_n$, $n \geqslant 2$. Then $f \in H_\alpha$. Hence $\{H_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution of $C_p(X, Y)$. By Proposition 11.2.3 the space $C_p(X, \mathbb{R}^{\mathbb{N}}$ is angelic, so its subset $C_p(X, Y)$ is angelic, too. Finally we deduce that $C_p(X, Y)$ is K-analytic. □

Example 11.2.6 If X is a separable and normed space and E is a Fréchet–Montel space, the space $\mathcal{L}_c(X, E)$ of all continuous linear maps from X into E endowed with the compact-open topology is K-analytic.

Proof Let B the unit ball in X, and let $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset of E (every Fréchet Montel space is separable; see Corollary 6.4.5). Set

$$K_\alpha := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{n_k} B(x_j, k^{-1}),$$

where $B(x_j, k^{-1})$ is the closed ball in E with the centre at the point x_j and radius k^{-1} for $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ and all $j, k \in \mathbb{N}$. Then $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution on E, and each compact set K in E is contained in some K_α. Set

$$D_\alpha := \{f \in \mathcal{L}(X, E) : f(B) \subset K_\alpha\}.$$

By Ascoli's theorem the family $\{D_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a relatively compact resolution on $\mathcal{L}_c(X, E)$. We apply Proposition 11.2.3 to show that $C_p(X, E_\sigma)$ is angelic. The inclusion $C_p(X, E) \subset C_p(X, E_\sigma)$ and Theorem 4.1.1 imply that $C_c(X, E)$ is angelic, too. Then $\mathcal{L}_c(X, E) \subset C_c(X, E)$ is angelic. Corollary 3.2.9 shows that $\mathcal{L}_c(X, E)$ is K-analytic. □

Example 11.2.7 If I is uncountable, $\mathcal{L}_p(\ell^2(I), \ell^2(\mathbb{N}))$ does not admit a compact resolution and $\mathcal{L}_c(\ell^2(I), \ell^2(\mathbb{N})_\sigma)$ admits a compact resolution.

Proof Since $C_p(\ell^2(I)_\sigma, \ell^2(\mathbb{N})_\sigma)$ is angelic (by second part of Proposition 11.2.3), we note that the space $\mathcal{L}_p(\ell^2(I), \ell^2(\mathbb{N}))$ is angelic as it is isomorphic to the angelic space $\mathcal{L}_p(\ell^2(I)_\sigma, \ell^2(\mathbb{N})_\sigma)$. By Talagrand [576] the space $\mathcal{L}_p(\ell^2(I), \ell^2(\mathbb{N}))$ is not K-analytic, so $\mathcal{L}_p(\ell^2(I), \ell^2(\mathbb{N}))$ does not have a compact resolution (since the

last space is angelic). By Theorem 3.3.12 there exists on $\ell^2(\mathbb{N})$ a quasi-(LB)-representation $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that every Banach disc in $\ell^2(\mathbb{N})$ is contained in some A_α. For each $\alpha \in \mathbb{N}$ let B_α be the closure of A_α in $\ell^2(\mathbb{N})$. Set

$$K_\alpha := \{f \in \mathcal{L}(\ell^2(I), \ell^2(\mathbb{N})_\sigma) : f(B) \subset B_\alpha\},$$

where B is the unit ball in $\ell^2(I)$. Applying Ascoli's theorem we deduce that $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution on $\mathcal{L}_c(\ell^2(I), \ell^2(\mathbb{N})_\sigma)$ of relatively compact sets; hence $\mathcal{L}_c(\ell^2(I), \ell^2(\mathbb{N})_\sigma)$ admits a compact resolution. □

A similar argument yields the following: If E is a separable normed space and F is a reflexive Fréchet space, $\mathcal{L}_c(E, F_\sigma)$ is K-analytic. By Proposition 11.2.3 the space $C_p(X, C[0,1]_\sigma)$ is angelic, where X is the closed unit ball in $\ell^2(I)$ endowed with the weak topology. Canela [115] proved that, if I is uncountable, $C_p(X, C[0,1]_\sigma)$ is not K-analytic. Hence we have:

Example 11.2.8 Let X be the closed unit ball in $\ell^2(I)$ endowed with the weak topology for uncountable I. Then $C_p(X, C[0,1]_\sigma)$ does not admit a compact resolution.

Chapter 12
Weakly Realcompact Locally Convex Spaces

Abstract In this chapter, we continue the study of spaces in the class $\mathfrak{G}$. We prove that the weak * dual $(E', \sigma(E', E))$ of a lcs E in the class $\mathfrak{G}$ is K-analytic if and only if $(E', \sigma(E', E))$ is Lindelöf if and only if $(E', \sigma(E', E))$ has countable tightness. We show that every quasibarrelled space in the class $\mathfrak{G}$ has countable tightness both for the weak and the original topologies. This extends a classical result of Kaplansky for a metrizable lcs. Although (DF)-spaces belong to the class $\mathfrak{G}$, concrete examples of (DF)-spaces without countable tightness are provided. On the other hand, there are many Banach spaces E for which E endowed with the weak topology is not Lindelöf. We show, however (following Khurana), that every WCG Fréchet space E is weakly K-analytic. An example due to Pol showing that there exists a Banach space $C(X)$ over a compact scattered space X such that $C(X)$ is weakly Lindelöf and not WCG is presented. We show (after Amir and Lindenstrauss) that every non-separable reflexive Banach space contains a complemented separable subspace. Several consequences are provided.

12.1 Tightness and Quasi-Suslin Weak Duals

This section deals with lcs E whose *weak** dual $(E', \sigma(E', E))$ is K-analytic (or analytic, or at least quasi-Suslin).

A classical result of Kaplansky, see [240, Theorem, p. 37], states that if X is a σ-compact space and Z is a metric space, the space $C_p(X, Z)$ has countable tightness. This applies to show that the weak topology of a metrizable lcs E has countable tightness. Indeed, since E is metrizable, $(E', \sigma(E', E))$ is σ-compact, so $C_p((E', \sigma(E', E))$ has countable tightness by Kaplansky's result. On the other hand, as

$$(E, \sigma(E, E')) \subset C_p((E', \sigma(E', E)),$$

we deduce that the space $(E, \sigma(E, E'))$ has also countable tightness.

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Recall that for a lcs E by $\tau_{pc}(E', E)$ we denote the topology on E' of the uniform convergence on precompact sets of E. Clearly $\sigma(E', E) \leq \tau_{pc}(E', E)$, and if E is quasi-complete, then also $\tau_{pc}(E', E) \leq \mu(E', E)$. Although the Mackey dual of an analytic space E need not be analytic, see Theorem 6.5.3, we know from Proposition 6.7.5 that $(E', \tau_{pc}(E', E))$ is analytic for each separable (LF)-space E. We show that the *weak** dual of any lcs in the class $\mathfrak{G}$ is a quasi-Suslin space; see Theorem 12.1.1.

Valdivia [611] proved that, if E is a Fréchet space, and F is the strong dual of E, the space $(F', \sigma(F', F))$ is quasi-Suslin, and $(F', \sigma(F', F))$ is K-analytic if and only if $(F, \mu(F, F'))$ is barrelled. Clearly F, as a (DF)-space, has a closed $\mathfrak{G}$-representation $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, i.e. each set K_α is $\sigma(F', F)$-closed. This motivates Theorem 12.1.1 (proved in [221, Theorem 4]) that extends [117, Proposition 1]. We say that a subset $A \subset E$ is *full* if it contains all adherent points in E of sequences from A.

Theorem 12.1.1 *Let E be a lcs in the class $\mathfrak{G}$. Then $(E', \sigma(E', E))$ is a quasi-Suslin space.*

Proof Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation for E. For each $\alpha \in \mathbb{N}^{\mathbb{N}}$ define

$$B_\alpha := \bigcup \{S^{\circ\circ} : S \subset A_\alpha, |S| \leq \aleph_0\}.$$

As countable unions of countable sets are countable, each sequence $(u_n)_n$ in B_α belongs to a bipolar of a sequence $(v_n)_n$ in A_α which is clearly equicontinuous by the assumptions on $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. By the Alaoglu–Bourbaki theorem, see [322, Theorem 8.5.2], the set $\{v_n : n \in \mathbb{N}\}^{\circ\circ}$ is absolutely convex, equicontinuous, and $\sigma(E', E)$-compact. Hence B_α is absolutely convex and $(u_n)_n$ has cluster points which belong to $\{v_n : n \in \mathbb{N}\}^{\circ\circ} \subset B_\alpha$. This proves that B_α is absolutely convex, weakly countably compact and full. Also the family $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation of E. From Proposition 3.2.5 it follows that E is quasi-Suslin. □

This implies the following corollary; see [127].

Corollary 12.1.2 *Let E be a lcs in the class $\mathfrak{G}$ such that $(E, \sigma(E, E'))$ is web-compact. Then $(E', \sigma(E', E))$ is K-analytic. In particular, every separable lcs in class $\mathfrak{G}$ has its precompact dual (E', τ_{pc}) analytic.*

Proof Since $(E', \sigma(E', E))$ is a subspace of $C_p(E, \sigma(E, E'))$ (the last space is angelic by Theorem 4.2.1), the space $(E', \sigma(E', E))$ is angelic. By Theorem 12.1.1 the space $(E', \sigma(E', E))$ is quasi-Suslin. Apply Corollary 3.2.9. For the particular case see Proposition 6.1.4. □

Corollary 12.1.2 extends Proposition 6.7.5 since (LF)-spaces belong to the class $\mathfrak{G}$. We need the following characterization of weakly realcompact lcs due to Corson; see [611, Page 137] for the proof.

Proposition 12.1.3 *Let (E, E') be a dual pair. Let $\{F_i : i \in I\}$ be the family of all separable closed subspaces of $(E', \sigma(E', E))$. Then the following statements are equivalent:*

(i) $(E, \sigma(E, E'))$ *is realcompact.*
(ii) $E = \{z \in (E')^* : z|F_i$ *is* $\sigma(E', E) -$ *continuous for each* $i \in I\}$.

We are ready to prove the following important result [118]; the implication (i) $\Rightarrow$ (iv) is true in general; see Lemma 9.4.20.

Theorem 12.1.4 (Cascales–Kąkol–Saxon) *Let E be a lcs in the class $\mathfrak{G}$. The following statements are equivalent:*

(i) $(E, \sigma(E, E'))$ *has countable tightness.*
(ii) For each space Y a function from E into Y that is $\sigma(E, E')$-continuous restricted to $\sigma(E, E')$-closed and separable subsets of E is $\sigma(E, E')$-continuous on E.
(iii) Every linear functional on E that is $\sigma(E, E')$-continuous restricted to $\sigma(E, E')$-closed and separable subspaces of E is $\sigma(E, E')$-continuous on E.
(iv) $(E', \sigma(E', E))$ *is realcompact.*
(v) $(E', \sigma(E', E))$ *is K-analytic.*
(vi) $(E', \sigma(E', E))^n$ *is Lindelöf for every* $n \in \mathbb{N}$.
(vii) $(E', \sigma(E', E))$ *is Lindelöf.*

Proof (i) $\Rightarrow$(ii): Let $f : E \to Y$ be a map $\sigma(E, E')$-continuous when restricted to the $\sigma(E, E')$-closed and separable subsets of E. It is enough to show that for any set $A \subset E$ and $x \in \overline{A}^{\sigma(E,E')}$ we have $f(x) \in \overline{f(A)}$. By the assumption there is countable $D \subset A$ such that $x \in \overline{D}^{\sigma(E,E')}$; thus $f|_{\overline{D}^{\sigma(E,E')}}$ is continuous. Hence $f(x) \in \overline{f(D)} \subset \overline{f(A)}$.

(ii) $\Rightarrow$(iii): Obvious.

(iii) $\Rightarrow$(iv): This follows from Proposition 12.1.3.

(iv) $\Rightarrow$(v): By Theorem 12.1.1 the space $(E', \sigma(E', E))$ is quasi-Suslin, so by Theorem 3.1.6 $(E', \sigma(E', E))$ admits a resolution of relatively countably compact sets. By the assumption $(E', \sigma(E', E))$ is realcompact, so every relatively countably compact set is relatively compact. Now it is enough to apply Proposition 3.2.7 (v) to deduce that $(E', \sigma(E', E))$ is K-analytic.

(v) $\Rightarrow$(vi): Since countable products of K-analytic spaces are K-analytic, and K-analytic spaces are Lindelöf, the conclusion follows.

(vi) $\Rightarrow$(vii): is clear.

(vi) $\Rightarrow$(i): Since $(E', \sigma(E', E))^n$ is a Lindelöf space for all $n \in \mathbb{N}$, by Proposition 9.4.1, the space $C_p(E', \sigma(E', E))$ has countable tightness. Since subspaces of spaces with countable tightness have countable tightness, we note that $(E, \sigma(E, E'))$ has countable tightness.

Finally, since Lindelöf spaces are realcompact (Proposition 3.2.6), we have (vii)$\Rightarrow$(iv). □

There is another approach to prove Theorem 12.1.4 by using Proposition 9.4.17. Indeed, if $E \in \mathfrak{G}$, then by Theorem 12.1.1 the space $(E', \sigma(E', E))$ is quasi-Suslin. Consequently, the space $\upsilon(E', \sigma(E', E))$ is K-analytic by Proposition 9.4.17. Since the countable tightness of $(E, \sigma(E, E'))$ of a lcs E implies that $(E', \sigma(E', E))$ is realcompact (by applying Lemma 9.4.20), the space $(E', \sigma(E', E))$ is a K-analytic space if $E \in \mathfrak{G}$ and $(E, \sigma(E, E'))$ has countable tightness.

A topological space X is said to have the *countable Suslin number* ($\mathrm{c}(X) \leq \aleph_0$) if the cardinality of every pairwise disjoint family of open sets in X does not exceed $\aleph_0$. It is known that every regular Lindelöf space is a paracompact space; this is a result due to Morita; see [455]. Moreover, every paracompact space having the countable Suslin number is a Lindelöf space; see [38, Chapter 2, exc.393]. By [75, 516] the weak topology $\sigma(E, E')$ of a Banach space E is Lindelöf if and only if $\sigma(E, E')$ is paracompact if and only if $\sigma(E, E')$ is normal. We note also the following fact which supplements Theorem 12.1.4.

Corollary 12.1.5 *Let E be a lcs. Then $(E, \sigma(E, E'))$ $(resp.(E', \sigma(E', E)))$ is a Lindelöf space if and only if $(E, \sigma(E, E'))$ $(resp.(E', \sigma(E', E)))$ is a paracompact space.*

Proof Every regular Lindelöf space is paracompact. Now assume that the space $(E', \sigma(E', E))$ is paracompact. Since $(E', \sigma(E', E))$ is dense in the product $\mathbb{R}^I$ for some set I, and the Suslin number $\mathrm{c}(\mathbb{R}^I)$ of $\mathbb{R}^I$ is countable for any set I, see, for example, [38, Chapt. 2, No. 383], we note that $\mathrm{c}(E', \sigma(E', E)) = \aleph_0$. Then, by the mentioned result in [38, Chapt. 2, No. 393] we have that $(E', \sigma(E', E))$ is Lindelöf. Since $(E, \sigma(E, E')) \subset C_p(E', \sigma(E', E)) \subset \mathbb{R}^{E'}$, the remaining case concerning $(E, \sigma(E, E'))$ one proves similarly. □

12.2 A Kaplansky-Type Theorem about Tightness

We provide a large subclass in $\mathfrak{G}$ of lcs having countable tightness. The following theorem due to Cascales, Kąkol, and Saxon [118, Proposition 4.8] extends Kaplansky's result [240] stating that the weak topology of a metrizable lcs has countable tightness. Recall again that E is [quasi]barrelled, if every $\sigma(E', E)$-bounded [every $\beta(E', E)$-bounded] set is equicontinuous. Every (LM)-space [every (LF)-space] is quasibarrelled [barrelled].

Theorem 12.2.1 (Cascales–Kąkol–Saxon) *Every quasibarrelled space E in $\mathfrak{G}$ has countable tightness and the same also holds true for $(E, \sigma(E, E'))$.*

Proof Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of E. Since E is quasibarrelled and the condition (c) holds, each A_α is equicontinuous. Replacing each A_α by its $\sigma(E', E)$-closed absolutely convex hull we may assume that each A_α is a $\beta(E', E)$-Banach disc (the strong dual of quasibarrelled spaces must be quasicomplete). This implies that the space $(E', \beta(E', E))$ is a quasi-(LB)-space, and therefore, using

Theorem 3.3.12 there exists a family of $\beta(E', E)$-Banach discs of E' (that we again denote by $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$) such that:

(i) $E' = \cup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.
(ii) $A_\alpha \subset A_\beta$ if $\alpha \leq \beta$ in $\mathbb{N}^{\mathbb{N}}$.
(iii) For every $\beta(E', E)$-Banach disc $B \subset E'$ there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $B \subset A_\alpha$.

Consider a web $\mathcal{W} = \{C_{n_1,n_2,\ldots,n_k}\}$, where each $C_{n_1,n_2,\ldots,n_k}$ is defined as usual. Then $\mathcal{W}$ is a web having the following properties:

$C_{n_1,n_2,\ldots,n_k} \subset C_{m_1,m_2,\ldots,m_k}$ for $n_j \leq m_j$ if $k \in \mathbb{N}$ and $1 \leq j \leq k$.

For every $\alpha = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$ and every $\beta(E', E)$-neighbourhood of zero $U \subset E'$ there exist $n_U \in \mathbb{N}$, $p_U \geq 0$, such that $C_{n_1,n_2,\ldots,n_U} \subset p_U U$.

The order condition follows from the definitions. The remaining condition we check as follows:

Every set A_α (as a Banach disc) is $\beta(E', E)$-bounded. Note that the web $\mathcal{W}$ is bounded. Indeed, assume that this does not hold. Then we find $\alpha = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$ and a $\beta(E', E)$-neighbourhood U of 0 in E' such that

$$C_{n_1,n_2,\ldots,n_k} \not\subset kU,$$

for all $k \in \mathbb{N}$. For every positive integer k there is $\alpha_k = (a_n^k)_n \in \mathbb{N}^{\mathbb{N}}$ with $\alpha_k|k = (n_1, n_2, \ldots, n_k)$, such that $A_{\alpha_k} \not\subset kU$. We define now

$$a_n = \max\{a_n^k : k \in \mathbb{N}\}$$

for all $n \in \mathbb{N}$ and $\gamma = (a_n)_n$. It is clear that $\gamma \geq \alpha_k$ and $A_\gamma \not\subset kU$ if $k \in \mathbb{N}$, which contradicts the boundedness of A_γ.

Given positive integers $k, n_1, n_2, \ldots, n_k$ we define

$$D_{n_1,n_2,\ldots,n_k} := \overline{C_{n_1,n_2,\ldots,n_k}}^{\sigma(E',E)}.$$

Since $\beta(E', E)$ has a basis of neighbourhoods of zero consisting of $\sigma(E', E)$-closed sets, and as the web $\mathcal{W}$ is bounded, for every $\alpha = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$ and every $\beta(E', E)$-neighbourhood U of zero in E' there exist $n_U \in \mathbb{N}$, $p_U \geq 0$ such that $D_{n_1,n_2,\ldots,n_U} \subset p_U U$.

If we re-label

$$A_\alpha := \bigcap_{k=1}^{\infty} D_{n_1,n_2,\ldots,n_k},$$

the new family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ still satisfies the desired properties. Since $(E', \beta(E', E))$ is quasi-complete, every $\beta(E', E)$-bounded set is contained in a $\beta(E', E)$-Banach disc, which means that the family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental family of equicontinuous subsets of E'. Taking polars in the dual pair (E, E'), we note that the family $\{A_\alpha^\circ : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a basis of neighbourhoods of zero in E. On

the other hand, since $D_{n_1,n_2,\ldots,n_U} \subset p_U U$, it follows that for every $\alpha = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$ the increasing sequence

$$D^\circ_{n_1} \subset D^\circ_{n_1,n_2} \subset \cdots \subset D^\circ_{n_1,n_2,\ldots,n_k} \subset \cdots$$

is bornivorous. Then, by Proposition 2.4.8, we have for every $\varepsilon > 0$ the inclusion

$$A^\circ_\alpha = \overline{\bigcup_{k=1}^{\infty} D^\circ_{n_1,n_2,\ldots,n_k}}^{\sigma(E,E')} \subset (1+\varepsilon) \bigcup_{k=1}^{\infty} D^\circ_{n_1,n_2,\ldots,n_k}. \tag{12.1}$$

Summarizing all facts we deduce that, if we define for $\alpha = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$

$$U_\alpha := \bigcup_{k=1}^{\infty} D^\circ_{n_1,n_2,\ldots,n_k},$$

the family $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a basis of neighbourhoods of zero in E.

We need to show that the tightness of E is countable: Take arbitrary set $A \subset E$ with $0 \in \overline{A}$. Then the set B of elements $x_{n_1,n_2,\ldots,n_k}$ is countable and $0 \in \overline{B}$, where $x_{n_1,n_2,\ldots,n_k}$ is a chosen point in $D^\circ_{n_1,n_2,\ldots,n_k} \cap A$, if

$$D^\circ_{n_1,n_2,\ldots,n_k} \cap A \neq \emptyset, \quad k, n_1, n_2, \ldots, n_k \in \mathbb{N}.$$

Finally, we prove that $(E, \sigma(E, E'))$ has countable tightness. Similarly as we did in the proof of (i)⇒(ii)⇒(iii) in Theorem 12.1.4, every linear functional on E that is continuous on each separable and closed subspace of E is continuous. The families of closed and separable subspaces of E and $\sigma(E, E')$-closed and separable subspaces of E, respectively, coincide. Hence the countable tightness of E implies condition (iii) from Theorem 12.1.4, and so $(E, \sigma(E, E'))$ has countable tightness. □

Theorem 12.1.4 fails if E is not in the class $\mathfrak{G}$. To prove this we need the following additional fact.

Proposition 12.2.2 *The space $C_p(X)$ belongs to the class $\mathfrak{G}$ if and only if X is countable.*

Proof Since $C_p(X)$ is a dense subspace of the product $\mathbb{R}^X$, we apply Proposition 3.2.2. Another proof: Assume $C_p(X)$ belongs to the class $\mathfrak{G}$. Then $\mathbb{R}^X$ is a space in the class $\mathfrak{G}$ (which clearly is a Baire space). Now Theorem 12.2.1 applies (since $\mathbb{R}^X$ has countable tightness if and only if X is countable). □

Example 12.2.3 There exists a lcs whose weak topology has countable tightness and its *weak** dual is not K-analytic.

Proof Let X be an uncountable regular Lindelöf P-space, i.e. every G_δ set in X is open. Since X^n is a Lindelöf space for any $n \in \mathbb{N}$, the space $C_p(X)$ has countable tightness, Theorem 9.4.1. By Proposition 12.2.2 the space $C_p(X)$ is not in the class $\mathfrak{G}$. Assume that $F := C_p(X)'_\sigma$ is K-analytic. Then F has a compact resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in F. Since X is a normal P-space (see Lemma 6.1.3), every topologically bounded set in X is finite, and by Proposition 2.4.16 the space $C_p(X) = C_c(X)$ is barrelled. Hence every set A_α is equicontinuous, so $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation. Consequently, $C_p(X)$ belongs to $\mathfrak{G}$, a contradiction. □

Example 12.2.4 (Cascales–Kąkol–Saxon) There exist (DF)-spaces with uncountable tightness whose weak topology has countable tightness.

Proof Let Λ be an uncountable indexing set. For each $S \subset \Lambda$ define $E_S = \{u \in \ell^2(\Lambda) : u(x) = 0 \text{ for } x \notin S\}$. Let E be the Banach space $\ell^2(\Lambda)$ endowed with the coarsest topology ξ such that the projection of E onto the Banach space E_S along $E_{\Lambda\setminus S}$ is continuous for every countable $S \subset \Lambda$. A basis of neighbourhoods of zero for E consists of the sets U of the form

$$U = V + E_{\Lambda\setminus S},$$

where V is a positive multiple of the unit ball in the Banach space $\ell^2(\Lambda)$ and S is a countable subset of Λ. The space $(E, \tau) := \ell^2(\Lambda)$ admits the (DF)-space topology ξ that is not quasibarrelled, $\sigma(E, E') < \xi < \tau$, and such that (E, ξ) does not have countable tightness. Note that $\sigma(E', E) = \sigma(E', E'')$ is K-analytic (since $\ell^2(\Lambda)$ is reflexive) and not analytic (as non-separable). Since (E, τ) is metrizable, the topology $\sigma(E, E')$ has countable tightness. On the other hand, the space (E, ξ) has uncountable tightness. Indeed, the set B of the characteristic functions of the singleton subsets of Λ has 0 in its closure but not in the closure of any countable subset of B. We see from this example that a (DF)-space may fail to have countable tightness even when the weak topology does. □

Every (WCG) Banach space E is weakly K-analytic. We prove this result in Theorem 12.4.6. In particular, every reflexive Banach space E is weakly K-analytic. This follows also from Theorem 12.1.4. Indeed, for the strong dual $(E', \beta(E', E))$ Theorem 12.2.1 applies to derive that $\sigma(E', E)$ has countable tightness. $(E, \sigma(E, E'))$ is K-analytic by Theorem 12.1.4.

Let E be a lcs in the class $\mathfrak{G}$. We call a $\mathfrak{G}$-representation $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ *bornivorous* if every $\beta(E', E)$-bounded set is contained in some A_α. It is easy to see that every dual metric space (hence any (DF)-space), every (LM)-space, and every quasibarrelled space in the class $\mathfrak{G}$ have a bornivorous $\mathfrak{G}$-representation; see [119, Lemma 2], [213, Propositions 1 and 2], [221, Theorem 9]. A general fact will be proved.

Theorem 12.2.5 (Cascales–Kąkol–Saxon) *Let E be a lcs in the class $\mathfrak{G}$. Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a bornivorous $\mathfrak{G}$-representation. The following assertions are equivalent.*

(i) *The space* $\left(E, \sigma\left(E, E'\right)\right)$ *has countable tightness.*
(ii) *The space* $\left(E, \mu\left(E, E'\right)\right)$ *is quasibarrelled.*
(iii) *The space* $\left(E, \mu\left(E, E'\right)\right)$ *has countable tightness.*

Proof (i) $\Rightarrow$ (ii): Theorem 12.1.4 implies that $\left(E', \sigma\left(E', E\right)\right)$ is realcompact and thus so is the closed subset $A_\alpha^{\circ\circ}$. The latter is also countably compact due to the bornivorous condition (which implies $A_\alpha^{\circ\circ} \subset A_\beta$) and the equicontinuity of sequences (in A_β). Hence $A_\alpha^{\circ\circ}$ is $\sigma\left(E', E\right)$-compact, and $A_\alpha^{\circ\circ\circ} = A_\alpha^{\circ}$ is a neighbourhood of zero in the Mackey topology. Each $\beta\left(E', E\right)$-bounded set is contained in some A_α and thus is $\mu\left(E, E'\right)$-equicontinuous. This proves that the Mackey topology is quasibarrelled.
(ii) $\Rightarrow$ (iii): Clearly $\left\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ is a $\mathfrak{G}$-representation for $\left(E, \mu\left(E, E'\right)\right)$, as well, and now (iii) follows from Theorem 12.2.1.
(iii) $\Rightarrow$ (i): Countable tightness of $E := (E, \mu(E, E'))$ implies the condition (iii) from Theorem 12.1.4 (see the proof of Theorem 12.2.1, second part). By Theorem 12.1.4 the space $(E, \sigma(E, E'))$ has countable tightness. □

If $[X, 1] := \{f \in C(X) : |f(x)| \leqslant 1 \text{ for all } x \in X\}$ is absorbing in $C(X)$, i.e. X is *pseudocompact*, by $C_u(X)$ we denote the space $C(X)$ endowed with the uniform norm topology having the unit ball $[X, 1]$. It is well known that, if X is pseudocompact and $C_c(X)' = C_u(X)'$, the space X is compact.

Theorem 12.2.6 *The following assertions are equivalent for a* (df)*-space* $C_c(X)$.

(1) X *is compact.*
(2) $C_c(X)$ *coincides with the Banach space* $C_u(X)$.
(3) $C_c(X)$ *has countable tightness.*
(4) The weak topology of $C_c(X)$ *has countable tightness.*
(5) The Mackey topology $\mu(C_c(X), C_c(X)')$ *has countable tightness.*

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Obvious.
(3) $\Rightarrow$ (4) follows from the proof in Theorem 12.2.5 (iii) $\Rightarrow$ (i) with $E := C_c(X)$.
(4) $\Leftrightarrow$ (5) from Theorem 12.2.5.
(5) $\Rightarrow$ (1): By Theorem 12.2.5 condition (5) implies that the topology $\mu(C_c(X), C_c(X)')$ is quasibarrelled, and therefore the bornivorous barrel $[X, 1]$ (see Theorem 2.6.4) is a $\mu(C_c(X), C_c(X)')$-neighbourhood of zero. Hence $C_c(X)' = C_u(X)'$, which implies that X is compact. □

Example 2.6.13 provided examples of (df)-spaces $C_c(X)$ that are not (DF)-spaces. Thus, the following corollary provides a class of spaces which are quasi-Suslin and not K-analytic.

Corollary 12.2.7 *Let* $E := C_c(X)$ *be any* (df)*-space that is not a* (DF)*-space. Then the weak* dual of* E *is quasi-Suslin and not K-analytic.*

Proof By Theorem 12.1.1 $(E', \sigma(E', E))$ is quasi-Suslin. By Theorems 12.2.6 and 12.1.4 the space X is compact if and only if $(E', \sigma(E', E))$ is K-analytic. □

12.3 K-Analytic Spaces in the Class $\mathfrak{G}$

In [115, Proposition 7] Canela proved that a weakly K-analytic lcs E satisfies $\operatorname{dens}(E) \leq \operatorname{dens}(E', \sigma(E', E))$. Moreover, if E is additionally metrizable, the equality $\operatorname{dens}(E) = \operatorname{dens}(E', \sigma(E', E))$ holds, where as usual $\operatorname{dens}(E)$ denotes the density of E. This result extended Talagrand's [579, Theorem 6.1] stating the same for (WCG) Banach spaces. In [128, Theorem 13] Cascales and Orihuela extended Canela's result to weakly Lindelöf Σ-spaces in the class $\mathfrak{G}$. Our next result extends the above results.

Proposition 12.3.1 *Let E be a lcs such that weak* dual $(E', \sigma(E', E))$ is a quasi-Suslin space and $(E, \sigma(E, E'))$ is a Lindelöf Σ-space. Then we have* $\operatorname{dens}(E', \sigma(E', E)) = \operatorname{dens}(E)$.

Proof Since $(E, \sigma(E, E'))$ is a Lindelöf Σ-space, the space $C_p(E, \sigma(E, E'))$ is angelic by applying Theorem 4.2.1. As the space $(E', \sigma(E', E))$ is included in $C_p(E, \sigma(E, E'))$, we note that $(E', \sigma(E', E))$ is angelic. Now Corollary 3.2.9 applies to conclude that $(E', \sigma(E', E))$ is K-analytic. Let B be a dense subset of E of cardinality at most $\aleph$. Then $\sigma(E', B)$ is Hausdorff, $\sigma(E', B) \leq \sigma(E', E)$, and $(E', \sigma(E', B))$ has a basis of neighbourhoods of zero of cardinality at most $\aleph$. $(E', \sigma(E', E))$ is Lindelöf. Hence the space $(E', \sigma(E', B))$ has a basis of open sets of cardinality at most $\aleph$. By Lemma 3.1.13 we have $\operatorname{dens}(E', \sigma(E', E)) \leq \aleph$. If B is a dense subset in $(E', \sigma(E', E))$ with $|B| \leq \aleph$, the Lindelöf property yields $\omega(E, \sigma(E, B)) \leq \aleph$. We apply Lemma 3.1.13 and conclude $\operatorname{dens}(E, \sigma(E, E')) \leq \aleph$. From the equality $\operatorname{dens}(E, \sigma(E, E')) = \operatorname{dens}(E)$ it follows that $\operatorname{dens}(E) \leq \aleph$. □

Corollary 12.3.2 *Let E be a lcs in the class $\mathfrak{G}$ such that $(E, \sigma(E, E'))$ is a Lindelöf Σ-space. Then* $\operatorname{dens}(E', \sigma(E', E)) = \operatorname{dens}(E)$.

Corollary 12.3.3 *If E is a separable lcs such that $(E', \sigma(E', E))$ is quasi-Suslin, the space $(E', \sigma(E', E))$ is analytic. In particular, then the weak* dual of a separable lcs in the class $\mathfrak{G}$ is separable.*

Proof The separability of E combined with Corollary 4.1.3 yields that the space $(E', \sigma(E', E))$ is angelic. Consequently, it is K-analytic by Corollary 3.2.9. Finally, by Proposition 6.1.4 we know that $(E', \sigma(E', E))$ is analytic. □

If for a compact space X the space $C(X)$ is a (WCG) Banach space, X is Talagrand compact. According to theorem of Amir–Lindenstrauss [8], see also [198, Theorem 12.12], [466], and Theorem 9.1.1, for a compact space X, the space $C(X)$ is a (WCG) Banach space if and only if X is Eberlein compact. The following simple observation motivates Theorem 12.3.5.

Proposition 12.3.4 *Let E be a metrizable lcs. Then a compact set in $(E, \sigma(E, E'))$ is Talagrand compact.*

Proof Since E is metrizable, $(E', \sigma(E', E))$ is σ-compact. Hence the space $(E', \sigma(E', E))$ is web-compact with $\Sigma = \mathbb{N}^{\mathbb{N}}$. By Corollary 4.4.2 every compact set in $C_p((E', \sigma(E', E))$ is Talagrand compact. Now the conclusion holds since $(E, \sigma(E, E')) \subset C_p((E', \sigma(E', E))$. □

It turns out, see [127, Theorem 12], that Talagrand compact sets can be characterized as weakly compact sets in a lcs in the class $\mathfrak{G}$. The implication (ii) $\Rightarrow$ (i) in Theorem 12.3.5 follows also from Corollary 4.4.4.

Theorem 12.3.5 *For a compact space X the following conditions are equivalent.*

(i) X is Talagrand compact.
(ii) There exists a lcs E in the class $\mathfrak{G}$ such that X is homeomorphic to a $\sigma(E, E')$-compact subset of E.

Proof (ii) $\Rightarrow$ (i): Let X be a $\sigma(E, E')$-compact subset of a lcs E in the class $\mathfrak{G}$. Let $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of E. If $A_\alpha := \{f | X : f \in K_\alpha\}$, $\alpha \in \mathbb{N}^{\mathbb{N}}$, then every A_α is a subset of $C(X)$. Moreover, $A_\alpha \subset A_\beta$ if $\alpha \leq \beta$. Also, since A_α is relatively countably compact in $C_p(X)$, it is relatively compact in $C_p(X)$ (as $C_p(X)$ is angelic). Set $S = \bigcup_\alpha \overline{A_\alpha}$. Since $S \subset C_p(X)$ separates points of X, we apply Corollary 3.2.9 and Theorem 9.1.3 (ii) to ensure that $C_p(X)$ is K-analytic. Hence X is Talagrand compact.

(i) $\Rightarrow$ (ii): Assume X is Talagrand compact. Then the Banach space $C(X)$ is weakly K-analytic, so $C(X)$ admits a compact resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in the weak topology; see Theorem 9.1.3 (i). We may assume that the sets K_α are absolutely convex (by using the Krein theorem; see [322]). Let $(C(X)', \xi)$ be the dual of $C(X)$ endowed with the topology of the uniform convergence on the sets K_α. Then $(C(X)', \xi)$ belongs to $\mathfrak{G}$, and the dual $(C(X)', \xi)'$ equals $C(X)$. Note that X is weakly compact in $(C(X)', \xi)$. □

Proposition 12.3.6 *Let $C_p(X)$ be a separable web-bounded space with countable tightness. Then $L_p(X)$ is separable.*

Proof Since $C_p(X)$ has countable tightness, X is realcompact by Proposition 9.4.1. The space $L_p(X)$ is a Lindelöf Σ-space by Theorem 9.4.19. Let σ be the original topology of $L_p(X)$. Since $C_p(X)$ is separable, $L_p(X)$ admits a weaker metrizable topology $\xi \leq \sigma$. As the weight of ξ is countable, by Lemma 3.1.13 the density of σ is countable. Hence $L_p(X)$ is separable. □

The *weak** dual of a separable lcs in the class $\mathfrak{G}$ is a separable space by Corollary 12.3.3. This and Proposition 12.3.6 motivate the following

Problem 12.3.7 *Let E be a separable web-bounded lcs. Is the weak* dual of E separable?*

12.4 Every (WCG) Fréchet Space Is Weakly K-Analytic

We know already (Theorems 12.1.4, 12.2.1) that the *weak** dual of a quasibarrelled lcs in the class $\mathfrak{G}$ is K-analytic. In particular, every reflexive Fréchet space is weakly K-analytic. In this section we prove that every weakly compactly generated Fréchet space E is weakly K-analytic; this result is due to Khurana [366]. For the same result for (WCG) Banach spaces, see [575], and see also [198, 482].

A Banach space E is *weakly compactly generated* (WCG) if there exists a weakly compact subset K in E whose linear span is a dense subspace of E. One may assume that K is absolutely convex by the Krein theorem.

In [482] Orihuela used the method of constructing projections in (WCG) Banach spaces (this method is due to Valdivia, see [484, 615, 617–620]), to provide a direct proof that the weak topology of a (WCG) Banach space is Lindelöf. Orihuela [482, Corollary 6] followed this method to prove also that a dual Banach space is weakly Lindelöf if and only if its *weak** dual unit ball is a Corson compact space. Moreover, if E is a dual Banach space which is weakly Lindelöf, the product $E \times E$ is weakly Lindelöf, [482, Theorem C]. We refer the reader to [17, 120, 123, 124, 483] (and references) concerning weakly countably determined (WCD) Banach spaces and weakly Lindelöf determined (WLD) Banach spaces (providing larger classes of weakly Lindelöf spaces than the class of (WCG) Banach spaces).

The following approach, using arguments from [484], was suggested to the authors by Montesinos; see also [8, 198].

Lemma 12.4.1 *Let E be a Banach space. Let $\Delta \subset E$ be a set such that $\overline{\Delta}$ is a linear subspace, and let $\nabla \subset E'$ be a set that 1-norms Δ, i.e. for all $x \in \Delta$, $\|x\| = \sup_{b' \in B_\nabla} |\langle x, b' \rangle|$, where $B_\nabla := \{b' \in \nabla; \|b'\| = 1\}$. Then $\overline{\Delta} \oplus \nabla_\perp$ is a topological direct sum, and $\|P\| = 1$, where $P : \overline{\Delta} \oplus \nabla_\perp \to \overline{\Delta}$ is the canonical projection.*

Proof Obviously $\overline{\Delta} \cap \nabla_\perp = \{0\}$, so $\overline{\Delta} \oplus \nabla_\perp$ is an algebraic direct sum. Moreover, given $x \in \overline{\Delta}$ and $y \in \nabla_\perp$, one has the following:

$$\|P(x+y)\| = \|x\| = \sup_{b' \in B_\nabla} |\langle x, b' \rangle| = \sup_{b' \in B_\nabla} |\langle x+y, b' \rangle| \leq \|x+y\|,$$

and hence $\|P\| = 1$ and the direct sum is topological (in particular, closed). □

Lemma 12.4.2 *Let E be a Banach space, Δ, and ∇ be as in Lemma 12.4.1, with $\overline{\nabla}^{w^*}$ a linear subspace. Then $\overline{\Delta} \oplus \nabla_\perp = E$ if and only if $\Delta^\perp \cap \overline{\nabla}^{w^*} = \{0\}$, where $\overline{\nabla}^{w^*}$ denotes the closure of ∇ in $\sigma(E', E)$.*

Proof The necessary condition is clear. Assume now that $\Delta^\perp \cap \overline{\nabla}^{w^*} = \{0\}$. Let $x' \in E'$ be such that $x'\big|_{\overline{\Delta} \oplus \nabla_\perp} = 0$. Then $x' \in \Delta^\perp \cap \overline{\nabla}^{w^*}$, so $x' = 0$. It follows that $\overline{\Delta} \oplus \nabla_\perp$ is dense in E; since, by Lemma 12.4.1, it is closed, we obtain the conclusion. □

For a subset $S \subset E$ of a Banach space E denote

$$\operatorname{span}_{\mathbb{Q}} S := \{x : x = \sum_{i=1}^{n} a_i z_i,\ a_i \in \mathbb{Q}, z_i \in S, 1 \le i \le n,\ n \in \mathbb{N}\}.$$

We say that $Y \subset E$ is $\mathbb{Q}$-*linear* if $\operatorname{span}_{\mathbb{Q}} Y = Y$.

Lemma 12.4.3 *Let E be a Banach space, $\Phi_0 : E' \to 2^E$, and $\Psi_0 : E \to 2^{E'}$ be two at most countably valued mappings. Suppose $A_0 \subset E$, $B_0 \subset E'$ and* card (A_0), *and* card $(B_0) \le \Gamma$ *for some infinite cardinal Γ. Then there exist $\mathbb{Q}$-linear sets A, B, $A_0 \subset A \subset E$, $B_0 \subset B \subset E'$, such that* card (A), card $(B) \le \Gamma$, *and $\Phi_0(B) \subset A$, $\Psi_0(A) \subset B$.*

Proof We construct by induction two sequences of sets

$$A_0 \subset A_1 \subset A_2 \cdots \subset E,\ \ B_0 \subset B_1 \subset B_2 \cdots \subset E'$$

as follows. Having constructed $A_0 \ldots A_n$, $B_0 \ldots B_n$, we set

$$A_{n+1} := \operatorname{span}_{\mathbb{Q}}(A_n \cup \Phi_0(B_n)), \quad B_{n+1} := \operatorname{span}_{\mathbb{Q}}(B_n \cup \Psi_0(A_n)).$$

Finally, we set

$$A := \bigcup_{n=0}^{\infty} A_n,\ \ B := \bigcup_{n=0}^{\infty} B_n.$$

That A and B satisfy the required properties is obvious. □

Proposition 12.4.4 *Let E be a (WCG) Banach space generated by an absolutely convex and weakly compact set K. Let A_0, B_0, Ψ_0, Φ_0, and Γ be as in Lemma 12.4.3.Then, there exist sets A and B such that* card$(A) \le \Gamma$, card$(B) \le \Gamma$, *$\overline{A}$ is linear, $\overline{B}^{w^*}$ is also linear, $E = \overline{A} \oplus B_\perp$, the canonical projection $P : E \to \overline{A}$ satisfies $\|P\| = 1$, and $P'b' = b'$ for all $b' \in \overline{B}^{w^*}$.*

Proof Given $x' \in E'$, let $\varphi_1(x')$ be an element in K such that

$$\langle \varphi_1(x'), x' \rangle = \sup |\langle K, x' \rangle|.$$

Put

$$\Phi(x') = \Phi_0(x') \cup \{\varphi_1(x')\}$$

for every $x' \in E'$. Given $x \in E$, let $\psi_1(x) \in B_{E'}$ such that $|\langle x, \psi_1(x) \rangle| = \|x\|$. Put

$$\Psi(x) := \Psi_0(x) \cup \{\psi_1(x)\}$$

for every $x \in E$. We apply Lemma 12.4.3 to the sets A_0 and B_0 and Ψ and Φ. We obtain sets A and B as in the proof of Lemma 12.4.3.

First of all, from the construction in Lemma 12.4.3 it follows that $\overline{A}$ and $\overline{B}^{w'}$ are linear sets. The set B 1-norms A. Indeed, let $a \in A$. There exists $n \in \mathbb{N}$ such that $a \in A_n$. Then $\psi_1(a) \in B_n$ satisfies $|\langle a, \psi_1(a)\rangle| = \|a\|$. This proves the assertion. By Lemma 12.4.1 we get that $A \oplus B_\perp$ is a direct sum. Let

$$x' \in A^\perp \cap \overline{B}^{w^*}.$$

By the Mackey–Arens theorem, we have $\overline{B}^{w^*} = \overline{B}^{\mu(E',E)}$, where $\mu(E', E)$ is the Mackey topology on E'. Therefore, for any $\epsilon > 0$ there exists $y' \in B$ such that

$$\sup |\langle K, x' - y'\rangle| < \varepsilon.$$

Then

$$\begin{aligned}\sup\langle K, x'\rangle &\le \sup\langle K, x' - y'\rangle + \sup\langle K, y'\rangle < \varepsilon + \sup\langle K, y'\rangle \\ &= \varepsilon + \langle \varphi_1(y'), y'\rangle = \varepsilon + \langle \varphi_1(y'), y' - x'\rangle \le \varepsilon + \sup\langle K, y' - x'\rangle < 2\varepsilon.\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we get $x'|_K \equiv 0$, and so $x' = 0$. This proves that $A^\perp \cap \overline{B}^{w^*} = \{0\}$, and so, by Lemma 12.4.2 we have $E = \overline{A} \oplus B_\perp$, and the canonical projection $P : E \to \overline{A}$ satisfies $\|P\| = 1$. Let

$$b' \in \overline{B}^{w^*} \ (= (B_\perp)^\perp).$$

Put $x = a + b$, where $a \in \overline{A}$ and $b \in B_\perp$. Then for all $b' \in \overline{B}^{w^*}$ we have

$$\langle x, P'b'\rangle = \langle Px, b'\rangle = \langle a, b'\rangle = \langle a + b, b'\rangle = \langle x, b'\rangle,$$

hence $P'b' = b'$. □

Now we are ready to prove the following result due to Preiss and Talagrand.

Theorem 12.4.5 (Preiss–Talagrand) *Every (WCG) Banach space is a weakly Lindelöf space.*

Proof First observe that it is enough to choose countable A_0 to get A countable, clearly dense in $\overline{A}$ and such that, for every element $x' \in \Psi_0(A)$ we have $P'x' = x'$.

Let $\mathfrak{U} := \{V_t : t \in T\}$ be an open cover of $(E, \sigma(E, E'))$. For each $x \in E$ let p_x be the supremum of all positive numbers p such that the open ball $B(x, p) \subset V_t$ for some $t \in T$. Choose $V_x \in \mathfrak{U}$ such that $B(x, p_x 2^{-1}) \subset V_x$. Assume that V_x is defined by a finite set $K_x \subset E'$. Set $\Psi_0(x) := K_x$. We apply Proposition 12.4.4, where $\Phi_0 : E' \to 2^E$ is a countably valued map. We get a countable dense set A in $P(E)$ $(= \overline{A})$ and such that, for every element $x' \in \Psi_0(A)$ we have $P'x' = x'$. We prove that the countable family $\{V_x : x \in A\}$ is a covering of E.

Fix arbitrary $x \in E$. Let $B(P(x), p) \subset V_{P(x)}$. Choose $z \in A$ such that

$$z \in B(P(x), 10^{-1}p).$$

This implies that $p_z > 9(10)^{-1}p$. Hence

$$B(z, 2(5)^{-1}p) \subset V_z.$$

This yields that $P(x) \subset V_z$. Hence for some $\epsilon > 0$ we have

$$|f(P(x) - z)| < \epsilon$$

for each $f \in \Psi_0(z)$. Then

$$|f(z - x)| = |P'(f)(z - x)| = |f(P(x) - P(z))| = |f(P(x) - z)| < \epsilon.$$

Hence $x \in V_z$. □

Since every regular Lindelöf space is paracompact (Morita), Theorem 12.4.5 implies that the weak topology of a (WCG) Banach space is paracompact.

We complete this section with the following theorem due to Khurana, [366] extending Talagrand's corresponding theorem [582] for (WCG) Banach spaces.

Theorem 12.4.6 (Khurana) *Let E be a Fréchet space which admits an increasing sequence of $\sigma(E, E')$-compact sets whose union is dense in E. Then $(E, \sigma(E, E'))$ is K-analytic. Moreover, E is a Borel subset of $(E'', \sigma(E'', E'))$, where E'' is the bidual of E.*

Proof Since for every metrizable lcs E the space $(E, \sigma(E, E'))$ is angelic, to prove that $(E, \sigma(E, E'))$ is a K-analytic space it is enough to show (by Corollary 3.2.9) that $(E, \sigma(E, E'))$ has a compact resolution.

In a natural way, we identify $(E, \sigma(E, E'))$ with a subspace of $\mathbb{R}^{E'}$ endowed with the product topology. Therefore $x = (g(x))_{g \in E'}$ for each $x \in E$. For each $f \in E'$ the map $P_f : \mathbb{R}^{E'} \to \mathbb{R}$ defined by

$$P_f((\alpha_g)_{g \in E'}) := \alpha_f$$

satisfies

$$P_f(x) = P_f((g(x))_{g \in E'}) = f(x)$$

for any $x = (g(x))_{g \in E'} \in E$, and therefore the restriction of P_f to E is f.

Let $(V_n)_n$ be a basis of closed absolutely convex neighbourhoods of zero in E such that

$$(n + 1)V_{n+1} \subset V_n$$

for each $n \in \mathbb{N}$. Let $\overline{V}_n$ be the closure of V_n in $\mathbb{R}^{E'}$. Then, given $n, p \in \mathbb{N}$, $f \in V_n^0$, and $z_{n+p} \in \overline{V}_{n+p}$, we have

$$\left|P_f(z_{n+p})\right| \leq \sup\left\{\left|P_f(x)\right| : x \in \overline{V}_{n+p}\right\} = \sup\left\{\left|P_f(x)\right| : x \in V_{n+p}\right\}.$$

Hence

$$\left|P_f(z_{n+p})\right| \leq \sup\left\{|f(x)| : x \in \frac{V_n}{n+p}\right\} \leq \frac{1}{n+p} \tag{12.2}$$

for each $f \in V_n^0$ and $z_{n+p} \in \overline{V}_{n+p}$. Let $(A_n)_n$ be an increasing sequence of weakly compact absolutely convex subsets of E such that $\bigcup_n A_n = H$ is dense in E. Since H is dense in E and V_n is a neighbourhood of zero in E, we have

$$E \subset H + V_n \subset H + \overline{V}_n$$

for each $n \in \mathbb{N}$. If

$$x \in \bigcap\left\{H + \overline{V}_n : n \in \mathbb{N}\right\}$$

there exists a sequence

$$(x = h_n + z_n)_n \tag{12.3}$$

with $h_n \in H$ and $z_n \in \overline{V}_n$. Fix an $n \in \mathbb{N}$. Then for each $f \in V_n^0$ and $p, q \in \mathbb{N}$ we note, by (12.2) and (12.3), that

$$\left|f(h_{n+p} - h_{n+q})\right| = \left|P_f(x - z_{n+p}) - P_f(x - z_{n+q})\right| = \left|P_f(z_{n+q}) - P_f(z_{n+p})\right|,$$

and therefore (12.2) yields

$$\left|f(h_{n+p} - h_{n+q})\right| \leq \left|P_f(z_{n+p})\right| + \left|P_f(z_{n+q})\right| \leq 2n^{-1}$$

for each $f \in V_n^0$. The uniformity implies that the sequence $(h_s)_s$ is Cauchy in the Fréchet space E; hence it has a limit $h \in E$. Then $\lim_{s\to\infty} h_s = h$. From (12.2) it follows that

$$P_f(x) = \lim_{s\to\infty} P_f(h_s + z_s) = \lim_{s\to\infty}\left\{f(h_s) + P_f(z_s)\right\} = f(h) = P_f(h).$$

Hence $x = h$ in $\mathbb{R}^{E'}$. Since $x = h \in E$,

$$E = \bigcap\left\{H + \overline{V}_n : n \in \mathbb{N}\right\},$$

and then

$$E = \bigcap \left\{ \bigcup \left[A_m + \overline{V_n} : m \in \mathbb{N} \right] : n \in \mathbb{N} \right\}.$$

Therefore, E admits a resolution $\left\{ K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}} \right\}$ with

$$K_\alpha = \bigcap \left\{ A_{m_n} + \overline{V}_n : n \in \mathbb{N} \right\}$$

for each $\alpha = (m_n) \in \mathbb{N}^{\mathbb{N}}$. In $\mathbb{R}^{E'}$ the sets $A_{m_n} + \overline{V}_n$ for $n \in \mathbb{N}$, are closed.

We claim that the closed set K_α is bounded in $\mathbb{R}^{E'}$. Indeed, if $f \in E'$, there exists $n \in \mathbb{N}$ such that $f \in V_n^0$ and

$$\sup \{ |f(x)| : x \in K_\alpha \} \leq \sup \left\{ \left| P_f(x) \right| : x \in A_{m_n} + \overline{V}_n \right\} \leq$$

$$\leq \sup \left\{ |f(x)| : x \in A_{m_n} \right\} + \sup \left\{ \left| P_f(x) \right| : x \in \overline{V}_n \right\} =$$

$$= \sup \left\{ |f(x)| : x \in A_{m_n} \right\} + \sup \left\{ \left| P_f(x) \right| : x \in V_n \right\} \leq k_{m_n} + 1,$$

where

$$k_{m_n} := \sup \left\{ |f(x)| : x \in A_{m_n} \right\}.$$

This proves that the closed set K_α is compact in $\mathbb{R}^{E'}$, and it is also compact in $(E, \sigma(E, E'))$. We proved that $\left\{ K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}} \right\}$ is a compact resolution in $(E, \sigma(E, E'))$. Finally, since $A_{m_n} + \overline{V}_n$ is closed in $\mathbb{R}^{E'}$, the set $\left(A_{m_n} + \overline{V}_n \right) \bigcap E''$ is closed in $(E'', \sigma(E'', E'))$, and so

$$\left(H + \overline{V}_n \right) \cap E''$$

is a Borel set in $(E'', \sigma(E'', E')$. Since

$$E = \bigcap \left\{ \left(H + \overline{V}_n \right) \cap E'' : n \in \mathbb{N} \right\},$$

we deduce that E is a Borel set in $(E'', \sigma(E'', E'))$. □

Theorem 12.4.6 combined with Theorem 9.4.24 applies to provide the following:

Proposition 12.4.7 *Let E be a (WCG) Baire lcs. Then E is a Fréchet space if and only if $(E, \sigma(E, E'))$ is K-analytic.*

Proof Assume E is a Fréchet space. By Theorem 12.4.6 the space E is weakly K-analytic. Conversely, assume $(E, \sigma(E, E'))$ is K-analytic. Hence $(E, \sigma(E, E'))$ admits a compact resolution. By Theorem 9.4.24 the space E is metrizable. Since

E has a $\sigma(E, E')$-compact resolution and the original topology of E has a basis of neighbourhoods consisting of $\sigma(E, E')$-closed sets, the space E admits a complete resolution. By Corollary 7.1.5 the space E is complete. □

By Proposition 12.4.7 we deduce that the separable space $X := \mathbb{R}^{\mathbb{R}}$ is not K-analytic (since X is (WCG) Baire and non-metrizable). Theorem 12.4.6 can be also deduced from Talagrand's theorem [582] for (WCG) Banach spaces; see [114]. Indeed, let E be a (WCG) Fréchet space and $(p_n)_n$ a fundamental sequence of continuous seminorms defining the original topology of E. For each $n \in \mathbb{N}$ define $E_n := E/p_n^{-1}(0)$ endowed with the normed topology $\|x + p_n^{-1}(0)\| := p_n(x)$ for each $x \in E$. Let F_n be the completion of E_n for each $n \in \mathbb{N}$. Clearly E is linearly homeomorphic to a closed vector subspace of the product $F := \prod_n F_n$. Since each E_n is a continuous image of E, each E_n is a (WCG) space; hence F_n is a (WCG) Banach space. Now by Talagrand's theorem each F_n is weakly K-analytic. Hence F is weakly K-analytic. Consequently E is weakly K-analytic.

12.5 Amir–Lindenstrauss's Theorem

This short section, motivated by the previous one, provides a theorem of Amir–Lindenstrauss [408] stating that every non-separable reflexive Banach space contains a complemented separable subspace. This result motivates us to study the property called *controlled separable projection property* (CSPP).

The proof of Theorem 12.5.1 is due to Yost and uses some ideas from the proof of a more general result due to Valdivia [617, Lemma 1].

Theorem 12.5.1 (Amir–Lindenstrauss) *Let E be a (WCG) Banach space and A_0 and B_0 two countable subsets of E and E', respectively. Then there exists a norm one projection $P : E \to E$ with a separable range such that $A_0 \subset P(E)$ and $B_0 \subset P'(E')$. Consequently, for every separable subspace F in E there exists a closed separable subspace G of E containing F and such that there is a norm one projection P of E onto G.*

Proof Let K be an absolutely convex weakly compact subset of E whose linear span is dense in E. By the Hahn–Banach theorem for each $x \in E$ there exists a functional $f_x \in E'$ such that

$$\|f_x\| = 1, \quad f_x(x) = \|x\|.$$

Next, using the fact that K is $\sigma(E, E')$-compact, for every $f \in E'$ there exists $x_f \in K$ such that

$$|f(x_f)| = \max_{x \in K} |f(x)|.$$

By $[C]$ we denote the $\mathbb{Q}$-linear span of a set C (in E or E'). Define

$$A_1 := [A_0 \cup \{x_f : f \in B_0\}], \quad B_1 := [B_0 \cup \{f_x : x \in A_1\}],$$

and next

$$A_2 := [A_1 \cup \{x_f : f \in B_1\}], \quad B_2 := [B_1 \cup \{f_x : x \in A_2\}],$$

etc. Then, set

$$A := \bigcup_{n=0}^{\infty} A_n, \quad B := \bigcup_{n=0}^{\infty} B_n.$$

Clearly A and B are countable $\mathbb{Q}$-linear subspaces of E and E', respectively. Note also that for $x \in A$ one has $f_x \in B$, and for $f \in B$ the corresponding x_f belongs to A. If $x \in A$ and $y \in B^{\perp} \subset E$,

$$\|x\| = f_x(x) = f_x(x+y) \leq \|x+y\|.$$

Then $\|x\| \leq \|x+y\|$ for each $x \in \overline{A}$ and $y \in B^{\perp}$. This implies that $\overline{A} \cap B^{\perp} = \{0\}$, so the map

$$P : \overline{A} + B^{\perp} \to \overline{A}, \ x + y \mapsto x,$$

is a surjective linear projection of norm one. Hence $\overline{A} + B^{\perp}$ is a closed linear subspace of E. To complete the proof we need to show that $\overline{A} + B^{\perp} = E$. It is enough to show that $\overline{A} + B^{\perp}$ is a (weakly) dense subspace of E. Choose arbitrary $f \in E'$ and assume that $f|(\overline{A} + B^{\perp}) = 0$. Then $f|A = 0$ and $f|B^{\perp} = 0$. Then $f \in B^{\perp\perp} \subset E'$. Since $B^{\perp\perp} = \overline{B}$, where the closure is taken in the Mackey topology $\mu(E', E)$ of E', for $\epsilon > 0$ there exists $g \in B$ such that $|f(x) - g(x)| < \epsilon$ for each $x \in K$. Since $x_g \in A \cap K$ and $f|A = 0$, so

$$\max_{x \in K} |g(x)| = |g(x_g)| = |f(x_g) - g(x_g)| < \epsilon.$$

This implies that $|g(x))| < \epsilon$ for all $x \in K$. Consequently, $|f(x)| < 2\epsilon$ for all $x \in K$. Since the linear span of K is dense in $\sigma(E, E')$, so $f = 0$. □

A Banach space E such that for each sequence $(x_n)_n \subset E$ and $(x'_n)_n \subset E'$ there exists a continuous projection $P : E \to E$ with separable $P(E)$ such that $(x_n)_n$ and $(x'_n)_n$ are contained in $P(E)$ and $P'(E')$, respectively, will be said to have the *controlled separable projection property* (CSPP); see [642], or [66, 234].

By the results of Sect. 24.12 it will follow that every (WLD) Banach space satisfies the (CSPP) and the Banach space $C[0, \omega_1]$ provides a concrete example of a space with the (CSPP) not being the (WLD); see Corollary 12.5.10. The last property follows from the fact that $[0, \omega_1]$ is not Corson compact. Recall here that

a Banach space E is called *weakly Lindelöf determined* (WLD) if its dual unit ball is Corson compact when endowed with the *weak** dual topology. This is a weaker property than being *weakly countably determined* (WCD) . Indeed, a Banach space E is (WCD) if E with the weak topology $\sigma(E, E')$ is a Lindelöf Σ-space; see Proposition 3.1.8 and [18, 196]. Note that every (WCG) Banach space is (WCD).

Below we provide a couple of Banach spaces $C(K)$ which do not satisfy the (CSPP), [234]; hence they are not (WLD).

We shall say that a compact space K is *countably measure determined* if there exists a sequence $(\mu_n)_n$ of Radon probabilities on K in $C(K)'$ such that $\bigcap_n \ker \mu_n = \{0\}$. Clearly every separable compact space enjoys this property. We note the following result from [234].

Proposition 12.5.2 *Let K be a compact space and assume that $C(K)$ satisfies the (CSPP). Then K does not contain a non-metrizable countably measure determined closed subset.*

Proof Assume that D is a closed non-metrizable subset of K that is countably measure determined. Let $(\mu_n)_n$ be a sequence of Radon probabilities on D with $\bigcap_n \ker \mu_n = \{0\}$. Let $(\lambda_n)_n$ be a sequence of Radon measures on K such that each λ_n extends μ_n and supp $\mu_n =$ supp λ_n for each $n \in \mathbb{N}$. By the (CSPP) there exists a continuous projection $P : C(K) \to C(K)$ such that $P(C(K))$ is separable and $P'(C(K)')$ $(= (\ker P)^\perp)$ contains the sequence $(\lambda_n)_n$. Define the linear surjective map $T : P(C(K)) \to C(D)$ by the formula $T(P(f)) := f|D$, $f \in C(K)$. Note that the map T is well defined. Indeed, if $f, g \in C(K)$ with $P(f) = P(g)$, $f - g \in \ker P$. This yields that

$$(\mu_n, (f-g)|D) = \int_D (f-g)d\mu_n = \int_K (f-g)d\lambda_n = (\lambda_n, (f-g)) = 0,$$

for each $n \in \mathbb{N}$. Hence $(f-g)|D = 0$, so T is well defined. Note that T is continuous. Indeed, if $f \in C_c(K)$, we have $f - P(f) \in \ker P$ and then $f|D = (P(f))|D$. Hence

$$\|T(P(f))\| = \|f|D\| = \|(P(f))|D\| \leq \|P(f)\|.$$

This applies to show that the separable quotient $P(C(K))/\ker T$ is isomorphic to a non-separable space $C(D)$, yielding a contradiction. □

Corollary 12.5.3 *If K is a compact space which contains a non-metrizable separable subset, the space $C(K)$ does not have the (CSPP).*

Corollary 12.5.4 *If a compact space K contains $\beta\mathbb{N}$, the space $C(K)$ does not have the (CSPP).*

Since every metric compact scattered space is countable, we have

Corollary 12.5.5 *Let K be a scattered compact space that contains a separable uncountable subset. Then $C(K)$ does not have the (CSPP).*

Summarizing we note the following:

(A) Every (WLD) Banach space satisfies the (CSPP). Therefore, the space $C(K)$ in Corollary 12.5.5 is not (WLD).
(B) The Banach space $C([0,\omega_1])$ has the (CSPP) and is not (WLD); see Corollary 12.5.10.

Let K be a compact scattered space and let μ be a non-negative Radon measure on K. Let $A \subset K$ be a non-empty Borel subset of K. Since μ is regular, there exists a compact set $B \subset A$ with $\mu(B) > 0$. Let $\lambda := \mu|B$. Then there exist an open set $U \subset K$ and $t \in \operatorname{supp}\lambda$ such that

$$U \cap \operatorname{supp}\lambda = \{t\}.$$

This implies that $\lambda(U \cap B) > 0$. Since

$$U \cap B = \{t\} \cup (U \cap B \setminus (\operatorname{supp}\lambda)),$$

one has

$$\mu(\{t\}) = \lambda(\{t\}) = \lambda(U \cap B).$$

This shows that $\{t\}$ is an atom in μ. Hence μ has a countable set of atoms.

Let L be a locally compact scattered Hausdorff space and let $K := L \cup \{\infty\}$ be the one-point compactification of L.

We have the following proposition.

Proposition 12.5.6 ([234]) *Assume that $K := L \cup \{\infty\}$ satisfies the condition (x), i.e. the closure of every countable subset of L is countable. Then for every Radon measure μ on K there exists a countable subset $N \subset L$ such that* $\operatorname{supp}\mu \subset N \cup \{\infty\}$.

We need also the following two additional easy facts; see [234].

Lemma 12.5.7 *Let G be an almost-clopen subset of $K := L \cup \{\infty\}$, i.e. an open set such that $\overline{G} \subset G \cup \{\infty\}$. Then for each $f \in C(K)$ the function $f \cdot 1_G + f(\infty) \cdot 1_{K \setminus G} \in C(K)$.*

Lemma 12.5.8 *If $K := L \cup \{\infty\}$ satisfies the property (xx), i.e. every closed subset of K contained in L is countable, for each $f \in C(K)$ there exists a countable set $N \subset L$ such that $f(K \setminus N) = \{f(\infty)\}$.*

We are ready to prove the following general result [234].

Proposition 12.5.9 *If $K := L \cup \{\infty\}$ satisfies the conditions (xx) and (xxx), i.e. for each countable $N \subset L$, there exists a countable almost-clopen set G with $N \subset G$, the space $C(K)$ satisfies the (CSPP).*

Proof We may assume that L is uncountable; otherwise K would be countable and metrizable (every compact countable space is metrizable!), so $C(K)$ would be separable, so satisfying the (CSPP).

Let $(f_n)_n$ and $(\mu_n')_n$ be sequences in $C(K)$ and $C(K)'$, respectively. By Lemma 12.5.8 and Proposition 12.5.6 there exists a countable set $N \subset L$ such that

$$f_n(K \setminus N) = \{f_n(\infty)\}, \quad \operatorname{supp} \mu_n \subset N \cup \{\infty\}$$

for each $n \in \mathbb{N}$. By the condition (xxx) there is a countable almost-open set G containing N. We define a linear and continuous map $P : C(K) \to C_(K)$ by the formula $P(f) := f \cdot 1_G + f(\infty) \cdot 1_{K \setminus G}$ for each $f \in C(K)$. Note also that

$$P^2(f) = P(f) \cdot 1_G + P(f)(\infty) \cdot 1_{K \setminus G} =$$

$$(f \cdot 1_G + f(\infty) \cdot 1_{K \setminus G}) \cdot 1_G + f(\infty) \cdot 1_{K \setminus G} = P(f).$$

Case 1. $\overline{G} = G \cup \{\infty\}$. Then $P(C(K))$ is linearly homeomorphic to $C(\overline{G})$. Since $\overline{G}$ is metrizable, $C(\overline{G})$ is separable; hence $P(C(K))$ is separable, too.

Case 2. $\overline{G} = G$. In that case $C(G)$ is linearly homeomorphic to a closed separable hyperplane of $P(C(K))$. Hence also $P(C(K))$ is separable.

Finally, note that $(f_n)_n \subset P(C(K))$ and

$$(\mu_n)_n \subset P'(C(K)') = (\ker\ P)^{\perp}$$

for each $n \in \mathbb{N}$. The first claim is clear. If $f \in \ker\ P$, then

$$\mu_n(f) = \int_{\operatorname{supp}\ \mu_n} f d\mu_n = \int_{G \cup \{\infty\}} f d\mu_n = 0,$$

since $f|G = 0$ and $f(\infty) = 0$. □

Now we are ready to show that (CSPP) does not imply the Lindelöf property for the weak topology of a Banach space. For a more general statement concerning (CSPP) and compact lines; see Corollary 12.5.10.

Corollary 12.5.10 *The space $C([0, \omega_1])$ satisfies the (CSPP) and is not (WLD).*

Proof The space $C([0, \omega_1])$ is not weakly Lindeöf. Indeed assume that $C([0, \omega_1])$ is weakly Lindelöf. Then, by Theorem 9.4.2 the space $[0, \omega_1]$ has countable tightness. This is impossible, since the closure of the set $A := [0, \omega_1] \setminus \{\omega_1\}$ contains ω_1 and the closure of any countable subset of A does not contain ω_1. This provides a contradiction. Clearly $[0, \omega_1]$ is the one-point compactification of the locally compact space $[0, \omega_1)$. It is enough to prove that $[0, \omega_1]$ satisfies the conditions (xx) and (xxx), and then apply Proposition 12.5.9.

$[0, \omega_1]$ satisfies the condition (xx). Indeed, let $A \subset [0, \omega_1)$ be closed in $[0, \omega_1]$. Since $\omega_1 \notin A$, there exists $\alpha < \omega_1$ such that $A \cap [\alpha, \omega_1] = \emptyset$. This shows that A is countable, since $A \subset [0, \alpha]$.

$[0, \omega_1]$ satisfies the condition (xxx). Indeed, let N be a countable subset of $[0, \omega_1)$. Choose $\alpha < \omega_1$ such that $\overline{N} \subset [0, \alpha]$. Then $G := [0, \alpha]$ is a clopen countable set containing N. □

12.6 An Example of Pol

Last results may suggest the following natural question.

Problem 12.6.1 *Is a weakly Lindelöf Banach space a (WCG) Banach space?*

This problem has been announced by Corson [147]; see also [409]. In [579] Talagrand gave an example of a compact space X such that $C(X)$ is weakly K-analytic and $C(X)$ is not a (WCG) Banach space. Another example of this type was obtained by Pol. In [502] Pol showed that there exists a Banach space $C(X)$ over a compact scattered space X such that $C(X)$ is weakly Lindelöf and $C(X)$ is not a (WCG) Banach space; see Theorem 12.6.8. This example answers also (in the negative) some questions of Corson [147], Problem 7 posed by Benyamini, Rudin, and Wage from [78].

Theorem 12.6.8 will be used (in the next section) to show that (gDF)-spaces are not in the class $\mathfrak{G}$. The results of this section are due to Pol [502].

A space E is said to have the *strong condensation property*, if for each uncountable subset A of E there exists an uncountable subset $C \subset A$ which is concentrated around a point $c \in E$, i.e. the set $C \setminus V$ is at most countable whenever V is a neighbourhood of c.

Let $\Omega = [1, \omega_1[$ be the set of all countable ordinals endowed with the usual order. Then, for each $\alpha \in \Omega$ the interval $[1, \alpha] := \{\beta \in \Omega : \beta \leq \alpha\}$ is countable. Therefore, if D is an uncountable subset of $[1, \omega_1[$, there exists an injective map $\varphi : [1, \omega_1[\to D$ such that $\beta < \varphi(\beta) < \varphi(\alpha)$ for each countable ordinals α, β such that $\beta < \alpha$; the map φ can be determined by transfinite induction, where $\varphi(\alpha)$ is the first ordinal of the set $\{\gamma \in [\delta, \omega_1[: x_\gamma \in D\}$, and $\delta := \sup\{\alpha, \sup\{\varphi(\gamma) : \gamma < \alpha\}\}$.

Note the following:

Proposition 12.6.2 *A topological space X has the strong condensation property if and only if each net $\{x_\alpha : \alpha \in [1, \omega_1[\}$ in X admits a convergent subnet $\{x_{\varphi(\alpha)} : \alpha \in [1, \omega_1[\}$ directed by Ω. Additionally, we may assume that the map $\varphi : [1, \omega_1[\to [1, \omega_1[$ satisfies $\alpha < \varphi(\alpha) < \varphi(\alpha')$ if $\alpha < \alpha'$.*

Proof Using transfinite induction, we note that an uncountable set A contains an uncountable subset

$$A' = \bigcup \{x_\alpha : \alpha \in [1, \omega_1[\ \},$$

where $x_\alpha \in A \setminus \cup\{x_\beta : \beta \in [1, \alpha[\}$. Then, if $C = \bigcup\{x_\alpha : \alpha \in D\}$ is an uncountable subset of A', we have that C contains an uncountable set

$$C' = \bigcup\{x_{\varphi(\alpha)} : \alpha \in [1, \omega_1[\ \},$$

where $\varphi : [1, \omega_1[\to D$ is the map as above. $\{x_{\varphi(\alpha)} : \alpha \in [1, \omega_1[\ \}$ is a subnet of $\{x_\alpha : \alpha \in [1, \omega_1[\ \}$, and C' is concentrated around a point $c \in E$ if and only if $\lim_{\alpha \in [1, \omega_1[} x_{\varphi(\alpha)} = c$.

Therefore, we proved that a net $\{x_\alpha : \alpha \in [1, \omega_1[\ \}$ in X with the strong condensation property has a convergent subset $\{x_{\varphi(\alpha)} : \alpha \in [1, \omega_1[\ \}$ if the set

$$\bigcup\{x_\alpha : \alpha \in [1, \omega_1[\ \}$$

is uncountable. If

$$\bigcup\{x_\alpha : \alpha \in [1, \omega_1[\ \}$$

is countable, there exists an uncountable subset D in $[1, \omega_1[$ such that $x_\alpha = x_\beta$, where $\alpha, \beta \in D$, and the map $\varphi : [1, \omega_1[\to D$ is as above. This proves that the constant net $\{x_{\varphi(\alpha)} : \alpha \in [1, \omega_1[\ \}$ is a convergent subnet of the net $\{x_\alpha : \alpha \in [1, \omega_1[\ \}$. The rest of the proof is clear. □

If a topological space X has the strong condensation property and the weight $w(X)$ of X is at most $\aleph_1$, the space X is Lindelöf. This follows from the fact stating that, if $\{B_\alpha : \alpha \in [1, \omega_1[\ \}$ is a basis of X and X is not Lindelöf, there exists a net $\{x_\alpha : \alpha \in [1, \omega_1[\ \}$ in X, with

$$x_\alpha \in Y \setminus \{B_\beta : \beta \leq \alpha\}$$

for each $\alpha \in [1, \omega_1[$ (since X is not covered by $\bigcup\{B_\beta : \beta \leq \alpha\}$ for each $\alpha \in [1, \omega_1[$). As each B_α only contains a countable number of points of $\{x_\alpha : \alpha \in [1, \omega_1[\ \}$, the net $\{x_\alpha : \alpha \in [1, \omega_1[\ \}$ does not admit a convergent subnet. Then X does not have the strong condensation property.

Let $\Gamma \subset \Omega$ be the set of all non-limit ordinals, and let $\Lambda = \Omega \setminus \Gamma$. Attach to each $\gamma \in \Lambda$ an increasing sequence $(s_\lambda(n))_n$ in Γ such that

$$\lim_{n \to \infty} s_\lambda(n) = \lambda.$$

Endow the set Ω with the following topology: The points from Γ are isolated, and the basic neighbourhoods of a point $\gamma \in \Lambda$ are of the form

$$W_\gamma(n) = \{\gamma\} \cup \{s_\gamma(m) : m \geqslant n\}.$$

Therefore, if K is a compact subset of Ω, the set $K \cap \Lambda$ is finite, because for each $\gamma \in K \cap \Lambda$ the set $W_\gamma(n) \cap \Lambda$ is unitary and the points of Γ are isolated.

Let $X := \Omega \cup \{\omega_1\}$ be the one-point compactification of the locally compact space Ω, where now the first uncountable ordinal ω_1 is the *point at infinity*. The compact space X is scattered. Our goal is to prove that the Banach space $C(X)$ is weakly Lindelöf. Recall that for a compact scattered space X the weak topology of $C(X)$ coincides on the unit ball with the topology of $C_p(X)$. Therefore, the space $C(X)$ is weakly Lindelöf if and only if $C_p(X)$ is Lindelöf. As usual $|A|$ denotes the cardinal of the set A and 1_X is the constant function on X with $1_X(X) = \{1\}$. To avoid misunderstanding the space $C_p(X)$ will be sometimes denoted by $C_p(X, \mathbb{R})$.

Let $D = \{0, 1\}$ be endowed with the discrete topology. Then $C_p(X, D)$ is the subset of $C_p(X, \mathbb{R})$ determined by the functions with the range in D.

We need the following:

Lemma 12.6.3 *Assume that $s(\alpha) \in \Gamma$ for each $\alpha \in [1, \omega_1[$ and that $\chi_{s(\alpha)}$ is the characteristic function of $\{s(\alpha)\}$. Then the net $\{\chi_{s(\alpha)} : \alpha \in [1, \omega_1[\}$ admits a convergent subnet $\{\chi_{s(\varphi(\alpha))} : \alpha \in [1, \omega_1[\}$ in $C_p(X, D)$.*

Proof Indeed, by Proposition 12.6.2 we note that, if

$$D := \bigcup \{s(\alpha) : \alpha \in [1, \omega_1[\}$$

is uncountable, there exists an injective map $\varphi : [1, \omega_1[\to D$ such that $\alpha < \varphi(\alpha) < \varphi(\beta)$ if $\alpha < \beta$. Then, the limit of the net $\{\chi_{s(\varphi(\alpha))} : \alpha \in [1, \omega_1[\ \}$ in $C_p(X, D)$ is the null function. From Proposition 12.6.2 it follows that, if

$$\bigcup \{s(\alpha) : \alpha \in [1, \omega_1[\ \}$$

is countable, there exists an uncountable subset D in $[1, \omega_1[$ such that $s(\alpha) = s(\beta)$ if $\alpha, \beta \in D$. Therefore there exists a constant map $\varphi : [1, \omega_1[\to D$ with $\varphi(\alpha) = \zeta$ for each $\alpha \in [1, \omega_1[$. Then, the limit of the net

$$\{\chi_{s(\varphi(\alpha))} : \alpha \in [1, \omega_1[\ \}$$

in $C_p(X, D)$ is the characteristic function χ_ζ. □

This yields the following:

Corollary 12.6.4 *Let $G_0 := \{f \in C_p(X, D) : f^{-1}(1) \subset \Gamma\}$ be endowed with the induced topology of $C_p(X, D)$. Then G_0 has the strong condensation property.*

Proof We prove that the net $\{f_\alpha : \alpha \in [1, \omega_1[\ \}$ in G_0 contains a subnet $\{f_\alpha : \alpha \in [1, \omega_1[\ \}$ which converges in G_0. By the compactness of $f_\alpha^{-1}(1)$ (in the discrete subspace Γ) we deduce that the compact set $f_\alpha^{-1}(1)$ is finite. There exists $m \in \omega$ such that the set $\{\alpha \in [1, \omega_1[: \left|f_\alpha^{-1}(1)\right| = m\}$ is uncountable. Hence, taking a subnet if necessary, we may assume that

$$f_\alpha^{-1}(1) = \{s_1(\alpha), s_2(\alpha), \cdots, s_m(\alpha)\},$$

for each $\alpha \in [1, \omega_1[$.

If $m = 1$, the existence of the convergent subnet follows from Lemma 12.6.3. Assume, as the induction hypothesis, that a net $\{g_\alpha : \alpha \in [1, \omega_1[\ \}$ in G_0, such that $\left|g_\alpha^{-1}(1)\right| = m-1$ for each $\alpha \in [1, \omega_1[$, has a convergent subnet $\{g_{\varphi(\alpha)} : \alpha \in [1, \omega_1[\}$ in G_0. Then, if $\{f_\alpha : \alpha \in [1, \omega_1[\ \}$ is a net in G_0 such that

$$\left|f_\alpha^{-1}(1)\right| = \{s_1(\alpha), s_2(\alpha), \cdots, s_m(\alpha)\}$$

for each $\alpha \in [1, \omega_1[$, then, by Lemma 12.6.3, the net $\{\chi_{s_1(\alpha)} : \alpha \in [1, \omega_1[\}$ admits a convergent subnet $\{\chi_{s_1(\varphi(\alpha))} : \alpha \in [1, \omega_1[\ \}$ in G_0. By the induction hypothesis the net

$$\{f_{\varphi(\alpha)} - \chi_{s_1(\varphi(\alpha))} : \alpha \in [1, \omega_1[\ \}$$

has a convergent subnet

$$\{f_{\varphi(\psi(\alpha))} - \chi_{s_1(\varphi(\psi(\alpha)))} : \alpha \in [1, \omega_1[\ \}$$

in G_0. Therefore, the net $\{f_{\varphi(\psi(\alpha))} : \alpha \in [1, \omega_1[\ \}$ also converges in G_0. □

Now we prove the following:

Proposition 12.6.5 *Let $D = \{0, 1\}$ be endowed with the discrete topology. Then the space $C_p(X, D)$ has the strong condensation property.*

Proof For each $n \in \mathbb{N}$ set

$$G_n := \{f \in C_p(X, D) : \left|f^{-1}(1) \cap \Lambda\right| = n\}.$$

By the continuity of the map $\varphi : C_p(X, D) \to C_p(X, D)$ defined by $\varphi(f) := 1_X - f$, from the equality

$$C_p(X, D) = \{f \in C_p(X, D) : f(\omega_1) = 0\} \oplus \{f \in C_p(X, D) : f(\omega_1) = 1\},$$

and since

$$\{f \in C_p(X, D) : f(\omega_1) = 0\} = \bigcup_n G_n,$$

we note that $C_p(X, D)$ has the strong condensation property, provided for each $n \in \omega$ the following holds:

(*) Every net $\{f_\alpha : \alpha \in [1, \omega_1[\ \}$ in G_n has a subnet $\{f_{\varphi(\alpha)} : \alpha \in [1, \omega_1[\ \}$ that converges in $C_p(X, D)$.

In Corollary 12.6.4 we proved (*) for $n = 0$. We prove the condition (*) by induction. Assume (*) holds in G_n, and assume that $\{f_\alpha : \alpha \in [1, \omega_1[\ \}$ is a net in G_{n+1}. Then

$$f_\alpha^{-1}(1) \cap \Lambda = \{s_1(\alpha), s_2(\alpha), \cdots, s_{n+1}(\alpha)\}$$

for each $\alpha \in [1, \omega_1[$. Taking a subnet, if necessary, we may assume that $s_1(\alpha) = s_1$ for each $\alpha \in [1, \omega_1[$, or

$$\sup\{s_1(\beta) : \beta < \alpha\} < s_1(\alpha)$$

for each $\alpha \in [1, \omega_1[$. If $s_1(\alpha) = s_1$ for each $\alpha \in [1, \omega_1[$, and χ_{s_1} is the characteristic function of $\{s_1\}$, by the induction hypothesis the net

$$\{f_\alpha - \chi_{s_1} : \alpha \in [1, \omega_1[\ \}$$

has a convergent subnet

$$\{f_{\varphi(\alpha)} - \chi_{s_1} : \alpha \in [1, \omega_1[\ \}.$$

This implies that $\{f_{\varphi(\alpha)} : \alpha \in [1, \omega_1[\ \}$ converges. For the other case, i.e. if

$$\delta(\alpha) := \sup\{s_1(\beta) : \beta < \alpha\} < s_1(\alpha)$$

for each $\alpha \in [1, \omega_1[$, let $W(\alpha)$ be a neighbourhood of $s_1(\alpha)$ contained in $]\delta(\alpha), s_1(\alpha)]$ such that $f_\alpha(x) = 1$ for each $x \in W(\alpha)$, and let $\chi_{W(\alpha)}$ be the characteristic function of $W(\alpha)$. Since $\lim_{\alpha \in [1,\omega_1[} \chi_{W(\alpha)} = 0$, and

$$\{f_\alpha - \chi_{W(\alpha)} : \alpha \in [1, \omega_1[\} \subset G_n,$$

then by the induction hypothesis, we deduce that the net $\{f_\alpha : \alpha \in [1, \omega_1[\}$ has a convergent subnet $\{f_{\varphi(\alpha)} : \alpha \in [1, \omega_1[\ \}$. □

This yields the following:

Corollary 12.6.6 *$C_p(X, D^{\mathbb{N}})$ is a Lindelöf space.*

Proof Since $\omega(X) \leq \aleph_1$ we deduce that $\omega(C_p(X, D)) \leq \omega(X^2) \leq \aleph_1^2 = \aleph_1$. This and Proposition 12.6.5 imply that $C_p(X, D)^{\mathbb{N}}$ has the strong condensation property, and its weight is at most $\aleph_1$. Therefore $C_p(X, D)^{\mathbb{N}}$ is a Lindelöf space. Now corollary follows from the fact that $C_p(X, D)^\omega$ and $C_p(X, D^\omega)$ are homeomorphic. □

Corollary 12.6.7 *Let Q be a countable subset of $D^{\mathbb{N}}$ and $P = D^{\mathbb{N}} \backslash Q$. Then $C_p(X, P)$ is a Lindelöf space.*

Proof Set $Q = \{q_1, q_2, \cdots\}$, and for each $i \in \mathbb{N}$ let $(V_n(q_i))_n$ be a base of open neighbourhoods of q_i. As $C_p(X, D^{\mathbb{N}}$ has the strong condensation property, we deduce that for each $n \in \mathbb{N}$ the closed subspace

$$E_{i,n} := \{f \in C_p(X, D^{\mathbb{N}}) : f(X) \cap [D^{\mathbb{N}} \backslash V_n(q_i)] = \emptyset\}$$

has the strong condensation property. Then the union

$$E_i := \bigcup_n E_{i,n} = \{f \in C_p(X, D^{\mathbb{N}}) : f(X) \cap \{q_i\} = \emptyset\}$$

has also the strong condensation property. Also, the product $\prod_{i\in\mathbb{N}} E_i$ and its diagonal have the strong condensation property. This, and the fact that the space $C_p(X, P)$ has weight at most $\aleph_1$ completes the proof. □

It is known, see [502, page 281, proof of Lemma 1] and [394, 40, VII], that if a countable set Q is dense in $D^{\mathbb{N}}$, the space $C_p(X, \mathbb{R})$ is a continuous image of $C_p(X, P)$. On the other hand, for a compact space S the Banach space $C(S)$ is (WCG) if and only if S is Eberlein compact. Also, Wage proved [633, Example p.20] that the compact space $X = \Omega \cup \{\omega_1\}$ is not Eberlein compact. Hence the Banach space $C(X)$ is not (WCG).

Therefore, we have proved the following:

Theorem 12.6.8 (Pol) *The space $C(X, \mathbb{R})$ is Lindelöf and has the strong condensation property, and $C(X, \mathbb{R})$ is not a (WCG) Banach space.*

12.7 More about Banach Spaces $C(X)$ over Compact Scattered X

In this section we use the previous example of Pol to show that in general (gDF)-spaces do not belong to the class $\mathfrak{G}$. First we prove the following:

Proposition 12.7.1 *Let X be a scattered compact space such that $C(X)$ is weakly Lindelöf. Then the weak* dual of $C(X)$ has countable tightness.*

Proof Let τ_p and $\tau_\sigma := \sigma(C(X), C(X)')$ be the original topology of $C_p(X)$ and the weak topology of $C(X)$, respectively. Since $(C(X), \tau_\sigma)$ is Lindelöf, $C_p(X)$ is also Lindelöf. Let B be the closed unit ball in $C(X)$. Since X is scattered, we have $\tau_p|B = \tau_\sigma|B$ by [552, Corollary 19.7.7].

A direct proof: Clearly $\tau_p|B \leq \tau_\sigma|B$. The argument used in the proof of Theorem 9.1.3 (i) applies to show that every sequence in B that converges in τ_p converges also in τ_σ. Hence the identity map from (B, τ_p) onto (B, τ_σ) is sequentially continuous. On the other hand, since X is scattered, the space $C_p(X)$ is Fréchet–Urysohn; see Proposition 14.1.1. As (B, τ_p) is Fréchet–Urysohn, we deduce easily that the (sequentially continuous) identity map $(B, \tau_p) \to (B, \tau_\sigma)$ is continuous. We proved $\tau_p|B = \tau_\sigma|B$. Then $\tau_p^n|B^n = \tau_\sigma^n|B^n$ for each $n \in \mathbb{N}$, where $B^n := \prod_{1\leq i\leq n} B$ and τ_p^n, τ_σ^n denote the own product topologies on $C(X)^n := \prod_{1\leq i\leq n} C(X)$. Since X is compact and scattered, X is zero-dimensional, and then [32, Theorem IV.8.6] applies to deduce that $(C_p(X)^n, \tau_p^n)$ is a Lindelöf space for each $n \in \mathbb{N}$. The space B^n (as closed in τ_p^n) is also a Lindelöf space. Hence B^n is Lindelöf in $(C(X)^n, \tau_\sigma^n)$ for each $n \in \mathbb{N}$. We have $C(X)^n =$

$\bigcup_m mB^n$, and each mB^n is a Lindelöf space in τ_σ^n, so the space $(C(X)^n, \tau_\sigma^n)$ is a Lindelöf space for each $n \in \mathbb{N}$. Proposition 9.4.1 implies that $C_p((C(X), \tau_\sigma))$ has countable tightness. Hence the space $(C(X)', \sigma(C(X)', C(X)) \subset C_p(C(X), \tau_\sigma)$ has countable tightness. □

Also we need the following result due to Pol [502].

Proposition 12.7.2 *Every weakly K-analytic Banach space $C(X)$ over a compact scattered space X is a (WCG) Banach space.*

Let $X := \Omega \cup \{\omega_1\}$ be the compact scattered space (the one-point compactification of the locally compact space Ω) considered in Sect. 12.6. By Theorem 12.6.8 and the proof of Proposition 12.7.1 the Banach space $C(X)$ is weakly Lindelöf and not (WCG) by [633]. Since every weakly K-analytic Banach space $C(X)$ over compact scattered X is a (WCG) Banach space (Proposition 12.7.2), the space $C(X)$ is not weakly K-analytic.

Hence we have the following; see Proposition 12.7.1.

Proposition 12.7.3 *$C(X)$ is weakly Lindelöf and it is not weakly K-analytic. The weak* dual of $C(X)$ has countable tightness.*

On the other hand, combining Proposition 12.7.1 with Example 9.4.33 (and its proof) we note the following example.

Example 12.7.4 Assume CH. There exists an uncountable compact scattered space X such that $C(X)$ is weakly Lindelöf and $C(X)$ is not a weakly Lindelöf Σ-space. Nevertheless, the *weak** dual of $C(X)$ has countable tightness.

Note also that examples of compact X such that $C(X)$ are weakly Lindelöf and not weakly Lindelöf Σ (in other words, Corson compact not Gul'ko compact) can be found in [18]; see also [17, Theorem 3.3] and Sect. 24.17. Last examples will be used to show that the (gDF)-spaces need not be in the class $\mathfrak{G}$.

Following Ruess, a lcs E is a *(gDF)-space* if E has a fundamental sequence $(B_n)_n$ of bounded sets and E is *C-quasibarrelled*, i.e. for every sequence $(U_n)_n$ of absolutely convex closed neighbourhood of zero such that for any bounded set B in E there exists $p \in \mathbb{N}$ such that $B \subset U_n$ for each $n \geq p$, the set $U = \bigcap_n U_n$ is a neighbourhood of zero.

It is known that a lcs (E, ξ) is a (gDF)-space if and only if it has a fundamental sequence $(B_n)_n$ of bounded sets and ξ is the finest locally convex topology on E that agrees with $\xi|B_n$ on each set B_p; see [491].

Clearly every (DF)-space is a (gDF)-space. Every (DF)-space is a dual metric space. Corollary 2.6.12 showed that $C_c(X)$ is a (df)-space if and only if it has a fundamental sequence of bounded sets and $C_c(X)$ is ℓ^∞-barrelled. Hence a (df)-space $C_c(X)$ is dual metric, so it belongs to the class $\mathfrak{G}$ (we already showed that all dual metric spaces are in the class $\mathfrak{G}$).

We prove that there exist (gDF)-spaces which are not in the class $\mathfrak{G}$. We need the following observation; see [526, 527]; for a simple proof we refer to [491, Proposition 8.3.10].

Proposition 12.7.5 *Let E be a Fréchet space. Then $(E', \tau_{pc}(E', E))$ is a (gDF)-space.*

We complete this section with the following promise:

Proposition 12.7.6 *There exists a (gDF)-space which does not belong to the class $\mathfrak{G}$.*

Proof Fix $E := C(X)$, where $X := \Omega \cup \{\omega_1\}$ is the compact scattered space considered above. Assume $(E', \tau_{pc}(E', E))$ belongs to the class $\mathfrak{G}$. By Proposition 12.7.1 we know that $(E', \sigma(E', E))$, the *weak** dual of $C(X)$, has countable tightness. Then, by Theorem 12.1.4, the weak topology of $C(X)$ is K-analytic, a contradiction with Proposition 12.7.3. □

Proposition 12.7.6 provides also an example of a (gDF)-space that is not a dual metric space.

Chapter 13
Corson's Property (C) and Tightness

Abstract In this chapter, the class of Banach spaces having the property (C) (isolated by Corson) is studied. This property provides a large subclass of Banach spaces E whose weak topology need not be Lindelöf. We collect some results of Corson, Pol, Frankiewicz, Plebanek and Ryll-Nardzewski.

13.1 The Property (C) and Weakly Lindelöf Banach Spaces

Let X be a topological space and choose $x \in X$. By the *tightness* of X at the point x, we mean (and denote $\mathrm{t}(x, X)$) the smallest infinite cardinal $\mathfrak{m}$ such that for each set $A \subset X$ with $x \in \overline{A}$, there exists a subset $B \subset A$ with $|B| \leq \mathfrak{m}$ such that $x \in \overline{B}$. A topological space X has tightness $\mathfrak{m}$ if $\mathfrak{m}$ is the smallest infinite cardinal such that $\mathrm{t}(x, E) \leq \mathfrak{m}$ for each $x \in E$.

Let X be a completely regular Hausdorff space. We know already that the space $C_p(X)$ has countable tightness if and only if each finite product X^n is a Lindelöf space; see Theorem 9.4.1. This result leads to the following problem.

Problem 13.1.1 Assume that X is a Lindelöf space. Is it true that every compact subset of $C_p(X)$ has countable tightness?

This interesting and difficult question has been answered in positive by Arkhangel'skii [32, Theorem IV.11.14] assuming the Proper Forcing Axiom. Problem 13.1.1 may suggest also another one formulated for Banach spaces.

Problem 13.1.2 Let E be a Banach space such that $(E, \sigma(E, E'))$ is a Lindelöf space. Is the unit ball in E' of countable tightness in $\sigma(E', E)$?

Problem 13.1.2 is strictly connected with Corson's property (C); see [147]. A convex closed subset M of a Banach space E is said to have the *property* (C) if M verifies one of the equivalent conditions:

(i) Every family $\mathcal{F}$ of complements of closed convex sets in M that covers M has a countable subfamily covering M.

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Developments in Mathematics 24, https://doi.org/10.1007/978-3-031-76062-4_13

(ii) For every family $\mathcal{A}$ of closed convex subsets of M with the empty intersection, there is a countable subfamily $\mathcal{B}$ of $\mathcal{A}$ with the empty intersection.
(iii) Every family $\mathcal{A}$ of non-empty closed convex subsets of M closed under countable intersections verifies $\bigcap \mathcal{A} \neq \emptyset$.

We shall say that a Banach space E has the property (C) if the above holds for $E = M$. If $(E, \sigma(E, E'))$ is a Lindelöf space, the space E has the property (C) (since closed convex sets in E and $\sigma(E, E')$ are the same). Hence, every (WCG) Banach space has the property (C). Every non-separable (WCG) Banach space has the property (C) and is not Lindelöf. We show, using Corson's [147, Example 2] and Pol's [504, Proposition 1], that there exist Banach spaces with the property (C) that are not weakly Lindelöf. See also [147, Examples 2,3,4]. Next, we show that the property (C) is a three-space property. We need the following results from [504].

Proposition 13.1.3 (Pol) *The following conditions are equivalent for a Banach space E:*

(i) E has the property (C).
(ii) If a family $\mathcal{K}$ of non-empty closed convex sets in E is closed under countable intersections, then for every $\sigma > 0$, there is $a \in E$ with $\operatorname{dist}(a, K) < \sigma$ for every $K \in \mathcal{K}$.
(iii) If a family $\mathcal{K}$ of non-empty closed convex sets in E is closed under countable intersections, then for every $\sigma > 0$, there is a closed convex subset M with the property C such that $\operatorname{dist}(M, K) < \sigma$ for every $K \in \mathcal{K}$.
(iv) If a family $\mathcal{K}$ of non-empty closed convex sets in the closed unit ball B_E in E is closed under countable intersections, then for every $\sigma > 0$ there is a closed convex subset M with the property C such that $\operatorname{dist}(M, K) < \sigma$ for every $K \in \mathcal{K}$.

Proof (i) $\Rightarrow$ (ii): If E has the property (C), we have $\bigcap \mathcal{A} \neq \emptyset$.

(ii) $\Rightarrow$ (i): We need to prove that, if $\mathcal{C}$ is a family of non-empty closed convex subsets of X closed under countable intersections, one has $\bigcap \mathcal{C} \neq \emptyset$. From (ii) we deduce that there exists $a_1 \in E$ with $\operatorname{dist}(a_1, C) < 2^{-1}$ for every $C \in \mathcal{C}$. Moreover, if for $1 \leq i \leq k$, we define $a_i \in E$ with $\operatorname{dist}(a_i, C) < 2^{-i}$ for every $C \in \mathcal{C}$ and

$$\operatorname{dist}(a_{i-1}, a_i) < 2^{-i+2}$$

for $2 \leq i \leq k$, (ii) applied for the family

$$\left\{C \cap \left(a_k + 2^{-k} B_E\right) : C \in \mathcal{C}\right\}$$

yields an element $a_{k+1} \in E$ with

$$\operatorname{dist}(a_{k+1}, C \cap (a_k + 2^{-k} B_E)) < 2^{-k-1}$$

for every $C \in \mathcal{C}$. It is clear that

$$\operatorname{dist}(a_{k+1}, C) < 2^{-k-1}, \quad \operatorname{dist}(a_{k+1}, a_k) < 2^{-k+1}.$$

Continuing this inductive procedure, we determine a Cauchy sequence $(a_k)_k$ whose limit $a \in C$ for every $C \in C$. This completes the proof since $a \in \bigcap \mathcal{C}$.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (ii): If (ii) fails, there exist a family $\mathcal{K}$ of non-empty closed convex subsets of E closed under countable intersections, a positive number ε such that for each $a \in E$ there exists $K_a \in \mathcal{K}$ with $\operatorname{dist}(a, K_a) \geq \varepsilon$. By (iii) there exists a closed convex subset M of E with the property (C) such that $\operatorname{dist}(M, K) < \varepsilon 2^{-1}$ for each $K \in \mathcal{K}$. As $\mathcal{K}$ is closed under countable intersections, and M has the property (C), we deduce that the intersection of the family

$$\{\overline{(K + \varepsilon 2^{-1} B_E)} \cap M : K \in \mathcal{K}\}$$

is non-empty. Then there exists

$$y \in \overline{K + \varepsilon 2^{-1} B_E}$$

for each $K \in \mathcal{K}$. This implies that $\operatorname{dist}(y, K) < \varepsilon$ for each $K \in \mathcal{K}$. Hence, $\operatorname{dist}(y, K_y) < \varepsilon$, a contradiction.

(iii) $\Rightarrow$ (iv) is clear.

(iv) $\Rightarrow$ (iii): Assume (iii) fails. Then there exist a family $\mathcal{K}$ of non-empty closed convex subsets of E closed under countable intersections, and a positive number ε such that for each closed convex subset M of E with the property (C), there exists $K_M \in \mathcal{K}$ with $\operatorname{dist}(M, K_M) \geq \varepsilon$. Since $\mathcal{K}$ is closed under countable intersections, there is a natural number $n \in \mathbb{N}$ such that $K \cap nB_E \neq \emptyset$, for every $K \in \mathcal{K}$. Since $\operatorname{dist}(M, K_M \cap nB_E) \geq \varepsilon$, then

$$\operatorname{dist}\left(n^{-1}M, n^{-1}[K_M \cap nB_E]\right) \geq \varepsilon n^{-1}.$$

Therefore (iv) fails for the family $\{n^{-1}[K \cap nB_E] : K \in \mathcal{K}\}$ and $\sigma = \varepsilon n^{-1}$. □

Now we are ready to prove the following.

Theorem 13.1.4 (Pol) *Let F be a closed subspace of a Banach space E. If the spaces F and E/F have the property (C), the space E has the property (C).*

Proof Assume F and E/F have the property (C). Let $q : E \to E/F$ be the quotient map. If E does not have the property (C), by Proposition 13.1.3 there exist $\varepsilon > 0$, a family $\mathcal{K}$ of non-empty closed convex subsets of E that are closed under countable intersections, such that for every closed convex subset M with the property (C) there exists $K_M \in \mathcal{K}$ with $\operatorname{dist}(M, K_M) \geq \varepsilon$. For each $z \in E$, the set $z + F$ is closed, is convex, and has the property (C) (by the assumption F does have

has the property (C)). Then there exists $K_{z+F} \in \mathcal{K}$ with

$$\operatorname{dist}(z + F, K_{z+F}) \geq \varepsilon.$$

This implies that $q(z) \notin \overline{q(K_{z+F})}$. Hence

$$\bigcap \left\{ \overline{q(K)} : K \in \mathcal{K} \right\} = \emptyset,$$

contradicting that E/F has the property (C). □

Theorems 8.1.5 and 13.1.4 provide an example of a Banach space with the property (C) such that $(E, \sigma(E, E'))$ is not a Lindelöf space (with an aid of Lemma 6.1.3).

Next, Corollary 13.1.5 will be used to characterize the property (C).

Corollary 13.1.5 *Assume that a Banach space E does not have the property (C). Then, there exist a subset A' of the unit ball $B_{E'}$ of E', $\epsilon > 0$ such that for every linear subspace $M \subset E$ with the property (C) there exists $f \in A'$ vanishing on M, and for each countable $C' \subset A'$ there is $x \in E$ with $\|x\| \leq 1$ and $f(x) \geq \epsilon$ for every $f \in C'$.*

Proof Let $\mathfrak{F}$ be the family of all subspaces of E with the property (C). Since E does not have the property (C), by Proposition 13.1.3 (iii) there exist a family $\mathcal{K}$ of non-empty closed convex sets in E closed under countable intersections, $\epsilon > 0$, such that for each $F \in \mathfrak{F}$ there exists $K_F \in \mathcal{K}$ with $\operatorname{dist}(F, K_F) \geq \epsilon$. Let B be the open unit ball in E. Since $(F + \epsilon B) \cap K_F = \emptyset$, by the Hahn–Banach theorem, there exists a continuous linear functional $f_F \in E'$ such that $\|f_F\| = 1$ and

$$\sup_{x \in F + \epsilon B} f_F(x) \leq \inf_{y \in K_F} f_F(y).$$

Then $f_F(x) = 0$ for all $x \in F$ and $f_F(y) \geq \epsilon$ for every $y \in K_F$. Set $A' := \{f_F : F \in \mathfrak{F}\}$, and let $C' = \{f_{F_1}, f_{F_2}, \ldots, \}$ be a countable subset of A'. Then, the desired conditions are satisfied for $\epsilon > 0$ chosen above and x equal to a point of the non-void set $\bigcap \{K_{F_n} : n \in \mathbb{N}\}$. □

We know (see Theorem 12.1.4) that, if E is a metrizable lcs, its *weak** dual $(E', \sigma(E', E))$ is a Lindelöf space. We characterize Banach spaces with the property (C) in terms of some tightness-type property for the *weak** dual $(E', \sigma(E', E))$. The condition (ii) below can be formulated for the closed unit ball in E' (instead of taking the whole dual space E').

Theorem 13.1.6 (Pol) *For a Banach space E, the following assertions are equivalent:*

(i) E has the property (C).
(ii) For each set $A \subset E'$ and each $f \in \overline{A}$ (the closure in $\sigma(E', E)$), there exists countable $B \subset A$ such that $f \in \overline{\operatorname{conv}(B)}$.

Proof (i) $\Rightarrow$(ii): Fix $f \in \overline{A}$. Set:

$$C_g := \{x \in E : g(x) \geq f(x) + 1\}$$

for each $g \in A$. Clearly, the sets C_g are closed, convex, and $\bigcap_{g\in A} C_g = \emptyset$. Since E has the property (C), there exists a countable subset $B \subset A$ with $\bigcap_{g\in B} C_g = \emptyset$. Then $f \in \overline{\text{conv}(B)}$. Indeed, otherwise by the Hahn–Banach theorem, there exists $x \in E$ such that $h(x) \geq f(x) + 1$ for each $h \in \overline{\text{conv}(B)}$. Hence $x \in \bigcap_{g\in B} C_g$, a contradiction.

(ii) $\Rightarrow$(i): Assume E does not have the property (C). Note that $0 \in \overline{A'}$, where A' is defined in the proof of Corollary 13.1.5. By this corollary we deduce that 0 does not belong to $\overline{\text{conv}(C')}$ for each countable $C' \subset A'$. This yields a contradiction with the assumption (ii). □

Combining Proposition 12.1.3 with Theorem 13.1.6, we deduce that for a Banach space with the property (C), the weak topology $\sigma(E, E')$ is realcompact. This fact has been already observed by Corson [147, Lemma 9]. Also, the following fact is clear; see [504, Theorem 5.1].

Proposition 13.1.7 *If K is a compact space and the Banach space $C(K)$ has the property (C), the space K has countable tightness.*

Proof Fix $A \subset K$ and $x \in \overline{A}$. Set $C_B := \{g \in C(K) : g|B = 0, g(x) = 1\}$ for each $B \in A^{\mathbb{N}}$, where $A^{\mathbb{N}}$ denotes the family of all countable subsets of A. Since $\bigcap_{B\in A^{\mathbb{N}}} C_B = \emptyset$, and each set C_B is closed and convex, the property (C) provides a sequence $(B_n)_n$ in $A^{\mathbb{N}}$ such that $\bigcap_n C_{B_n} = \emptyset$. Then $x \in \overline{B}$, where $B := \bigcup_n B_n$. □

We complete this section by showing that for Banach spaces E with the property (C), we have $\text{ck}(H) = \text{k}(H)$ for every bounded set $H \subset E$; see [13, Proposition 2.6] (cf. also Chap. 4). We need the following lemma from [13].

Lemma 13.1.8 *Let $x^{**} \in E'' \setminus E$ and $b \in \mathbb{R}$ with $d(x^{**}, E) > b > 0$. Then $0 \in \overline{\{x^* \in B_{E'} : x^{**}(x^*) > b\}}$, where the closure is taken in $\sigma(E', E)$.*

Proof Let $\epsilon > 0$ and $x_1, x_2, \ldots x_n \in E$. We may assume that $b + \epsilon < d(x^{**}, E)$. Let

$$V := \{y^* \in E' : \sup_{1\leq i\leq n} |y^*(x_i)| < \epsilon\}.$$

It is enough to show that the $\sigma(E', E)$-neighbourhood of zero V intersects the set:

$$S(x^{**}, b) := \{x^* \in B_{E'} : x^{**}(x^*) > b\}.$$

By the Hahn–Banach theorem, there exists $\xi \in E'''$ with $\xi(x) = 0$ for each $x \in E$ and $\|\xi\| = 1$ with $\xi(x^{**}) = d(x^{**}, E)$. By the classical Goldstein theorem (see [322]), there exists $x^* \in B_{E'}$ $(\subset B_{E'''})$ such that

$$|\xi(x_i) - x^*(x_i)| = |x^*(x_i)| < \epsilon$$

for $1 \leq i \leq n$ and $|\xi(x^{**}) - x^{**}(x^*)| < \epsilon$. Then x^* and $-x^*$ belong to V. Since

$$|x^{**}(x^*)| = |x^{**}(x^*) - \xi(x^{**}) + \xi(x^{**})| \geq$$

$$|\xi(x^{**})| - |x^{**}(x^*) - \xi(x^{**})| > b + \epsilon - \epsilon = b,$$

either x^*, or $-x^*$, belongs to $V \cap S(x^{**}, b)$. □

Now we prove the following result [121].

Theorem 13.1.9 (Cascales–Marciszewski–Raja) *If E is a Banach space with the property (C), then for every bounded set $H \subset E$, we have* $\mathrm{ck}(H) = \mathrm{k}(H)$.

Proof By Theorem 4.6.19 we know that $\mathrm{ck}(H) \leq \mathrm{k}(H)$. Then, if $\mathrm{k}(H) = 0$, the required equality holds. Therefore, it is enough to show that for each $0 < b < \mathrm{k}(H)$, we have $b \leq \mathrm{ck}(H)$. This will prove the equality $\mathrm{ck}(H) = \mathrm{k}(H)$. Fix $0 < b < \mathrm{k}(H)$, and choose:

$$x^{**} \in \overline{H}^{\mathbb{N}^*} \setminus E$$

such that $d(x^{**}, E) > b$. Set:

$$S(x^{**}, b) := \{x^* \in B_{E'} : x^{**}(x^*) > b\}. \tag{13.1}$$

By Lemma 13.1.8 we have $0 \in \overline{S(x^{**}, b)}^{\mathbb{N}^*}$. By Theorem 13.1.6 there exists a countable subset $C \subset S(x^{**}, b)$ such that $0 \in \overline{\mathrm{conv}}^{\mathbb{N}^*} C$. As $S(x^{**}, b)$ is convex, there exists a countable set $D \subset S(x^{**}, b)$ with $0 \in \overline{D}^{\mathbb{N}^*}$. Since $\overline{H}^{\mathbb{N}^*}$ is pseudometrizable in the pointwise topology on D, there exists a sequence $(h_n)_n$ in H with $h_n \to x^{**}$ on D. If h^{**} is any $\mathbb{N}^*$-cluster point of $(h_n)_n$, we have:

$$h^{**}|D = x^{**}|D.$$

Consequently, $h^{**}(x^*) = x^{**}(x^*) > b$ for each $x^* \in D$. Since $0 \in \overline{D}^{\mathbb{N}^*}$, for fixed arbitrary $y \in E$ and $\epsilon > 0$, there exists $x^* \in D$ with $|x^*(y)| < \epsilon$. This applies to get

$$\|h^{**} - y\| \geq h^{**}(x^*) - x^*(y) > b - \epsilon.$$

Hence $d(h^{**}, E) \geq b$ for each $\mathbb{N}^*$-cluster point h^{**} of $\varphi = (h_n)_n$. Consequently,

$$\mathrm{ck}(H) \geq d(\mathrm{clust}_{E''}(\varphi), E) \geq b.$$

□

Theorem 13.1.9 provides a large class of Banach spaces for which the numbers $\mathrm{ck}(H)$ and $\mathrm{k}(H)$ coincide, for example, every separable Banach space has this property.

13.2 The Property (C) for Banach Spaces $C(K)$

If K is a compact space, the dual $C(K)'$ is identified with $M(K)$, the space of signed Radon measures on K of finite variation (see Chap. 19 for detailed information on $C(K)'$). Let $M_0(K)$ be the unit ball in $M(K)$. By $P(K)$ we denote the set of the probabilistic Radon measures on K endowed with the topology $\sigma(P(K), C(K))$.

We need the following property; see [195, Problem 3.12.8 (a),(c),(f)].

Lemma 13.2.1 *For a compact space K, the following holds:*

(i) $\mathrm{t}(P(K)) = \mathrm{t}(M_0(K))$.
(ii) If $T \subset K$ is a closed G_δ-subset and $x \in T$, then $\mathrm{t}(x, T) = \mathrm{t}(x, K)$.

Proof (i) By the definition $\mathrm{t}(P(K)) \leq \mathrm{t}(M_0(K))$. Now set:

$$A := \{(x, y) : |x| + |y| \leq 1\}.$$

Since the map

$$U : P(K) \times P(K) \times A \to M_0(K)$$

defined by $U(\mu, \nu, x, y) := x\mu - y\nu$ is a continuous surjection, we have:

$$\mathrm{t}(M_0(K)) \leq \mathrm{t}(P(K))$$

by using [195, 3.12.8 (a),(f)]. For the part (ii), we refer to [195, 3.12.8 (c)]. □

Motivated by the above facts, we show the following result; see [243, Theorem 2.3]; the proof uses some ideas from [496].

Theorem 13.2.2 (Frankiewicz–Plebanek–Ryll-Nardzewski) *If K is a compact space with countable tightness, the space $C(K)$ is realcompact in the weak topology.*

Proof By Proposition 12.1.3 it is enough to show that, if $\xi \in (C(K)')^*$ is a functional that is $\sigma(C(K)', C(K))$-continuous on each $\sigma(C(K)', C(K))$-separable subspace, then $\xi \in C(K)$.

Define φ on K by $\varphi(x) := \xi(\delta_x)$, where δ_x is the Dirac measure at x. Since ξ is $\sigma(C(K)', C(K))$-continuous on the subspace spanned on $\{\delta_x : x \in S\}$, where S is an arbitrary separable subspace of K, the map φ is continuous on S. Since K has countable tightness, the same argument used in the proof of Theorem 12.1.4 applies to show that φ is continuous on K, i.e. $\varphi \in C(K)$. To show that $\xi \in C(K)$, it is enough to prove that $\xi(\mu) = \mu(\varphi)$ for every $\mu \in P(K)$; hence, we need to show that

$\vartheta := \xi - \varphi = 0$. Assume, by contradiction, that $\vartheta(\mu) > 0$ for some $\mu \in P(K)$. By the Radon–Nikodym theorem [439, Theorem 13.12] to the μ-continuous measure $\gamma(B) := \vartheta(\mu_B)$, where $B \subset K$ is Borel set and $\mu_B := \mu|B$, there exists a μ-measurable function g on K such that

$$\vartheta(\mu_B) = \int_B g d\mu$$

for each B. Fix $a > 0$ and closed $M \subset \{x \in K : g(x) \geq a\}$ with $\mu(M) > 0$. If $\eta := (\mu(M))^{-1}\mu_M$, we have $\vartheta(\eta_B) \geq a\eta(B)$ for all μ-measurable B contained in M. Observe that we may assume $\mu(M \cap V) > 0$ if $V \subset K$ is open with $V \cap M \neq \emptyset$.

Finally, there exist $x \in M$, a sequence $(U_n)_n$ of open subsets of M such that for each neighbourhood U_x of x, there exists $m \in \mathbb{N}$ such that $U_m \subset U_x$; this can be checked in [554]. Set:

$$\eta_n := (\eta(U_n))^{-1}\eta_{U_n}, \quad \mathfrak{F} := \{\eta_n : n \in \mathbb{N}\}.$$

Then $\delta_x \in \overline{\mathfrak{F}}$, $\vartheta(\eta_n) \geq a$, and $\vartheta(\delta_x) = 0$. Since ϑ is $\sigma(C(K)', C(K))$-continuous on $\overline{\mathfrak{F}}$, we reach a contradiction. □

In [505, Corollary 4.2] Pol showed that for compact K such that each $\mu \in P(K)$ is *countably determined*, i.e. there exists a countable family $\mathfrak{F}$ of compact subsets of K such that $\mu(B) = \sup\{\mu(F) : F \subset B, F \in \mathfrak{F}\}$ for every open B, the space $C(K)$ has the property (C) if and only if $P(K)$ has countable tightness. This result was generalized in [243, Theorem 3.2] to compact spaces K such that every $\mu \in P(K)$ is *separable*, i.e. the Banach space $L_1(\mu)$ is separable.

Theorem 13.2.3 (Frankiewicz–Plebanek–Ryll-Nardzewski) *Let K be a compact space:*

(i) If $P(K)$ has countable tightness, then $C(K)$ has the property (C).
(ii) If every $\mu \in P(K)$ is separable and $C(K)$ has the property (C), then $P(K)$ has countable tightness.

Proof Assume $P(K)$ has countable tightness. If $C(K)$ does not have the property (C), the space $M_0(K)$ does not have countable tightness, because the set A' considered in Theorem 13.1.6 (see (ii) → (i)) is a subset of $M_0(K)$. Lemma 13.2.1 implies that $P(K)$ does not have countable tightness, a contradiction. Thus, (i) holds true.

To prove (ii), assume that every $\mu \in P(K)$ is separable and the space $C(K)$ has the property (C). Assume also that there exists separable $\mu \in P(K)$ such that the tightness $t(\mu, P(K))$ is uncountable. By the assumption on μ, let $(f_n)_n$ be a sequence in $C(K)$ which is dense in $L_1(\mu)$. Fix:

$$\mathfrak{U} := \bigcap_n \{\nu \in P(K) : \nu(f_n) = \mu(f_n)\}.$$

Applying Lemma 13.2.1 (ii) to the zero-set $\mathfrak{U}$, we note that $\mathrm{t}(\mu, \mathfrak{U})$ is uncountable. Hence, there exists an uncountable family $\mathfrak{W} \subset \mathfrak{U}$ such that $\mu \in \overline{\mathfrak{W}}$ and $\mu \notin \overline{\mathfrak{N}}$ for each countable $\mathfrak{N} \subset \mathfrak{W}$. Fix $\epsilon > 0$. For each scalar t and each $g \in C(K)$ the product $tg \in C(K)$. Therefore for each countable $\mathfrak{N} \subset \mathfrak{W}$, there are $n \in \mathbb{N}$ and $g_1, \ldots, g_n \in C_c(K)$ such that

$$\mathfrak{N} \cap V(g_1, \ldots, g_n, 3\epsilon) = \emptyset, \tag{13.2}$$

where

$$V(g_1, \ldots, g_n, 3\epsilon) := \bigcap_{1 \le j \le n} \{\nu \in C(K)' : |\nu(g_j) - \mu(g_j)| < 3\epsilon\}$$

is a $\sigma(C(K)', C(K))$-neighbourhood of μ. For $\nu \in P(K)$ the set $T(\nu) := \{g \in C(K) : \nu(g) \ge 2\epsilon, \mu(g) \le \epsilon\}$ is convex, closed in $C(K)$, and $\bigcap_{\nu \in \mathfrak{W}} T(\nu) = \emptyset$, since $\mu \in \overline{\mathfrak{W}}$.

We prove that for each countable $\mathfrak{N} \subset \mathfrak{W}$, the set $\bigcap_{\nu \in \mathfrak{N}} T(\nu)$ is non-empty. This will yield a contradiction, since we show that $C(K)$ does not have the property (C). Fix a countable family $\mathfrak{N} \subset \mathfrak{W}$, and choose functions $g_1, g_2, \ldots, g_n$ in $C(K)$ satisfying (13.2). Next, choose functions $f_{j_1}, \ldots f_{j_n}$ from the sequence $(f_n)_n$ such that

$$\int_K |f_{j_l} - g_l| d\mu < n^{-1}\epsilon$$

for each $l = 1, 2, \ldots n$. If $h := \sum_{1 \le l \le n} |f_{j_l} - g_l|$, we have $\mu(h) \le \epsilon$. For $\nu \in \mathfrak{N}$ there exists $1 \le l \le n$ such that $|\nu(g_l) - \mu(g_l)| \ge 3\epsilon$. Hence

$$\nu(h) \ge |\nu(g_l) - \nu(f_{j_l})| \ge |\nu(g_l) - \mu(g_l)| - |\nu(f_{j_l}) - \mu(g_l)| \ge$$

$$3\epsilon - |\mu(f_{j_l}) - \mu(g_l)| \ge 2\epsilon.$$

This shows that $h \in \bigcap_{\nu \in \mathfrak{N}} T(\nu)$ proving that $C(K)$ does not have the property (C), a contradiction. □

The first part of Theorem 13.2.3 asserts that the countable tightness of $P(K)$ yields always the property (C) for $C(K)$. One may ask whether the assumption that every $\mu \in P(K)$ is separable can be removed from the second part of the theorem, thus obtaining the equivalence in the first part. In [249] Fremlin proved that under the assumption of Martin's axiom MA(ω_1) and the negation of the Continuum Hypothesis ¬CH, the following assertions are equivalent:

(i) Every measure $\mu \in P(K)$ is separable.
(ii) There does not exist a continuous surjection from K onto $[0, 1]^{\omega_1}$.

This yields the following.

Proposition 13.2.4 *Under MA + ¬CH, the following assertions are equivalent for a compact space K:*

(i) The space P(K) has countable tightness.
(ii) The Banach space C(K) has the property (C).

Proof (i) ⇒ (ii): See the first part of the proof of Theorem 13.2.3.

(ii) → (i): By Proposition 13.1.7 we know $t(K) \leq \aleph_0$. Since the space $[0, 1]^{\omega_1}$ does not have countable tightness, Fremlin's result applies to deduce that every measure in $P(K)$ is separable and Theorem 13.2.3 applies. □

From [243, Lemma 3.5] it follows that, if K is a compact zero-dimensional space and $C(K)$ is weakly Lindelöf, every measure $\mu \in P(K)$ is separable. This yields the following result of [243, Theorem 3.6].

Theorem 13.2.5 *If K is a compact zero-dimensional space and C(K) is weakly Lindelöf space, the space P(K) has countable tightness.*

13.3 The Property (C) for Banach Spaces $C(K \times K)$

In the present section, we focus on the following problem.

Problem 13.3.1 Suppose that $P(K)$ has countable tightness. Does this imply that every $\mu \in P(K)$ has countable Maharam type (i.e. $L_1(\mu)$ is separable)?

The *Maharam type* of a measure $\mu \in P(K)$ can be defined as the least cardinal number κ for which there exists a family $\mathcal{C} \subseteq Bor(K)$ of cardinality κ and such that the condition $\inf\{\mu(B \triangle C) : C \in \mathcal{C}\} = 0$ is satisfied for every $B \in Bor(K)$ (see [248, 425]). Equivalently, μ has Maharam type κ if the space of all μ-integrable functions $L_1(\mu)$ has density κ as a Banach space. A measure $\mu \in P(K)$ is *homogeneous* if its type is the same on every set $B \in Bor(K)$ of positive measure.

As we mentioned in the previous section, assuming Martin's axiom MA(ω_1) and the negation of the Continuum Hypothesis ¬CH, Fremlin [249] showed that if a compact space K admits a measure of uncountable type, then K can be continuously mapped onto $[0, 1]^{\omega_1}$, so in particular K must have uncountable tightness. Since $P(K)$ contains a subspace homeomorphic to K, it follows that Problem 13.3.1 has a positive solution under MA(ω_1).

Talagrand [581] showed that if K admits a measure of type ω_2, then $P(K)$ can be continuously mapped onto $[0, 1]^{\omega_2}$. Thus, the following analogue of Problem 13.3.1 holds true: if $\tau(P(K)) \leq \omega_1$, then every measure $\mu \in P(K)$ is of type $\leq \omega_1$.

We shall prove below the theorem of Plebanek and Sobota [501] asserting that for every compact space K, if $P(K \times K)$ has countable tightness, then K carries only measures of countable type (Theorem 13.3.10). This does not solve Problem 13.3.1 completely, but seems to be a substantial step forward.

Let us first recall some standard notation and definitions. In this section K always stands for a compact Hausdorff space. $Bor(K)$ stands for the σ-algebra of all Borel

subsets of K. Every $\mu \in P(K)$ is treated as a Borel measure on K, which is inner regular with respect to compact sets.

Remark 13.3.2 Take an open set $U \subseteq K$ and a closed set $F \subseteq K$. Note that the set of the form $V_{U,a} = \{\nu \in P(K) : \nu(U) > a\}$ is open in $P(K)$ with respect to the topology $\sigma(P(K), C(K))$ for every $a \in \mathbb{R}$.

Let $M \subseteq P(K)$ and $\nu_0 \in \overline{M}$. It follows that:

(a) $\nu_0(U) \leq a$ provided $\nu(U) \leq a$ for every $\nu \in M$;
(b) $\nu_0(F) \geq a$ provided $\nu(F) \geq a$ for every $\nu \in M$.

The following fact is well known; see, e.g. Plebanek [497, Lemma 2] or Fremlin [249, Introduction].

Lemma 13.3.3 *If a compact space K carries a regular measure of uncountable type, then there is $\mu \in P(K)$ which is homogeneous of type ω_1.*

Let $\mu \in P(K)$ and denote its measure algebra by

$$Bor(K)/_{\mu=0} = Bor(K)/\{A \in Bor(K) \colon \mu(A) = 0\}.$$

For $B \in Bor(K)$, let $B^\bullet$ stand for the corresponding element of $Bor(K)/_{\mu=0}$. We shall use the following standard result.

Lemma 13.3.4 *Let $\mu \in P(K)$ be a homogeneous measure of type ω_1 and let $\mathcal{C}$ be a countable family of Borel subsets of K. Then there is $B \in Bor(K)$ such that $\mu(B) = 1/2$ and B is μ-independent of $\mathcal{C}$, i.e. $\mu(B \cap C) = \frac{1}{2}\mu(C)$ for every $C \in \mathcal{C}$.*

Proof By the Maharam theorem (see Maharam [425] or Fremlin [248]), there is a measure-preserving isomorphism of measure algebras $\varphi : Bor(K)/_{\mu=0} \to \mathfrak{A}$, where $\mathfrak{A}$ is the measure algebra of the usual product measure λ on 2^{ω_1}. Let $\mathcal{C}^\bullet = \{C^\bullet : C \in \mathcal{C}\}$.

Recall that for every $a \in \mathfrak{A}$, there is a set $A \subseteq 2^{\omega_1}$ depending on coordinates in a countable set $I_A \subseteq \omega_1$ such that $A^\bullet = a$; see Fremlin [250, Section 8]. Therefore, there is a countable set $I \subseteq \omega_1$ such that for every $C \in \mathcal{C}$, there is $A \subseteq 2^{\omega_1}$ such that $A = A' \times 2^{\omega_1 \setminus I}$ for some $A' \in Bor(2^I)$ and $\varphi(C^\bullet) = A^\bullet$.

Take $\xi < \omega_1$ such that $\xi > \sup I$, and $B \in Bor(K)$ for which $B^\bullet = \varphi^{-1}(c_\xi^\bullet)$, where $c_\xi = \{x \in 2^{\omega_1} : x(\xi) = 0\}$. Then, B has the required property. □

The following corollary can be easily obtained using Lemma 13.3.4 and regularity of μ.

Corollary 13.3.5 *Let $\mu \in P(K)$ be a homogeneous measure of type ω_1. For every $\varepsilon > 0$, there exist sequences $\langle B_\xi \in Bor(K) : \xi < \omega_1\rangle$ and $\langle U_\xi \in Open(K) : \xi < \omega_1\rangle$ such that:*

(i) $\mu(B_\xi) = 1/2$,
(ii) $B_\xi \subseteq U_\xi$ *and* $\mu(U_\xi \setminus B_\xi) < \varepsilon$,
(iii) B_ξ *is μ-independent of the algebra generated by* $\mathcal{C}_\xi = \{B_\eta, U_\eta : \eta < \xi\}$.

Fix for a moment a measure $\mu \in P(K)$. Given two Boolean algebras $\mathcal{A}$ and $\mathcal{B}$ of sets, we write:

$$\mathcal{A} \otimes \mathcal{B} = alg(\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}),$$

for their product algebra; here $alg(\cdot)$ denotes the algebra of sets generated by a given family. Let $\mathcal{R}$ denote the Borel rectangle algebra in $K \times K$, i.e.

$$\mathcal{R} = Bor(K) \otimes Bor(K).$$

The following notation is crucial for our considerations: given an algebra $\mathcal{A} \subseteq Bor(K)$, we write $P(\mathcal{A} \otimes \mathcal{A}, \mu)$ for the family of all finitely additive probability measures ν on $\mathcal{A} \otimes \mathcal{A}$ such that

$$\nu(A \times K) = \nu(K \times A) = \mu(A) \text{ for every } A \in \mathcal{A}.$$

By a result due to Marczewski and Ryll-Nardzewski [431], every $\nu \in P(\mathcal{R}, \mu)$ is automatically countably additive and can be extended to a (regular) measure on the product σ-algebra $\sigma(Bor(K) \otimes Bor(K))$. In turn, such a measure can be extended to a regular measure on $Bor(K \times K)$. We outline below a relatively short argument for completeness (cf. Plebanek [495, Theorem 4]).

Theorem 13.3.6 *Every $\nu \in P(\mathcal{R}, \mu)$ can be extended to a regular Borel measure on $K \times K$.*

Proof Let $\mathcal{L}$ denote the family of finite unions of rectangles $F \times F'$ where $F, F' \subseteq K$ are closed. Using the fact that $\nu \in P(\mathcal{R}, \mu)$, it is easy to see that ν is $\mathcal{L}$-regular, i.e. for every $\varepsilon > 0$ and $A \in \mathcal{R}$, there exists $L \in \mathcal{L}$ contained in A and such that $\nu(A \setminus L) < \varepsilon$.

Let $\mathcal{F}$ be the lattice of all closed subsets of $K \times K$. By the main result from Bachman and Sultan [48], ν can be extended to an $\mathcal{F}$-regular finitely additive measure ν' on $alg(\mathcal{R} \cup \mathcal{F})$. By $\mathcal{F}$-regularity and compactness, ν' is continuous from above at $\emptyset$, and the standard Caratheodory extension procedure gives an extension to a regular measure on $\sigma(\mathcal{R}) = Bor(K \times K)$. □

For a subset $B \subseteq K$, we use below the following notation: $B^0 = B$ and $B^1 = B^c = K \setminus B$. We now prove two lemmas concerning extensions of measures on finite algebras with fixed marginal distributions.

Lemma 13.3.7 *If $\mathcal{A}$ is a finite subalgebra of $Bor(K)$, then every $\nu \in P(\mathcal{A} \otimes \mathcal{A}, \mu)$ can be extended to $\widehat{\nu} \in P(\mathcal{R}, \mu)$.*

Proof Let us fix a finite algebra $\mathcal{A} \subseteq Bor(K)$ and $\nu \in P(\mathcal{A} \otimes \mathcal{A}, \mu)$. Let $\mathcal{A}_1 = alg(\mathcal{A} \cup \{B\})$ where $B \in Bor(K) \setminus \mathcal{A}$. We shall check first that ν can be extended to $\nu_1 \in P(\mathcal{A}_1 \otimes \mathcal{A}_1, \mu)$.

Let $\{S_1, \ldots, S_l\}$ be the family of all atoms of $\mathcal{A}$ having positive measure. It is sufficient to define ν_1 only on atoms of $\mathcal{A}_1 \times \mathcal{A}_1$. For $i, j \leq l$, let

$$\alpha_{i,j} = \nu(S_i \times S_j)/(\mu(S_i)\mu(S_j)).$$

For $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$, put

$$\nu_1((S_i \times S_j) \cap (B^{\varepsilon_1} \times B^{\varepsilon_2})) = \mu(S_i \cap B^{\varepsilon_1})\mu(S_j \cap B^{\varepsilon_2})\alpha_{i,j}.$$

It is easy to check that this uniquely defines the required $\nu_1 \in P(\mathcal{A}_1 \otimes \mathcal{A}_1, \mu)$ (cf. the proof of the next lemma).

It follows that ν admits an extension to $\nu_{\mathcal{D}} \in P(\mathcal{D} \otimes \mathcal{D}, \mu)$ for every finite algebra $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{D} \subseteq Bor(K)$. Now the assertion follows by compactness argument as follows (cf. the proof of Lemma 13.3.9). Let $P(\mathcal{R})$ denote the set of all finitely additive probability measures on $\mathcal{R}$; clearly, $P(\mathcal{R})$ is a closed subset of $[0, 1]^{\mathcal{R}}$; hence, it is compact. Denote by $P_{\mathcal{D}}$ the set of all measures $\widetilde{\nu} \in P(\mathcal{R})$ extending ν and such that $\widetilde{\nu}|_{\mathcal{D}\otimes\mathcal{D}} \in P(\mathcal{D} \otimes \mathcal{D}, \mu)$. $P_{\mathcal{D}}$ is a closed subset of $P(\mathcal{R})$. Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be a sequence of finite extensions of $\mathcal{A}$ in $Bor(K)$. Then $P_{\mathcal{D}_1} \cap \ldots \cap P_{\mathcal{D}_n}$ is non-empty and hence, by finite intersection property, there exists:

$$\widehat{\nu} \in \bigcap \{P_{\mathcal{D}} : \mathcal{D} \text{ is a finite extension of } \mathcal{A} \text{ in } Bor(K)\}.$$

Clearly, $\widehat{\nu}$ is an element of $P(\mathcal{R}, \mu)$ extending ν. □

Lemma 13.3.8 *Let $\mathcal{A}$ be a finite subalgebra of $Bor(K)$, $\mathcal{A}_1 = alg(\mathcal{A} \cup \{B\})$, where $B \in Bor(K) \setminus \mathcal{A}$ is μ-independent of $\mathcal{A}$ and $\mu(B) = 1/2$.*

Then for every $\nu \in P(\mathcal{A} \otimes \mathcal{A}, \mu)$, there exists an extension $\nu_1 \in P(\mathcal{A}_1 \otimes \mathcal{A}_1, \mu)$ of ν such that $\nu_1(B \times B) = 1/2$.

Proof We extend ν to $\nu_1 \in P(\mathcal{A}_1 \times \mathcal{A}_1, \mu)$ in a similar way to the one presented in the proof of Lemma 13.3.7.

Let $T_1, \ldots, T_k$ be the list of all the atoms of $\mathcal{A}$. For all $i, j \leq k$ and $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$, put

$$\nu_1((T_i \times T_j) \cap (B^{\varepsilon_1} \times B^{\varepsilon_2})) = \tfrac{1}{2}\nu(T_i \times T_j)$$

if $\varepsilon_1 = \varepsilon_2$ and 0 otherwise. Then

$$\nu_1(B \times B) = \textstyle\sum_i \sum_j \nu_1((T_i \times T_j) \cap (B \times B)) = \frac{1}{2} \sum_i \sum_j \nu(T_i \times T_j) = \frac{1}{2}.$$

We now prove that $\nu_1 \in P(\mathcal{A}_1 \otimes \mathcal{A}_1, \mu)$. It is sufficient to check that $\nu_1(S \times K) = \nu_1(K \times S) = \mu(S)$ for every atom S of the algebra $\mathcal{A}_1$. We have:

$$\nu_1((T_i \cap B) \times K) = \textstyle\sum_j \nu_1((T_i \cap B) \times T_j) =$$

$$= \textstyle\sum_j \big(\nu_1((T_i \cap B) \times (T_j \cap B)) + \nu_1((T_i \cap B) \times (T_j \cap B^c))\big) =$$

$$= \sum_j \nu_1((T_i \cap B) \times (T_j \cap B)) = \tfrac{1}{2} \sum_j \nu(T_i \times T_j) =$$

$$= \tfrac{1}{2}\nu(T_i \times K) = \tfrac{1}{2}\mu(T_i) = \mu(T_i \cap B),$$

where the last identity follows from the μ-independence of B and $\mathcal{A}$. Similarly one checks remaining possibilities. □

Lemma 13.3.9 *Let $\mu \in P(K)$ be a homogeneous measure of type ω_1 and suppose that $\langle B_\xi \in Bor(K) : \xi < \omega_1\rangle$, $\langle U_\xi \in Open(K) : \xi < \omega_1\rangle$ and $\mathcal{C}_\xi$ are as in Corollary 13.3.5.*

For every $\xi < \omega_1$ there is $\nu_\xi \in P(\mathcal{R}, \mu)$ such that:

- $\nu_\xi(B_\eta \times B_\eta) = 1/2$ *for* $\eta \geq \xi$,
- $\nu_\xi(A \times A) = (\mu \otimes \mu)(A \times A)$ *for every* $A \in \mathcal{C}_\xi$.

Proof Fix $\xi < \omega_1$. Let $\mathcal{A}$ be a finite algebra generated by some elements of $\mathcal{C}_\xi$ and I be a finite subset of $\omega_1 \setminus \xi$. Then there is $\nu_{\mathcal{A},I} \in P(\mathcal{R}, \mu)$ such that:

– $\nu_\xi(B_\eta \times B_\eta) = 1/2$ for $\eta \in I$,
– $\nu_\xi(A \times A) = (\mu \otimes \mu)(A \times A)$ for every $A \in \mathcal{A}$.

Indeed, such $\nu_{\mathcal{A},I}$ can be first defined on $alg(\mathcal{A} \cup \{B_\eta : \eta \in I\})$ using Lemma 13.3.8 and induction on $|I|$ and then extended to a member of $P(\mathcal{R}, \mu)$ using Lemma 13.3.7.

Now the existence of ν_ξ with the required properties follows again by compactness argument: $P(\mathcal{R}, \mu)$ is clearly a closed subset of $[0, 1]^{\mathcal{R}}$, so it is compact in the topology of convergence on all elements of $\mathcal{R}$. Hence, the required measure ν_ξ can be defined as a cluster point of the net $\nu_{\mathcal{A},I}$ indexed by the pairs $(\mathcal{A}, I)$, where $\mathcal{A}$ is an algebra generated by a finite subset of $\mathcal{C}_\xi$ and I is a finite subset of $\omega_1 \setminus \xi$. □

We are now ready to prove the main result of [501].

Theorem 13.3.10 (Plebanek–Sobota) *Let $P(K \times K)$ have countable tightness. Then every $\mu \in P(K)$ has countable type.*

Proof Assume for the sake of contradiction that there exists $\mu \in P(K)$ of uncountable type. Without loss of generality, we can assume that μ is a homogeneous measure of type ω_1; see Lemma 13.3.3. Let $0 < \varepsilon < 1/16$.

Take the sequences $\langle B_\xi : \xi < \omega_1\rangle$ and $\langle U_\xi : \xi < \omega_1\rangle$ as in Corollary 13.3.5. For every $\xi < \omega_1$, take $\nu_\xi \in P(\mathcal{R}, \mu)$ as in Lemma 13.3.9 and extend it to $\widehat{\nu_\xi} \in P(K \times K)$ using Theorem 13.3.6. Let $\nu \in P(K \times K)$ be a cluster point of the sequence $\langle \widehat{\nu_\xi} : \xi < \omega_1\rangle$, i.e.

$$\nu \in \bigcap_{\xi < \omega_1} \overline{\{\widehat{\nu_\eta} : \eta \geq \xi\}}.$$

We shall show that $\nu \notin \overline{\{\widehat{\nu_\eta} : \eta \in I\}}$ for every countable $I \subseteq \omega_1$, which will contradict the assumption that the tightness of $P(K \times K)$ is countable.

Let $I \subseteq \omega_1$ be countable. Take $\xi > \sup I$. By regularity of μ, there exists closed $F \subseteq K$ such that $F \subseteq B_\xi$ and $\mu(B_\xi \setminus F) < \varepsilon$. For every $\eta \in I$, we have $\widehat{\nu}_\eta(B_\xi \times B_\xi) = 1/2$ and

$$\widehat{\nu}_\eta((B_\xi \times B_\xi) \setminus (F \times F)) \leq \widehat{\nu}_\eta((B_\xi \setminus F) \times K) + \widehat{\nu}_\eta(K \times (B_\xi \setminus F))$$

$$= 2\mu(B_\xi \setminus F) < 2\varepsilon.$$

Therefore $\widehat{\nu}_\eta(F \times F) > 1/2 - 2\varepsilon$ whenever $\eta \in I$.

On the other hand, if $\eta > \xi$, then

$$\nu_\eta(U_\xi \times U_\xi) = (\mu \otimes \mu)(U_\xi \times U_\xi) = \mu(U_\xi)^2 < (1/2 + \varepsilon)^2,$$

so by Remark 13.3.2(a) $\nu(F \times F) \leq \nu(U_\xi \times U_\xi) \leq (1/2 + \varepsilon)^2$.

As $\varepsilon < 1/16$, we have $(1/2 + \varepsilon)^2 < 1/2 - 2\varepsilon$. We conclude from Remark 13.3.2(b) that $\nu \notin \overline{\{\widehat{\nu}_\eta : \eta \in I\}}$ and the proof is complete. □

Let us remark that modifying our way to Theorem 13.3.10, one can prove the following more general result.

Theorem 13.3.11 (Plebanek–Sobota) *Suppose that K and L are compacta carrying measures of uncountable type. Then $\tau(P(K \times L)) \geq \omega_1$.*

It is not difficult to check that if every $\mu \in P(K)$ has countable type, then every $\nu \in P(K \times K)$ has countable type as well. In connection with Problem 13.3.1 and Theorem 13.3.10, it is natural to ask whether the countable tightness of $P(K)$ implies the countable tightness of $P(K \times K)$. As far as we know, the problem is open. Note that $P(K) \times P(K)$ embeds into $P(K \times K)$ and if $\tau(P(K)) = \omega$, then $\tau(P(K) \times P(K)) = \omega$, since the countable tightness is productive for compact spaces (see [195, Exercise 3.12.8]). However, the space $P(K \times K)$ seems to be far more complicated than $P(K) \times P(K)$.

Let us now provide several consequences of Theorem 13.3.10. Recall that a compact space K is said to be *Rosenthal compact* if K embeds into $B_1(X)$, the space of Baire-one functions on a Polish space X equipped with the topology of pointwise convergence. The class of Rosenthal compacta is stable under taking countable product and, by a result of Godefroy [278], if K is Rosenthal compact, then so is $P(K)$. Moreover, Rosenthal compacta are Fréchet–Urysohn spaces (see Bourgain, Fremlin, and Talagrand [101]); hence, they have countable tightness. This, together with Theorem 13.3.10, implies the following result of Bourgain ([99]) and Todorčević ([603]) (see also Marciszewski and Plebanek [427]).

Corollary 13.3.12 (Bourgain–Todorčević) *If K is Rosenthal compact, then every $\mu \in P(K)$ has countable type.*

Let us say that $P(K)$ has *convex countable tightness* if $P(K)$ fulfills condition (ii) of Theorem 13.1.6. Clearly countable tightness implies convex countable tightness.

Looking back at the proof of Theorem 13.3.10, it is easy to notice that in fact the following (formally) stronger result has been obtained.

Theorem 13.3.13 (Plebanek–Sobota) *Let $P(K \times K)$ have convex countable tightness. Then every $\mu \in P(K)$ has countable type.*

Theorem 13.3.13 gives the following important result ([501, Corollary 5.7]), very much related to Theorem 13.2.3.

Corollary 13.3.14 *For any compact space K, $\tau(P(K \times K)) = \omega$ if and only if $C(K \times K)$ has property (C).*

Proof Assume that $\tau(P(K \times K)) = \omega$. By Theorem 13.2.3.(i) $C(K \times K)$ has the property (C).

For the converse assume that $C(K \times K)$ has the property (C). By Theorem 13.1.6, $P(K \times K)$ has convex countable tightness, which by Theorem 13.3.13 implies that every $\mu \in P(K \times K)$ has countable type. Using Theorem 13.2.3.(ii), we conclude that $\tau(P(K \times K)) = \omega$. □

One can ask whether the property (C) of a space $C(K)$ implies the property (C) of the space $C(K \times K)$. Note that the converse holds true. Indeed, if X is a Banach space with the property (C) and Y is its closed subspace, then Y also has the property (C). Since $C(K)$ embeds isometrically into $C(K \times K)$ by the operator $C(K) \ni f \mapsto f \circ \pi \in C(K \times K)$, where $\pi : K \times K \to K$ is a projection, $C(K)$ is isometric to a closed subspace of $C(K \times K)$.

It is also worth of noting that if $C(K)$ has the property (C), then $C(K) \oplus C(K)$ has the property (C) as well, since the property (C) is a three-space property (see Theorem 13.1.4).

Chapter 14
Fréchet–Urysohn Spaces and Groups

Abstract This chapter deals with topological (vector) spaces satisfying some sequential conditions. We study Fréchet–Urysohn spaces (i.e., spaces E such that for each $A \subset E$ and each $x \in A^-$ there exists a sequence in A converging to x). The main result states that every sequentially complete Fréchet–Urysohn lcs is a Baire space. Since every infinite-dimensional Montel (DF)-space E is non-metrizable and sequential, the following question arises: Is every Fréchet–Urysohn space in the class G metrizable?

14.1 Fréchet–Urysohn Topological Spaces

Recall that a topological space E is called *Fréchet–Urysohn* if for every subset $A \subset E$ and every $x \in \overline{A}$, there exists a sequence from A converging to the point x. We say that a topological space E satisfies the Fréchet–Urysohn property if E is a Fréchet–Urysohn space. The Fréchet–Urysohn property is known to be highly non-multiplicative; the square of a compact Fréchet–Urysohn space need not be Fréchet–Urysohn; see [559]. Van Douwen [626] proved that the product of a metrizable space by a Fréchet–Urysohn space may not be (even) sequential. Recall that E is said to be *sequential* if every sequentially closed subset of E is closed. These concepts have been studied by topologists and analysts over the last half century; see, for example, [175, 188, 195, 245, 328, 346, 347, 472, 556, 626, 638, 643, 644]; see also the following recent survey paper [544].

It is known that a Fréchet–Urysohn lcs may not be metrizable. Probably the first example of such a space was presented in [49]. The following list of certain results provides many examples of non-metrizable Fréchet–Urysohn lcs. We refer to [273, Theorem 2], [32, Theorem II.3.7], [32, Theorem III.1.2], [122, Corollary 4.2], [553, Theorem 5.1], [308, 589, 600, Theorem 1, Theorem 2], respectively.

Proposition 14.1.1 *(i) (McCoy–Gerlits–Nagy) X is an ω-space, i.e. every open ω-cover of X has a countable ω-subcover, if and only if $C_p(X)$ is Fréchet–Urysohn (for the definition of an ω-cover, see Proposition 9.4.31).*

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(ii) (Pytkeev–Gerlits–Nagy) $C_p(X)$ is Fréchet–Urysohn if and only if $C_p(X)$ is sequential if and only if $C_p(X)$ is a k-space (for the definition of a k-space, see the text before Lemma 6.7.8).

(iii) (Arkhangel'skii–Pytkeev) If X is compact, $C_p(X)$ is Fréchet–Urysohn if and only if X is scattered.

(iv) (Pytkeev–Gerlits) $C_c(X)$ is Fréchet–Urysohn if and only if $C_c(X)$ is sequential if and only if $C_c(X)$ is a k-space.

(v) (Hernández–Mazón) If X is first countable, the space $C_c(X)$ is Fréchet–Urysohn if and only if X is hemicompact.

(vi) (Arkhangel'skii–Tkachuk) $C_p(X)^{\mathbb{N}}$ is Fréchet–Urysohn for Fréchet–Urysohn $C_p(X)$. There exist Fréchet–Urysohn spaces $C_p(X)$ and $C_p(Y)$ such that $C_p(X) \times C_p(Y)$ does not have countable tightness (Todorčević).

Proposition 14.1.2 is an unpublished result of Morishita; see [334, Theorem 2]; see also [31, Theorems 10.5, 10.7], [33, Theorem II.7.16]. Proposition 14.1.2 was extended in [122, Corollary 4.2] to K-analytic spaces X. For the definition of a k_R-space, see the text before Theorem 9.1.5.

Proposition 14.1.2 *For a Čech-complete and Lindelöf space X, the following assertions are equivalent:*

(i) X is scattered.

(ii) $C_p(X)$ is Fréchet–Urysohn.

(iii) $C_p(X)$ is a k_R-space.

Proof (i)⇒(ii) holds for any Lindelöf space, [32, Theorem II.7.16].

(ii)⇒ (iii) is obvious.

(iii) ⇒ (i): Assume X is not scattered. Then, X contains a compact space Y that can be continuously mapped onto the interval $[0, 1]$; see [122, Theorem 4.1] since every Čech-complete and Lindelöf space is K-analytic. Let φ_Y be the restriction map of $C_p(X)$ into $C_p(Y)$ defined by $\varphi_Y(f) = f|Y$ for any $f \in C_p(X)$. Clearly φ_Y is continuous, open, and $\varphi_Y(C_p(X)) = C_p(Y)$. Set $W := C_p(Y) \cap [0, 1]^Y$. Observe that W is not a k_R-space. Let μ be a finite non-negative non-atomic regular Borel measure on Y. Define a map $\psi : W \to [0, \mu(Y)]$ by the formula $\psi(f) = \int_Y f d\mu$ for any $f \in W$. Note that ψ is not continuous at zero. Indeed, for every $\epsilon > 0$, $f \in W$, and $\Gamma \subset Y$, set:

$$< f, \Gamma, \epsilon >= \{g \in W : |f(x) - g(x)| < \epsilon, x \in \Gamma\}.$$

Assume that ψ is continuous at zero. Then, there exist a finite subset $\Gamma \subset Y, \epsilon > 0$ such that

$$\psi(< 0, \Gamma, \epsilon >) \subset [0, 2^{-1}\mu(Y)).$$

Since $\mu(\Gamma) = 0$, by the regularity of μ, there exists a closed subset F in Y such that $\Gamma \cap F = \emptyset$ and $\mu(F) \geq 2^{-1}\mu(Y)$. There exists a function $f \in W$ such that $f|F = 1$ and $f|\Gamma = 0$. Finally, $\psi(f) \geq 2^{-1}\mu(Y)$, a contradiction with $f \in < 0, \Gamma, \epsilon >$.

Now we show that ψ is k_R-continuous, i.e. for every compact subset K of W, the map $\psi|K$ is continuous. Y is compact, so the space $C_p(Y)$ is *monolithic*, i.e. the closure of any countable subset of $C_p(Y)$ is a space with a countable network [32, Theorem II.6.19], and has countable tightness by Proposition 9.4.1. If K is a compact subset of $W \subset C_p(Y)$, then K is also monolithic (see [32, Theorem II.6.5]), and clearly K has countable tightness. We claim that K is a Fréchet–Urysohn space. Indeed, let $A \subset K$ be a subset of K and let $x \in \overline{A}$ (the closure in K). By the assumption there exists a countable set $B \subset A$ such that $x \in \overline{B}$. Since K is monolithic, $\overline{B}$ has a countable network. As every compact space with a countable network is a space with a countable base, there exists in B a sequence which converges to x. Hence to show that $\psi|K$ is continuous, it is enough to prove that $\psi|K$ is sequentially continuous. Let $(x_n)_n$ be a sequence in

$$K \subset [0,1]^Y \cap C_p(Y),$$

and assume that $x_n \to x$ in K. Therefore

$$0 \leqslant x_n(t) \leqslant 1, \;\; x_n(t) \to x(t), \;\; t \in Y, \;\; n \in \mathbb{N}.$$

By the Lebesgue dominating theorem, we have $\psi(x_n) \to \psi(x)$.

Finally, we show that $C_p(Y)$ is not a k_R-space. Let $r : \mathbb{R} \to [0,1]$ be a map defined by $r(x) = 1$ for $x \geqslant 1$, $r(x) = x$ for $0 \leqslant x \leqslant 1$, and $r(x) = 0$ for $x \leqslant 0$. Define a map $\Psi : C_p(Y) \to W$ by $\Psi(f) = rf$ for any $f \in C_p(Y)$. Then, Ψ is the retraction. Since W is not a k_R-space, the space $C_p(Y)$ is not a k_R-space. Since φ_Y is continuous and open, $C_p(X)$ is not a k_R-space. □

14.2 A Few Facts about Fréchet–Urysohn Topological Groups

Recall the concept of the *double sequence property*; see [23, 33].

Definition 14.2.1 (α_4) For any family $\{x_{m,n} : (m,n) \in \mathbb{N} \times \mathbb{N}\} \subset X$ with $\lim_n x_{m,n} = x \in X$, $m \in \mathbb{N}$, there exist a sequence $(i_k)_k$ of distinct natural numbers and a sequence $(j_k)_k$ of natural numbers such that $\lim_k x_{i_k,j_k} = x$.

Every Fréchet–Urysohn topological group satisfies the property (α_4). This property fails for topological spaces in general [472, Theorem 4].

By [43, Lemma 3.3] a Fréchet–Urysohn tvs satisfies the following (stronger) property (as); see Lemma 14.2.3.

Definition 14.2.2 We shall say that X satisfies the property (as) if for any family $\{x_{m,n} : (m,n) \in \mathbb{N} \times \mathbb{N}\} \subset X$ with $\lim_n x_{m,n} = x \in X$ for $m \in \mathbb{N}$, there exist two strictly increasing sequences of natural numbers $(i_k)_k$ and $(j_k)_k$, such that $\lim_k x_{i_k,j_k} = x$.

We need the following lemma from [135, Lemma 1.3].

Lemma 14.2.3 *A Fréchet–Urysohn Hausdorff topological group X satisfies the property (as), and hence the property (α_4), as well.*

Proof By 0 we denote the neutral element of X. It is enough to show the property (as) for any family

$$\{x_{m,n} : (m,n) \in \mathbb{N} \times \mathbb{N}\} \subset X$$

with $\lim_n x_{m,n} = 0$, $m \in \mathbb{N}$. Fix a sequence $(a_m)_m \subset X$ with $\lim_m a_m = 0$ and $a_m \neq 0$ for $m \in \mathbb{N}$ (if such a sequence does not exist, then X is discrete and the conclusion is trivial). Set:

$$y_{m,l} := a_m + x_{m,l+m}$$

if $a_m + x_{m,l+m} \neq 0$ and $y_{m,l} := a_m$, otherwise. Set:

$$M := \{y_{m,l} : (m,l) \in \mathbb{N} \times \mathbb{N}\}.$$

Then $0 \notin M$. Note that $0 \in \overline{M}$. Let U and U_0 be neighbourhoods of zero with $U + U \subset U_0$. Since $\lim_m a_m = 0$, there is $m \in \mathbb{N}$ such that $a_m \in U$, and by $\lim_n x_{m,n} = 0$, there is $l \in \mathbb{N}$ such that $x_{m,l+m} \in U$. Hence $y_{m,l} \in U + U \subset U_0$. Next, by the Fréchet–Urysohn property, and since $0 \in \overline{M}$, there exists a sequence $(m_k, l_k)_k$ such that $\lim_k y_{m_k,l_k} = 0$.

Case 1. The sequence $(l_k)_k$ is bounded. Taking a subsequence if necessary, we may assume that $l_k = r$ for some natural number r. Since

$$0 = \lim_k y_{m_k,l_k} = \lim_k y_{m_k,r},$$

and $y_{m_k,r} \neq 0$, we conclude that $\lim_k m_k = \infty$. We may also assume that $m_1 < m_2 < \dots$. Assume first that

$$N_1 = \{k \in \mathbb{N} : y_{m_k,r} = a_{m_k}\}$$

is infinite. Set $N_1 = \{p_1, p_2, \dots\}$ with $p_1 < p_2 < \dots$. Then $a_{m_{p_k}} + x_{m_{p_k},r+m_{p_k}} = 0$ for $k \in \mathbb{N}$. As $\lim_k a_{m_{p_k}} = 0$, we have $\lim_k x_{m_{p_k},r+m_{p_k}} = 0$. Now set:

$$i_k := m_{p_k}, \quad j_k := r + m_{p_k}$$

for $k \in \mathbb{N}$. Then, one obtains strictly increasing sequences $(i_k)_k$ and $(j_k)_k$, such that $\lim_k x_{i_k,j_k} = 0$. Suppose that

$$N_1 = \{k \in \mathbb{N} : y_{m_k,r} = a_{m_k}\}$$

is finite. Then

$$N_2 = \{k \in \mathbb{N} : y_{m_k,r} \neq a_{m_k}\}$$

is infinite, and hence $N_2 = \{q_1, q_2, \dots\}$ for $q_1 < q_2 < \dots$. Consequently,

$$y_{m_{q_k},l_{q_k}} = a_{m_{q_k}} + x_{m_{q_k},r+m_{q_k}}$$

for $k \in \mathbb{N}$. Since $\lim_k y_{m_{q_k},l_{q_k}} = 0$ and $\lim_k a_{m_{q_k}} = 0$, we note:

$$\lim_k x_{m_{q_k},r+m_{q_k}} = 0.$$

Set again $i_k := m_{q_k}$ and $j_k := r + m_{q_k}$ for $k \in \mathbb{N}$. Then, there exist strictly increasing sequences $(i_k)_k$ and $(j_k)_k$ such that $\lim_k x_{i_k,j_k} = 0$.

Case 2. The sequence $(l_k)_k$ is not bounded. We may assume that $(l_k)_k$ is strictly increasing. Then $\lim_k m_k = \infty$. Indeed, otherwise taking a subsequence if necessary, we may assume that $m_k = s$ for some s. The sequence $(l_k)_k$ is strictly increasing, so we have $\lim_k x_{s,s+l_k} = 0$. From $\lim_k y_{s,l_k} = 0$, it follows $a_{m_k} = a_s = 0$, a contradiction with the choice of $(a_m)_m$. Hence $\lim_k m_k = \infty$. There exists a strictly increasing sequence $(n_k)_k$ such that $m_{n_1} < m_{n_2} < \dots$. Set

$$i_k := n_k, \quad j_k := m_{n_k} + l_{n_k}$$

for $k \in \mathbb{N}$. Then $(i_k)_k$ and $(j_k)_k$ are strictly increasing sequences such that $\lim_k x_{i_k,j_k} = 0$. □

One of the interesting problems (due to Malyhin) concerning Fréchet–Urysohn groups asks whether it is consistent that every countable Fréchet–Urysohn topological group is metrizable [553]; see also [454] and [604] for some examples under various additional set-theoretic assumptions. Some possible approaches to attack this problem have been recently provided in [103].

We show that under Martin's axiom MA, there exist non-metrizable analytic (hence separable) Fréchet–Urysohn lcs. We observe that the *Borel conjecture* implies that separable Fréchet–Urysohn spaces $C_p(X)$ are metrizable.

On the other hand, Laver [399] proved that it is relatively consistent with the system ZFC (of Zermelo, Fraenkel, plus the Axiom of Choice) that the Borel conjecture is true. In fact there exist many important classes of lcs for which the Fréchet–Urysohn property implies the metrizablity; we will see that every lcs E in the class $\mathfrak{G}$ is metrizable if and only if E is Fréchet–Urysohn.

Example 14.2.4 There exists a σ-compact, hence K-analytic, non-metrizable Fréchet–Urysohn topological group that is not Baire.

Proof The direct sum X of the $\aleph_1$ copies of the circle group $(\mathbb{R}/\mathbb{Z}, +)$ is a σ-compact Fréchet–Urysohn group that is non-metrizable; see [454, Example 1.2]. Assume X is Baire. Then X is locally compact (since X is a Baire group). As

every locally compact Fréchet–Urysohn topological group is metrizable (Proposition 9.7.3), we reach a contradiction. □

We have also the following.

Example 14.2.5 The Σ-product X of the compact group $\{0,1\}^{\omega_1}$, i.e. the set of elements of the product $\{0,1\}^{\omega_1}$ with at most countably many non-zero coordinates, is Fréchet–Urysohn, locally compact, and not K-analytic. Nevertheless, under CH it has a compact resolution. On the other hand, under MA $+\neg$ CH the space X does not have a compact resolution.

Proof Note that X is Fréchet–Urysohn and locally compact; see [470, Theorem 2.1] and [69, Ch. IX, Exerc. 17]. Note also that X is not Lindelöf. Indeed, the following open cover $\{U_t : t \in [0,\omega_1)\}$ of X does not admit a countable subcover, where U_t is the set of all points of X whose t-coordinate is 0. Since every K-analytic space is a Lindelöf space, X is not K-analytic. If CH is assumed, the space $[0,\omega_1)$ has a compact resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ swallowing compact sets (Proposition 3.2.3). Define compact sets K_α for $\alpha \in \mathbb{N}^{\mathbb{N}}$ in X by the formula:

$$K_\alpha = \{x = x(t) \in X : x(t) = 0,\ t \notin A_\alpha\}.$$

Note that $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution swallowing compact sets of X. On the other hand, under MA $+\neg$ CH, the space $[0,\omega_1)$ does not have a compact resolution by Proposition 3.2.3, and the same property holds for X. □

We provided examples of non-metrizable Fréchet–Urysohn σ-compact topological groups. Next Example 14.2.6 extends Webb's theorem [635, Theorem 5.7]. Webb proved that only finite-dimensional Montel (DF)-spaces are Fréchet–Urysohn. Recall that every Fréchet–Urysohn Montel (DF)-space is a hemicompact tvs, and every locally compact tvs is finite-dimensional.

Example 14.2.6 Every Fréchet–Urysohn hemicompact topological group X is a locally compact Polish space.

Proof Let $(K_n)_n$ be an increasing sequence of compact sets covering X such that every compact set in X is contained in some K_m. Note that X is locally compact. Indeed, it is enough to show that there exists $n \in \mathbb{N}$ such that K_n contains a neighbourhood of the unit of X. Let $\mathfrak{F}$ be a basis of neighbourhoods of the unit of X. Assume that no K_n contains an element of $\mathfrak{F}$. For every $U \in \mathfrak{F}$ and $n \in \mathbb{N}$, choose $x_{U,n} \in U \setminus K_n$, and for each $n \in \mathbb{N}$, let

$$A_n = \{x_{U,n} : U \in \mathfrak{F}\}.$$

Since $0 \in \overline{A_n}$ for every $n \in \mathbb{N}$, there exists a sequence $(U_{m,n})_m$ in $\mathfrak{F}$ such that $x_{m,n} \to 0$, $m \to \infty$, where $x_{m,n} := x_{U_{m,n},n}$. By Lemma 14.2.3 there exist a sequence $(n_k)_k$ of distinct numbers in $\mathbb{N}$, a sequence $(m_k)_k$ in $\mathbb{N}$ such that $x_{m_k,n_k} \to 0$. As $\{x_{m_k,n_k} : k \in \mathbb{N}\} \cup \{0\}$ is contained in some K_p and $x_{m_k,n_k} \notin K_{n_k}$ for each $k \in \mathbb{N}$, we reach a contradiction. Hence X is a locally compact Fréchet–Urysohn

group. By Proposition 9.7.3 we know that X is metrizable. Hence X is analytic. Since any analytic Baire topological group is a Polish space (Theorem 7.1.7), the proof is finished. □

We complete this section with a simple characterization of Fréchet–Urysohn spaces $C_c(X)$ over locally compact metric spaces X; this supplements Theorem 9.7.1. First we note the following fact; see also Theorem 14.5.1.

Lemma 14.2.7 *If $X = (X, d)$ is a metric space and $C_c(X)$ is a Fréchet–Urysohn space, the space X is separable.*

Proof We may assume that X is non-compact, so there exists on X an equivalent unbounded metric. Let $\mathcal{K}(X)$ be the family of all compact sets in X. For $K \in \mathcal{K}(X)$, let V_K be an open neighbourhood of K such that

$$d(x, K) < (\max\{1, \delta(K)\})^{-1}$$

for each $x \in V_K$, where $\delta(A)$ means the diameter of A. Then

$$\delta(V_K) \leq \delta(K) + 2(\max\{1, \delta(K)\})^{-1}.$$

Let

$$f_K : X \to [0, \max\{1, \delta(K)\}]$$

be a continuous function such that $f_K(x) = 0$ for all $x \in K$ and $f_K(y) = \max\{1, \delta(K)\}$ for each $y \in X \setminus V_K$. Set

$$M := \{f_K : K \in \mathcal{K}(X)\}.$$

Then $0 \in \overline{M}$, where the closure is taken in $C_c(X)$. By the assumption there exists a bounded set $B \subseteq M$ whose closure contains 0. Observe that

$$\sup\{\delta(K) : f_K \in B\} = \infty.$$

Indeed, assume on the contrary that for some $n \in \mathbb{N}$, one has $\delta(K) \leq n$ for all $K \in \mathcal{K}(X)$ for which $f_K \in B$. Choose $Q \in \mathcal{K}(X)$ with $\delta(Q) > n + 2$, and set

$$U[Q] := \{f \in C(X) : \sup_{x \in Q} |f(x)| < 2^{-1}\}.$$

Since $0 \in \overline{B}$, there exists $f_K \in B \cap U[Q]$. So $Q \subseteq V_K$, and hence

$$\delta(Q) \leq \delta(V_K) \leq \delta(K) + 2(\max\{1, \delta(K)\})^{-1} \leq n + 2,$$

a contradiction.

For every $n \in \mathbb{N}$, choose $K_n \in \mathcal{K}(X)$ such that $f_{K_n} \in B$ and $\delta(K_n) \geq n$. If there are $y \in X$ and $k \in \mathbb{N}$ such that $y \notin V_{K_n}$ for each $n \geq k$, we have $f_{K_n}(y) = \max\{1, \delta(K_n)\} \geq n$ for each $n \geq k$, which leads to

$$\sup_{f \in B} |f(y)| = +\infty,$$

a contradiction with the boundedness of B. Therefore, for each $x \in X$, $k \in \mathbb{N}$, there exists $n_k \geq k$ (where n_k depends of x), such that $x \in V_{K_{n_k}}$. Hence

$$d(x, K_{n_k}) \leq (\max\{1, \delta(K_{n_k})\})^{-1} \leq n_k^{-1}.$$

Since each compact metric space is separable, for each $k \in \mathbb{N}$ there is a countable set A_k in K_{n_k} dense in K_{n_k}. Choosing

$$x_k \in A_k \cap B(x, 2/n_k),$$

where $B(x, 2/n_k)$ stands for the open ball of centre at x and radius $2/n_k$, we see that $x_k \to x$ in X. Hence the countable set $\bigcup_{k=1}^{\infty} A_k$ is dense in X, and then X is separable. □

Now we prove the following.

Proposition 14.2.8 *If X is a locally compact metric space, the following assertions are equivalent:*

(i) $C_c(X)$ *is a Fréchet space.*
(ii) $C_c(X)$ *is Fréchet–Urysohn.*
(iii) $C_c(X)$ *has countable tightness.*
(iv) X *is σ-compact.*

Proof Since every locally compact separable metric space is hemicompact, Lemma 14.2.7 proves (i)⇔ (ii). The implication (iv) ⇒ (i) follows from the fact any metric σ-compact space is separable, hence hemicompact.

To complete the proof, it suffices to show the implication (iii) ⇒ (iv): for every $K \in \mathcal{K}(X)$, there exists an open set U_K with the compact closure, such that

$$K \subseteq U_K \subseteq \overline{U_K}.$$

Choose $f_K \in C(X)$ such that $f_K(x) = 0$ for each $x \in K$ and $f_K(x) = 1$ for all $x \in X \setminus U_K$. Then, 0 belongs to the closure (in $C_c(X)$) of $\{f_K : K \in \mathcal{K}(X)\}$. By the assumption there exist a sequence $K_n \in \mathcal{K}(X)$, $n \in \mathbb{N}$, such that the closure of $\{f_{K_n} : n \in \mathbb{N}\}$ contains zero 0. Hence, for arbitrary $Q \in \mathcal{K}(X)$, there is $j \in \mathbb{N}$ with $\sup_{x \in Q} |f_{K_j}(x)| < 2^{-1}$, so that $Q \subseteq \overline{U_{K_j}}$. This shows that X is σ-compact. □

14.3 Sequentially Complete Fréchet–Urysohn Spaces Are Baire

In this section we prove the following main result from [345].

Theorem 14.3.1 *Every sequentially complete Fréchet–Urysohn lcs is a Baire space.*

To prove Theorem 14.3.1, we show first following Lemma 14.3.2 due to Burzyk [110]. Recall that a sequence $(x_n)_n$ in a topological additive group X is called a K-*sequence* if every subsequence of $(x_n)_n$ contains a subsequence $(y_n)_n$ such that the series $\sum_n y_n$ converges in X; see [15]; see also [111, 491]. Clearly, if X is a metric and complete additive topological group, every sequence in X converging to zero is a K-sequence.

Lemma 14.3.2 *Let $(X_k)_k$ be an increasing sequence of closed subsets of a topological additive group X covering X. Let $(x_n)_n$ be a K-sequence in X. Then there exists $m \in \mathbb{N}$ such that $x_n \in X_m + \{-\sum_{k\in A} x_k : A \subset \{1, 2, \dots m\}\}$ for every $n \in \mathbb{N}$.*

Proof Suppose the conclusion fails. Since $(X_n)_n$ is increasing, and subsequences of K-sequences are K-sequences, for each $n \in \mathbb{N}$ there exists $x_{n+1} \notin X_n + \{-\sum_{k\in A} x_k : A \subset \{1, 2, \dots n\}\}$, where $x_1 \neq 0$. Set

$$G_{n+1} := X_n + \{-\sum_{k\in A} x_k : A \subset \{1, 2, \dots n\}\}$$

and $G_1 := \{0\}$. The sets G_n are closed in X, so for each $n \in \mathbb{N}$ there exist a continuous pseudonorm q_n on X, and $\epsilon_n > 0$, such that

$$\inf_{z\in G_n} q_n(x_n - z) > \epsilon_n. \tag{14.1}$$

Since $x_n \to 0$, there exists a sequence $(k_n)_n$ in $\mathbb{N}$ such that

$$q_n(-x_{k_m}) < 2^{-n-m}\epsilon_n$$

for all $m \geq n$ and $n, m \in \mathbb{N}$. Then there exists a subsequence $(s_n)_n$ of $(k_n)_n$ such that $\sum_n x_{s_n} = x \in X$. The sequence $(X_n)_n$ covers X. There exists $m \in \mathbb{N}$ such that $x \in X_{s_{m-1}}$. If $u = x - \sum_{n=1}^{m-1} x_{s_n}$, we have $u \in G_{s_m}$ and

$$x_{s_m} - u = -\sum_{n=m+1}^{\infty} x_{s_m}.$$

We conclude that $q_{s_m}(x_{s_m} - u) \leq \epsilon_{s_m}$ providing a contradiction with (14.1). □

Corollary 14.3.3 *Let $(E_n)_n$ be an increasing sequence of closed subsets of a Fréchet–Urysohn sequentially complete lcs E covering E. If $(x_n)_n$ is a sequence in E such that $x_n \to 0$, there is a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ and $m \in \mathbb{N}$ such that $x_{n_k} \in E_{n_m} + \{-\sum_{j\in A} x_{n_j} : A \subset \{1, 2, \dots m\}\}$ for each $k \in \mathbb{N}$.*

Proof By τ we denote the original topology of E. For each $k, n \in \mathbb{N}$, set $x_{n,k} := kx_n$. Since for any fixed $k \in \mathbb{N}$ one has $x_{n,k} \to 0$ for $n \to \infty$, by Lemma 14.2.3 there exists a strictly increasing sequence $(n_k)_k$ such that $kx_{n_k} \to 0$ if $k \to \infty$. Let B be the closed absolutely convex hull of the bounded set $\{kx_{n_k} : k \in \mathbb{N}\}$. Then the linear span E_B of B endowed with the Minkowski functional norm

$$\|x\|_B := \inf\{\epsilon > 0 : \epsilon^{-1}x \in B\}$$

is a Banach space (see [374, 20.11(2)], and $\tau|E_B \le \tau_B$) where τ_B is the topology generated by the norm $\|x\|_B$. Clearly $x_{n_k} \to 0$ in τ_B. Then each subsequence of $(x_{n_k})_k$ contains a subsequence $(z_k)_k$ whose series $\sum_k z_k$ converges in τ_B, hence in τ. Since $E_B = \bigcup_k E_{n_k} \cap E_B$, and the sets $E_{n_k} \cap E_B$ are closed in τ_B, we apply Lemma 14.3.2 to get $m \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ we have

$$x_{n_k} \in E_{n_m} \cap E_B + \{-\sum_{j\in A} x_{n_j} : A \subset \{1, 2, \dots m\}\}.$$

□

We need also the following two additional lemmas found in [110].

Lemma 14.3.4 *Let X be a Fréchet–Urysohn additive topological group. Let $(X_n)_n$ be a decreasing sequence of dense subsets of X. Then there exists a sequence $(x_k)_k$ in X such that $x_k \to 0$ and $x_k \in X_k$ for each $k \in \mathbb{N}$.*

Proof For each $n \in \mathbb{N}$ there exists a sequence $(x_{n,m})_m$ in X_n such that $x_{n,m} \to 0$ if $m \to \infty$. By Lemma 14.2.3 there exist two strictly increasing sequences $(n_k)_k$, $(m_k)_k$, in $\mathbb{N}$ such that $x_{n_k,m_k} \to 0$ if $k \to \infty$. Clearly $x_{n_k,m_k} \in X_{n_k} \subset X_k$ for each $k \in \mathbb{N}$. To complete the proof, it is enough to set $x_k := x_{n_k,m_k}$ for all $k \in \mathbb{N}$. □

Lemma 14.3.5 *Let X be a Fréchet–Urysohn additive topological group. Let $(X_n)_n$ be an increasing sequence of closed subsets of X such that $\operatorname{int} X_n = \emptyset$ for each $n \in \mathbb{N}$. Then there exists a strictly increasing sequence $(t_n)_n$ in $\mathbb{N}$, and a sequence $x_n \to 0$ in X such that $x_n \notin X_{t_n} + \{-\sum_{k\in A} x_k : A \subset \{1, 2, \dots n\}\}$ for all $n \in \mathbb{N}$.*

Proof Take a sequence $(x_{1,j})_j$ in X such that $x_{1,j} \to 0$ if $j \to \infty$. Since the sets

$$W_{1,j} := X_j + \{-\sum_{(1,n)\in A} x_{1,n} : A \subset \{(1, p),\ 1 \le p \le n\}\}$$

are closed with the empty interior, the complements $X \setminus W_{1,j}$ compose a decreasing sequence of open dense subsets in X. By a simple induction, applying Lemma 14.3.4, we construct a matrix $(x_{i,j})_{i,j}$ in X such that

$$x_{i,j} \to 0, \ \ j \to \infty, \ \ i \in \mathbb{N}, \tag{14.2}$$

$$x_{i,j} \notin X_j + \{-\sum_{(n,m)\in A} x_{n,m} : A \subset \{(k,l) : 1 \le k \le i, 1 \le l \le j\}\} \tag{14.3}$$

for $i \ge 2, \ \ j \in \mathbb{N}$. By Lemma 14.2.3 there exist two strictly increasing sequences $(t_n)_n$, $(l_n)_n$ in $\mathbb{N}$ such that $x_{l_n,t_n} \to 0$ if $n \to \infty$. By (14.3) we conclude that

$$x_{l_n,t_n} \notin X_{t_n} + \{-\sum_{k\in A} x_{l_k,t_k} : A \subset \{1, 2, \dots n\}\}$$

for each $n \in \mathbb{N}$. Finally, it is enough to set $x_n := x_{l_n,t_n}$ for each $n \in \mathbb{N}$. □

Now we are ready to prove Theorem 14.3.1.

Proof Suppose E is not a Baire space. By Saxon's Theorem 2.2.1 there exists in E an absorbing closed and balanced set B with the empty interior. For each $n \in \mathbb{N}$, set $E_n := nB$. Then the sequence $(E_n)_n$ of closed sets with the empty interior is increasing and covers the whole space E. Let $(x_n)_n$ and $(t_n)_n$ be sequences in E and $\mathbb{N}$, respectively, constructed in Lemma 14.3.5, i.e.

$$x_n \notin E_{t_n} + \{-\sum_{k\in A} x_k : A \subset \{1, 2, \dots n\}\} \tag{14.4}$$

for each $n \in \mathbb{N}$ and $x_n \to 0$. Set $W_n := E_{t_n}$ for each $n \in \mathbb{N}$. Clearly the sequence $(W_n)_n$ covers E. From Lemma 14.3.3 it follows that there exist a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$, $m \in \mathbb{N}$, such that

$$x_{n_k} \in W_{n_m} + \{-\sum_{j\in A} x_{n_j} : A \subset \{1, 2, \dots m\}\}$$

for each $k \in \mathbb{N}$. This yields $x_{n_m} \in W_{n_m} + \{-\sum_{j\in A} x_{n_j} : A \subset \{1, 2, \dots m\}\}$ which contradicts (14.4). Hence E is a Baire space. □

Theorem 14.3.1 applies to get the following classical

Theorem 14.3.6 *Let $\{E_t : t \in T\}$ be a family of Fréchet spaces. Then the topological product $E := \prod_{t\in T} E_t$ is a Baire space.*

Proof Since the Σ_0-product E_0 of E is a Fréchet–Urysohn sequentially complete subspace of E, by Theorem 14.3.1 the space E_0, endowed with the topology induced from E, is a Baire space. E_0 is dense in E. Therefore E is Baire space. □

We complete this section with the following application of Theorem 14.3.1.

Corollary 14.3.7 *Let X be a Lindelöf P-space. Then $C_p(X)$ is a bornological Baire space.*

Proof Since X is a Lindelöf P-space, each finite product X^n is a Lindelöf space. This implies that X is an ω-space; see [273, Proposition, p.156–157]. Indeed, let $\mathcal{U}$ be an open ω-cover of X. Then $\mathcal{U}^n := \{A^n : A \in \mathcal{U}\}$ is an ω-cover of X^n for each $n \in \mathbb{N}$. Note that, if $\mathcal{U}_n \subset \mathcal{U}$ is countable and $\mathcal{U}_n^n$ covers X^n for each $n \in \mathbb{N}$, the set $\bigcup_n \mathcal{U}_n$ is a countable ω-subcover of $\mathcal{U}$. Now by Proposition 14.1.1(i) the space $C_p(X)$ is Fréchet–Urysohn. Since X is a P-space, $C_p(X)$ is sequentially complete; see [322, Theorem 3.6.7].

We apply Theorem 14.3.1 to deduce that $C_p(X)$ is a Baire space. The proof is completed, since every Fréchet–Urysohn lcs is bornological by Lemma 14.4.3. □

14.4 Three-Space Property for Fréchet–Urysohn Spaces

In [601] Todorčević constructed two Fréchet–Urysohn spaces $C_p(X)$ and $C_p(Y)$ whose product is not Fréchet–Urysohn, as having uncountable tightness. This implies that the Fréchet–Urysohn property is not a three-space property; see also Example 14.4.7.

In [135] it was proved that the product of a first-countable space X by a Fréchet–Urysohn topological group Y is a Fréchet–Urysohn space. In [446] Michael proved that, if E is a first-countable topological space and F is a Fréchet–Urysohn topological space with the property (as) in Definition 14.2.2, the product space $E \times F$ is Fréchet–Urysohn. For tvs we have the following.

Proposition 14.4.1 *If F is a closed metrizable subspace of a tvs E such that the quotient E/F is a Fréchet–Urysohn space, the space E is a Fréchet–Urysohn space.*

Proof Since F is a metrizable subspace of E, there exists a decreasing sequence $(V_n)_n$ of neighbourhoods of zero in E such that $V_{n+1} + V_{n+1} \subset V_n$ for all $n \in \mathbb{N}$ and the sequence $(V_n \cap F)_n$ is a basis of neighbourhood of zero in F. To prove that E is Fréchet–Urysohn, it is enough to show that, if $X \subset E$ is an arbitrary set and $0 \in \overline{X}$, there exists a sequence in X converging to 0. Let $Q : E \to E/F$ be the quotient map. Let $\mathcal{U}(E)$ be the set of all neighbourhoods of zero in E. Note that for each neighbourhood of zero $U \in \mathcal{U}(E)$ and each $n \in \mathbb{N}$, there exists $x_{U,n} \in X \cap U \cap V_n$. Hence $Q(0) \in \overline{A_n}$, where

$$A_n := \{Q(x_{U,n}) : U \in \mathcal{U}(E)\}.$$

Since E/F is a Fréchet–Urysohn tvs, for each $n \in \mathbb{N}$ there exists a sequence $(U_{k(n)})_k$ in $\mathcal{U}(E)$ such that

$$\lim_k Q(x_{U_{k(n)},n}) = Q(0).$$

By the property (as) (Lemma 14.2.3), there exist two increasing sequences $(k_p)_p$ and $(n_p)_p$ in $\mathbb{N}$ such that

$$Q(x_{U_{k_p(n_p),n_p}}) \to Q(0) \text{ if } n_p \to \infty.$$

Set $u_p := x_{U_{k_p(n_p),n_p}}$ for each $p \in \mathbb{N}$. Fix a balanced neighbourhood of zero $W \in \mathcal{U}(E)$. Then there exist $n \in \mathbb{N}$ such that $(V_n + V_n) \cap F \subset W$, and $m > n$ such that

$$[u_p + (W \cap V_n)] \cap F \neq \emptyset$$

for $p > m$. Therefore, there exist elements $y \in W \cap V_n, u \in F$, such that $u_p + y = u$. Since $u_p \in V_{n_p} \subset V_p \subset V_n$ and $y \in V_n$, we deduce that $u = u_p + y \in (V_n + V_n) \cap F \subset W$. Therefore $u_p = u - y \in W + W$. This implies that $(u_p)_p$ is a sequence in X which converges to 0 in E. Hence E is Fréchet–Urysohn. □

This yields the following result of Michael [446].

Corollary 14.4.2 (Michael) *Let E be a Fréchet–Urysohn tvs and F a metrizable tvs. Then the product $E \times F$ is a Fréchet–Urysohn tvs.*

We present another example showing that the Fréchet–Urysohn property is not a three-space property. We need the following two additional lemmas; see [347, Theorem 3.1] and [519, Lemma 1.4]. For Lemma 14.4.6 we omit the proof and refer to [519, Lemma 1.4].

Lemma 14.4.3 *Let E be a Fréchet–Urysohn lcs. Let $(A_n)_n$ be an increasing bornivorous sequence in E. Then, there exists $m \in \mathbb{N}$ such that A_m is a neighbourhood of zero in E. Hence, E is b-Baire-like and bornological.*

Proof Assume that none of the sets A_n is a neighbourhood of zero in E. By $\mathcal{U}(E)$ we denote the set of all absolutely convex neighbourhoods of zero in E. Then for each $U \in \mathcal{U}(E)$, $n \in \mathbb{N}$, there exists $x_{U,n} \in U \setminus nA_n$. For each $n \in \mathbb{N}$, set

$$B_n := \{x_{U,n} : U \in \mathcal{U}(E)\}.$$

Then $0 \in \overline{B_n}$ for each $n \in \mathbb{N}$. Since E is Fréchet–Urysohn, for each $n \in \mathbb{N}$, there exists a sequence $(U_n(k))_k$ in $\mathcal{U}(E)$ such that $x_{U_n(k),n} \to 0$ for each $n \in \mathbb{N}$. By Lemma 14.2.3 every Fréchet–Urysohn lcs satisfies the condition (as) from Definition 14.2.2. Hence there exist strictly increasing sequences $(k_p)_p$ and $(n_p)_p$ such that $x_{U_{n_p(k_p),n_p}} \to 0$ if $n_p \to \infty$. On the other hand, the bounded set

$$\{x_{U_{n_p(k_p),n_p}} : p \in \mathbb{N}\}$$

is not included in any set $n_p A_{n_p}$ for $p \in \mathbb{N}$, a contradiction, since $(A_n)_n$ is an increasing bornivorous sequence. □

Corollary 14.4.4 *Let (E, τ) be a cls. If $(E, \sigma(E, E'))$ is Fréchet–Urysohn, then $\tau = \sigma(E, E')$.*

Remark 14.4.5 Remind that Proposition 6.7.10 provides the following stronger result (due to Ferrando). Let $E = (E, \tau)$ be a lcs such that $(E, \sigma(E, E'))$ is a k-space, then $\tau = \sigma(E, E')$. This extends [375, Corollary 6.5].

Lemma 14.4.6 *Let H be a dense hyperplane in the Banach space ℓ^2. Fix $y \in \ell^2 \setminus H$. Let G be a tvs. Let $Z \subset G$ be a dense hyperplane. Fix $x \in G \setminus Z$. Then there exist a lcs E, a closed subspace $F \subset E$ such that F is isomorphic to the space Z, the quotient space E/F is isomorphic to the quotient space $\ell^2/[y]$, and E has a quotient isomorphic to the space $G/[x]$, where $[x]$ means* $\operatorname{span}\{x\}$.

Now we are ready to show the following.

Example 14.4.7 The Fréchet–Urysohn property is not a three-space property.

Proof Let Z be the $\sum_0$-product of the space $\mathbb{R}^{\mathbb{R}}$ at the point 0. The space Z is a Fréchet–Urysohn lcs. Fix $x \in \mathbb{R}^{\mathbb{R}} \setminus Z$. The space $G := Z + [x]$ is not a Fréchet–Urysohn space. Indeed, by Proposition 14.4.3 it is enough to show that G is not bornological in the induced topology: let $f : G \to \mathbb{R}$ be a linear functional on G defined by $f(y + \lambda x) := \lambda$, $y \in Z$, and $\lambda \in \mathbb{R}$. Observe that $Z = \ker(f) := \{y \in G : f(y) = 0\}$. Hence, as Z is dense and sequentially closed in $\mathbb{R}^{\mathbb{R}}$, $\ker(f)$ is dense and sequentially closed in G. Consequently, f is discontinuous and sequentially continuous (hence bounded). Indeed, let A be a countable subset of $\mathbb{R}$. Set $Z_A := \{(x_i) \in Z : x_i = 0, i \in \mathbb{R} \setminus A\}$. Then the restriction of f to $Z_A + [x]$ is continuous (since its kernel is the closed set Z_A). This shows that G is not bornological. Next, we apply Proposition 14.4.1 to deduce that the quotient space $G/[x]$ is not Fréchet–Urysohn. Let H be a dense hyperplane in the Banach space ℓ^2. Choose $y \in \ell^2 \setminus H$. By Lemma 14.4.6 there exists a lcs E containing a closed subspace F such that F is isomorphic to the Fréchet–Urysohn space Z, the quotient E/F is isomorphic to the Banach space $\ell^2/[y]$, and E has a quotient isomorphic to the quotient $G/[x]$ (which is not Fréchet–Urysohn). This implies that E is not Fréchet–Urysohn, since any Hausdorff quotient of Fréchet–Urysohn tvs is Fréchet–Urysohn. □

14.5 Topological Vector Spaces with Bounded Tightness

There is another interesting tightness-type condition, formally weaker than the Fréchet–Urysohn property. Let E be a tvs. If for every subset A of E, and $x \in \overline{A} \subseteq E$, there is a bounded set $B \subseteq A$ such that $x \in \overline{B}$, the space E is said to have *bounded tightness* [131, 214].

The concept of the bounded tightness was formally defined in [214], and next, in [131] used to the study the weak topology of normed spaces. Since every Fréchet–Urysohn tvs has bounded tightness, it is natural to ask about the converse implication. It turns out that the following general result [340] holds.

Theorem 14.5.1 (Kąkol–López-Pellicer–Todd) *For a tvs E (not necessarily Hausdorff), the following assertions are equivalent:*

(1) E is Fréchet–Urysohn.
(2) For every subset A of E such that $0 \in \overline{A}$, there exists a bounded subset B of A such that $0 \in \overline{B}$.

(3) For any sequence $(A_n)_n$ of subsets of E each with $0 \in \overline{A_n}$, there exists a sequence $B_n \subset A_n$, $n \in \mathbb{N}$, such that $\bigcup_n B_n$ is bounded and $0 \in \overline{\bigcup_{n\leq k} B_k}$ for each $n \in \mathbb{N}$.

Proof (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3): It is obvious that (3) holds if $0 \in A_n$ for infinitely many n or if $\overline{\{0\}} = E$. Therefore, we assume that $0 \in \overline{A_n}\backslash A_n$, for each $n \in \mathbb{N}$, and that there exists a null sequence $(x_n)_n$ in $E\backslash\overline{\{0\}}$. For each $n \in \mathbb{N}$, there exists a closed neighbourhood U_n of zero such that $0 \notin U_n + x_n$. Let each $C_n = U_n \cap A_n$. Clearly 0 is in each $\overline{C_n}\backslash C_n$ but not in the set:

$$A := \bigcup\nolimits_n (C_n + x_n)\,.$$

We claim that $0 \in \overline{A}$. Indeed, for U, an open neighbourhood of zero, there exist $k \in \mathbb{N}$ with $x_k \in U$ and, V, a neighbourhood of zero with $V + x_k \subset U$. Since there is $y \in V \cap C_k$, we have $y + x_k \in U \cap A$. Thus $0 \in \overline{A}\backslash A$. By the assumption there is $B \subset A$ with B bounded and $0 \in \overline{B}$. There exist subsets

$$B_n \subset C_n = U_n \cap A_n$$

such that $B = \bigcup_n (B_n + x_n)$. By the construction, 0 does not belong to the closed sets:

$$\bigcup\nolimits_{k<n} (U_k + x_k)\,.$$

Therefore, 0 does not belong to $\overline{\bigcup_{k<n} (B_k + x_k)}$. Hence $0 \in \overline{\bigcup_{n\leq k} (B_k + x_k)}$, for each $n \in \mathbb{N}$.

Let W and V be balanced neighbourhoods of zero such that $V - V \subset W$. Fix $n \in \mathbb{N}$. There exists $m \geq n$, in $\mathbb{N}$, such that $x_k \in V$ for all $k \geq m$. Since

$$0 \in \overline{\bigcup\nolimits_{m\leq k} (B_k + x_k)},$$

we see that there exist $k \geq m$ and $y \in B_k$ with $y + x_k \in V$. From

$$y \in V - x_k \subset V - V \subset W$$

it follows that for each $n \in \mathbb{N}$, the set W meets $\bigcup_{n\leq k} B_k$. As any neighbourhood of zero contains W and V as above, 0 is in the closure of each $\bigcup_{n\leq k} B_k$. We claim that $\bigcup_n B_n$ is bounded. Indeed, as

$$B = \bigcup\nolimits_n (B_n + x_n)$$

and $D := \{x_n : n \in \mathbb{N}\}$ are bounded, and since

$$\bigcup\nolimits_n B_n \subset \bigcup\nolimits_n (B_n + x_n) - \{x_m : m \in \mathbb{N}\} = B - D,$$

the set $\bigcup_n B_n$ is bounded, too.

(3) $\Rightarrow$ (1): Assume that $0 \in \overline{A}$, and set $A_n := nA$, for each $n \in \mathbb{N}$. Since $0 \in \overline{A_n}$ for each $n \in \mathbb{N}$, there exist $B_n \subset A_n$, as in (3). So each $\bigcup_{n \le k} B_k$ is non-empty, and, consequently, there exists a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ with B_{n_k} non-empty. For each $k \in \mathbb{N}$ select $y_k \in B_{n_k}$. There exists a sequence $(a_k)_k$ in A such that $y_k = n_k a_k$ for each $k \in \mathbb{N}$. As the sequence $(n_k)_k$ is strictly increasing and $(y_k)_k = (n_k a_k)_k$ is bounded, the sequence $(a_k)_k$ in A converges to zero in E. □

Chapter 15
Sequential Properties in the Class $\mathfrak{G}$

Abstract In this chapter, we prove that an lcs in the class $\mathfrak{G}$ is metrizable if and only if E is b-Baire-like if and only if E is Fréchet–Urysohn. Consequently, no proper (LB)-space is Fréchet–Urysohn. We prove that if a (DF)- or (LM)-space E is sequential, then E is either metrizable or Montel (DF). We distinguish a variant of the property C_3 (due to Webb), called property C3– (i.e., sequential closure of any vector subspace is sequentially closed), and characterize both (DF)-spaces and (LF)-spaces with the property C3– as being of the form M, ϕ, or $M \times \phi$, where M is metrizable.

15.1 Fréchet–Urysohn Spaces Are Metrizable in the Class $\mathfrak{G}$

In this section we prove that every Fréchet–Urysohn lcs in the class $\mathfrak{G}$ is metrizable; see [119]. Several applications will be provided.

Let us start with the following lemma.

Lemma 15.1.1 *Let E be a lcs in the class $\mathfrak{G}$. Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of E. For $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, set $C_{n_1 \dots n_k} = \bigcup\{A_\beta : \beta = (m_k) \in \mathbb{N}^{\mathbb{N}}, n_j = m_j,\ j = 1, 2, \dots k\}$, for $k \in \mathbb{N}$. Then, the sequence of polars $C^\circ_{n_1} \subset C^\circ_{n_1,n_2} \subset \cdots \subset C^\circ_{n_1,n_2,\dots,n_k} \subset \cdots$ is bornivorous in E.*

Proof Assume, by the contradiction, that there exists a bounded set B in E such that $B \not\subset kC^\circ_{n_1 \dots n_k}$ for every $k \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$, there exists $x_k \in B$ such that $k^{-1}x_k \notin C^\circ_{n_1 \dots n_k}$. Therefore, for every $k \in \mathbb{N}$, there exists $f_k \in C_{n_1 \dots n_k}$ such that

$$|f_k(k^{-1}x_k)| > 1.$$

For every $k \in \mathbb{N}$, there exists $\beta_k = (m^k_n)_n \in \mathbb{N}^{\mathbb{N}}$ such that $f_k \in A_{\beta_k}$, where $n_j = m^k_j$ for $j = 1, 2, \dots k$. Next, define $a_n = \max\{m^k_n : k \in \mathbb{N}\}$, $n \in \mathbb{N}$, and $\gamma = (a_n) \in \mathbb{N}^{\mathbb{N}}$. Since $\gamma \geq \beta_k$ for $k \in \mathbb{N}$, we have $A_{\beta_k} \subset A_\gamma$, so $f_k \in A_\gamma$ for all $k \in \mathbb{N}$. Finally, since by the definition any sequence in A_γ is equicontinuous,

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we note that the sequence $(f_k)_k$ is equicontinuous. Hence, the sequence $(f_k)_k$ is uniformly bounded on bounded sets in E, including B, a contradiction. □

We need also the following.

Lemma 15.1.2 *For a quasibarrelled space E, the following statements are equivalent:*

(i) *E is in $\mathfrak{G}$;*
(ii) *There is a family of absolutely convex closed subsets $\mathcal{F} := \{D_{n_1,n_2,\ldots,n_k} : k, n_1, n_2, \ldots, n_k \in \mathbb{N}\}$ satisfying:*
  (a) *$D_{m_1,m_2,\ldots,m_k} \subset D_{n_1,n_2,\ldots,n_k}$, whenever $n_i \leq m_i$, $i = 1, 2, \ldots, k$;*
  (b) *For every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, we have $D_{n_1} \subset D_{n_1,n_2} \subset \cdots \subset D_{n_1,n_2,\ldots,n_k} \subset \cdots$ and the sequence is bornivorous.*
  (c) *If $U_\alpha := \bigcup_k D_{n_1,n_2,\ldots,n_k}$, $\alpha = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$, then $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a basis of neighbourhoods of the origin in E.*
(iii) *E has a $\mathfrak{G}$-base, i.e. a basis of neighbourhoods of zero $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ satisfying the decreasing condition $U_\beta \subset U_\alpha$ whenever $\alpha \leq \beta$ in $\mathbb{N}^{\mathbb{N}}$.*
(iv) *The strong dual $(E', \beta(E', E))$ is a quasi-(LB)-space.*

Proof (i) $\Rightarrow$ (ii): Fix a $\mathfrak{G}$-representation $\{L_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of E. Since E is quasibarrelled, each set L_α is equicontinuous. Hence, each set $M_\alpha := L_\alpha^{\circ\circ}$ is weakly compact and $\beta(E', E)$-bounded. This shows that each M_α is a $\beta(E', E)$-Banach disc. Therefore, $\{M_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution on E' consisting of Banach discs in $(E', \beta(E', E))$. Hence, the strong dual $(E', \beta(E', E))$ is a quasi-(LB)-space.

By Theorem 3.3.12 there is a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in $(E', \beta(E', E))$ consisting of Banach disc such that for every $\beta(E', E)$-Banach disc $B \subset E'$, there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ with $B \subset A_\alpha$. The quasibarrelledness of E ensures that each set A_α is equicontinuous, and then the $\mathfrak{G}$-representation $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental family of equicontinuous subsets of E'. Hence, the family of polars $\{A_\alpha^\circ : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a basis of neighbourhoods of zero in E.

For $k, n_1, n_2, \ldots, n_k \in \mathbb{N}$ as usual, define: $C_{n_1,n_2,\ldots,n_k}$ and set

$$D_{n_1,n_2,\ldots,n_k} := C^\circ_{n_1,n_2,\ldots,n_k}$$

.

For each $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, we have:

$$V_\alpha := \overline{\bigcup_{k=1}^{\infty} D_{n_1,n_2,\ldots,n_k}}^{\sigma(E,E')} \subset (\bigcap_{k=1}^{\infty} C_{n_1,n_2,\ldots,n_k})^\circ \subset A_\alpha^\circ.$$

Each set V_α is a closed, absolutely convex and bornivorous set by Lemma 15.1.1. Then, V_α is a neighbourhood of zero. By Proposition 2.4.8 we deduce that for every $\varepsilon > 0$, we have:

$$V_\alpha = \overline{\bigcup_{k=1}^{\infty} D_{n_1,n_2,\ldots,n_k}}^{\sigma(E,E')} \subset (1+\varepsilon)\bigcup_{k=1}^{\infty} D_{n_1,n_2,\ldots,n_k} = (1+\epsilon)U_\alpha.$$

Thus $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base of neighbourhoods of zero in E.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): The family of polars $\{U_\alpha^\circ : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation of the space E.

(ii) $\Rightarrow$ (iv): If $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is an $\mathfrak{G}$-base in E, the sets U_α° provide a quasi-(LB)-space representation for $(E', \beta(E', E))$.

(iv) $\Rightarrow$ (ii): Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a quasi-(LB) representation for $(E', \beta(E', E))$. Since E is quasibarrelled, each set A_α is equicontinuous. Hence, E is in the class $\mathfrak{G}$ and (i) $\Rightarrow$ (ii) applies. □

Now we are ready to prove Theorem 15.1.3. This result, due to Cascales, Kąkol, and Saxon [119], generalizes parts of [328, Theorem 5.1], [346, Theorem 2.1], and [463, Theorem 3].

Theorem 15.1.3 (Cascales–Kąkol–Saxon) *For a lcs E in the class $\mathfrak{G}$, the following statements are equivalent:*

(i) E *is metrizable.*
(ii) E *is Fréchet-Uryshon.*
(iii) E *is b-Baire-like.*

Proof (i) $\Rightarrow$ (ii): is obvious.

(ii) $\Rightarrow$ (iii): See Lemma 14.4.3.

(iii) $\Rightarrow$ (i): If E is b-Baire-like, E is a quasibarrelled space, and therefore, we can obtain a countable family:

$$\mathcal{F} := \{D_{n_1,n_2,\ldots,n_k} : k, n_1, n_2, \ldots, n_k \in \mathbb{N}\}$$

as in the proof of Lemma 15.1.2.

Since the sequence $D_{n_1} \subset D_{n_1,n_2} \subset \cdots \subset D_{n_1,n_2,\ldots,n_k} \subset \cdots$ is bornivorous, for every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, we have $E = \bigcup_{k=1}^{\infty} kD_{n_1,n_2,\ldots,n_k}$ and, again, since E is b-Baire-like, some $D_{n_1,n_2,\ldots,n_m}$ is a neighbourhood of zero for certain $m \in \mathbb{N}$. Thus, by Lemma 15.1.2, the family

$$\mathcal{U} := \{D_{n_1,n_2,\ldots,n_k} \in \mathcal{F} : D_{n_1,n_2,\ldots,n_k} \text{ is neighbourhood of } 0\}$$

is a countable basis of neighbourhoods of zero for E. □

Corollary 15.1.4 *Let $\{E_t : t \in T\}$ be a family of non-zero lcs. If T is uncountable, the product $\prod_t E_t$ does not belong to the class $\mathfrak{G}$.*

Proof Assume that $\prod_{t\in T} E_t$ is in the class $\mathfrak{G}$ and T is an uncountable set. Then $\prod_{t\in T} E_t$ contains a subspace of the form $\mathbb{R}^A$ for some uncountable set A. Clearly

$\mathbb{R}^A$ is a Baire space in the class $\mathfrak{G}$. By Theorem 15.1.3 we deduce that $\mathbb{R}^A$ is metrizable, a contradiction. □

By Proposition 12.2.2 we know that $C_p(X)$ belongs to the class $\mathfrak{G}$ if and only if X is countable.

Corollary 15.1.5 *The following assertions are equivalent for a completely regular Hausdorff space X:*

(i) X is countable.
(ii) $C_p(X)$ is an (LM)-space.
(iii) $C_p(X)$ admits a $\mathfrak{G}$-base.

Proof (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (i): Since an (LM)-space belongs to the class $\mathfrak{G}$, we apply Proposition 12.2.2.

(i) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i): The space $C_p(X)$ is always quasibarrelled by [322, Corollary 11.7.3, Theorem 2]. We apply Lemma 15.1.2, and then again Proposition 12.2.2. □

Recall again that a bounded resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in a tvs E is *bornivorous* if every bounded set in E is contained in some K_α. Clearly, every (DF)-space admits a bornivorous bounded resolution. Also, every regular (LM)-space admits a bornivorous bounded resolution. Indeed, let E be an (LM)-space. Let $(E_n)_n$ be a defining sequence of E of metrizable lcs. For each $n \in \mathbb{N}$, let $(U_k^n)_k$ be a countable basis of absolutely convex neighbourhoods of zero in E_n such that

$$U_k^n \subset U_k^{n+1},\ 2U_{k+1}^n \subset U_k^n$$

for each $k \in \mathbb{N}$. For each $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, set $K_\alpha := \bigcap_k n_k U_k^{n_1}$. Then $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bounded resolution. Indeed, if $x \in E$, there exists $r_1 \in \mathbb{N}$ such that $x \in E_{r_1}$. Hence for each $k \in \mathbb{N}$, there exists $m_k \in \mathbb{N}$ such that $x \in m_k U_k^{r_1}$. Set $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ with $n_1 = max\{r_1, m_1\}$ and $n_k = m_k$ for all $k \geq 2$. Clearly, $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution on E, and $\bigcap_k n_k U_k^{n_1}$ is bounded in E_{n_1}, hence also in E. If, additionally, E is *regular*, i.e. for every bounded set B in E, there exists $m_1 \in \mathbb{N}$ such that B is contained and bounded in E_{m_1}, then for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $B \subset \bigcap_k n_k U_k^{m_1}$. This yields a sequence $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ such that $B \subset K_\alpha$. The following corollary follows from Theorem 15.1.3.

Corollary 15.1.6 *If a lcs E has a bornivorous bounded resolution, the strong dual $(E', \beta(E', E))$ has a $\mathfrak{G}$-base. For a lcs E, the strong dual $(E', \beta(E', E))$ is metrizable if and only if $(E', \beta(E', E))$ is Fréchet–Urysohn and E admits a bornivorous bounded resolution.*

Proof If $(E', \beta(E', E))$ is metrizable, it is Fréchet–Urysohn and E admits a bornivorous bounded resolution. Now assume E admits a bornivorous bounded resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. The polars K_α° of the sets K_α in E' form a basis of neighbourhoods of zero for $\beta(E', E)$. Since the sets $K_\alpha^{\circ\circ}$ in E'' compose a resolution

consisting of equicontinuous sets covering E'', the space $(E', \beta(E', E))$ belongs to the class $\mathfrak{G}$. If $(E', \beta(E', E))$ is Fréchet–Urysohn, Theorem 15.1.3 yields the metrizability of $(E', \beta(E', E))$. □

Corollary 15.1.7 *The spaces $D'(\Omega)$ of the distributions and $A(\Omega)$ of the real analytic functions for an open set $\Omega \subset \mathbb{R}^{\mathbb{N}}$, respectively, are not Fréchet–Urysohn; they have countable tightness, both for the original and the weak topologies.*

Proof Since $D'(\Omega)$ is non-metrizable and quasibarrelled, and is the strong dual of a complete (hence regular) (LF)-space $D(\Omega)$ of the test functions (see [439]), we apply Corollary 15.1.6 and Theorem 12.2.1. The same argument can be used to the space $A(\Omega)$ via [173, Theorem 1.7 and Proposition 1.7]. □

The following corollary, due to Cascales, Kąkol, and Saxon [119], extends [463, Theorem 3] since every (LF)-space is barrelled and belongs to the class $\mathfrak{G}$.

Corollary 15.1.8 *For a barrelled lcs in the class $\mathfrak{G}$, the following conditions are equivalent:*

(i) *E is metrizable.*
(ii) *E is Fréchet–Urysohn.*
(iii) *E is Baire-like.*
(iv) *E does not contain φ, i.e. an $\aleph_0$-dimensional vector space with the finest locally convex topology.*

Proof Since every barrelled b-Baire-like space is Baire-like (see Proposition 2.4.2), the conditions (i), (ii), and (iii) are equivalent by Theorem 15.1.3.

(iii) $\Rightarrow$ (iv): By Theorem 15.1.3 the space E is metrizable. Since φ is a non-metrizable (LF)-space, (iv) holds.

(iv) $\Rightarrow$ (i): Every barrelled space not containing φ is Baire-like by Corollary 2.4.4. □

Since the completion of a lcs $E \in \mathfrak{G}$ belongs to the class $\mathfrak{G}$, and the completion of a quasibarrelled space is barrelled, Corollary 15.1.8 yields the following.

Corollary 15.1.9 *A quasibarrelled lcs in the class $\mathfrak{G}$ is metrizable if and only if its completion does not contain φ.*

We know already that every Fréchet–Urysohn lcs in the class $\mathfrak{G}$ is metrizable, and only metrizable spaces $C_p(X)$ belong to the class $\mathfrak{G}$. On the other hand, under MA+$\neg$CH there exist non-metrizable Fréchet–Urysohn spaces $C_p(X)$. Indeed, by Proposition 14.1.1 and [435, Theorem 1], the space $C_p(X)$ is Fréchet–Urysohn if and only if X has the *γ-property*, i.e. if for any open cover $\mathcal{R}$ of X such that any finite subset of X is contained in a member of $\mathcal{R}$, there exists a countable infinite subfamily $\mathcal{R}'$ of $\mathcal{R}$ such that any element of X lies in all but finitely many members of $\mathcal{R}'$. Gerlits and Nagy [273] showed (under MA) that every subset of $\mathbb{R}$ of the cardinality smaller than $2^{\aleph_0}$ has the γ-property. Hence, under MA+$\neg$CH, there are uncountable γ-subsets Y of $\mathbb{R}$. Thus, for such Y the space $C_p(Y)$ is non-metrizable and Fréchet–Urysohn.

A tvs E is said to have a *superresolution* if E admits a bounded resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that for every finite tuple $(n_1, \dots n_p)$ of positive integers and every bounded set Q in E, there exists $\alpha = (m_k) \in \mathbb{N}^{\mathbb{N}}$ such that $m_j = n_j$, for $1 \leq j \leq p$, and K_α absorbs Q. This implies that for any finite tuple $(n_1, \dots n_p)$, the sequence $(C_{n_1,\dots,n_p,n})_n$ is bornivorous in E. Clearly, every metrizable tvs admits a superresolution. We note the following.

Proposition 15.1.10 *A b-Baire-like space E is metrizable if and only if E admits a superresolution.*

Proof Let $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a superresolution for E consisting of absolutely convex bounded sets. Then, the sets $C_{n_1 n_2,\dots,n_k}$ are also absolutely convex. Note that for every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$and every neighbourhood of zero U in E, there exists $k \in \mathbb{N}$ such that $C_{n_1,n_2,\dots,n_k} \subset 2^k U$, apply the proof of Proposition 7.1.3. If F is the completion of E, the space F is Baire-like. We prove that there exists $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ such that $\overline{C_{n_1,n_2,\dots,n_k}}$ is a neighbourhood of zero in F for each $k \in \mathbb{N}$.

Since, by the assumption, the sequence $(nC_n)_n$ is bornivorous in E, and E is quasibarrelled, we apply Proposition 2.4.8 to deduce that

$$F = \overline{E} = \overline{\bigcup_n nC_n} \subset (1+\epsilon)\bigcup_n n\overline{C_n}$$

for each $\epsilon > 0$. Since F is a Baire-like space, there exists $n_1 \in \mathbb{N}$ such that $\overline{C_{n_1}}$ is a neighbourhood of zero in F.

Now assume that for a finite tuple $(n_1, \dots n_p)$ of positive integers, the set $\overline{C_{n_1,\dots,n_k}}$ is a neighbourhood of zero for each $1 \leq k \leq p$. By the assumption, the sequence $(nC_{n_1,\dots,n_p,n})_n$ is bornivorous in E, and consequently, $E = \bigcup_n nC_{n_1,\dots,n_p,n}$. We apply the same argument as above to get $n_{p+1} \in \mathbb{N}$ such that $\overline{C_{n_1,\dots,n_p,n_{p+1}}}$ is a neighbourhood of zero. This completes the inductive step. We provided a countable basis $(2^{-k}\overline{C_{n_1,\dots,n_k}})_k$ of neighbourhoods of zero, so E is metrizable. □

Corollary 15.1.11 *$C_p(X)$ is a metrizable space if and only if $C_p(X)$ admits a superresolution.*

Proof First observe that for any X, the space $C_p(X)$ is b-Baire-like. Indeed, let $(A_n)_n$ be a bornivorous sequence of absolutely convex closed subsets of $C_p(X)$ covering $C_p(X)$. $C_p(X)$ is quasibarrelled [322, Corollary 11.7.3]. We apply Proposition 2.4.8 to deduce $\mathbb{R}^X = \overline{C_p(X)} = \bigcup_n \overline{A_n}$, where the closure is taken in $\mathbb{R}^X$. By the Baire category theorem, some $\overline{A_m}$ is a neighbourhood of zero in $\mathbb{R}^X$. Hence, A_m is a neighbourhood of zero in $C_p(X)$, so $C_p(X)$ is b-Baire-like. Now Proposition 15.1.10 completes the proof. □

15.2 Sequential (LM)-Spaces and the Dual Metric Spaces

In the previous section, we showed that any Fréchet–Urysohn lcs in the class $\mathfrak{G}$ is metrizable. One can ask if the same conclusion holds for any sequential lcs E; for the definition see the text before Corollary 9.8.7. Clearly every Fréchet–Urysohn space is sequential, and the converse implication fails in general. In [472] Nyikos showed that the $\aleph_0$-dimensional space φ is sequential and φ is not Fréchet–Urysohn. Since φ is the strong dual of the Fréchet space $\omega := \mathbb{K}^{\mathbb{N}}$ of all scalar sequences, and φ is an (LB)-space, the space φ is a Montel (DF)-space.

It should be pointed out that Webb [638] used the designations C_1 and C_2 for the Fréchet–Urysohn and sequential spaces, respectively. According to Webb [638], a topological space E is said to have the *property* C_3 if the sequential closure of any subset of E is sequentially closed. Clearly

$$\text{metrizable} \ \Rightarrow C_1 \Leftrightarrow [C_2 \wedge C_3].$$

The property C_3 has been used in [328] to show that an (LM)-space E is metrizable if and only if E has the property C_3. Nevertheless, in [346] it was proved that there exist non-metrizable (DF)-spaces with the property C_3. This and the next section are based on results from [346].

Proposition 15.2.1 *Any sequential topological space E has countable tightness.*

Proof Assume E is a sequential space. Fix an arbitrary set $M \subset E$. For any $x \in \overline{M}$, we need to obtain a countable subset, say $M_x \subset M$ such that $x \in \overline{M_x}$. Set:

$$N := \bigcup \{\overline{A} : A \subset M, \ \text{ and } \ |A| \leq \aleph_0\}.$$

Clearly $N \subset \overline{M}$. Observe that N is a sequentially closed set. Take $\{x_n, n \in \mathbb{N}\} \subset N$ such that $x_n \to y \in E$. For every $n \in \mathbb{N}$, there exists a countable set $A_n \subset M$ such that $x_n \in \overline{A_n}$. Therefore, $y \in \overline{\bigcup_n A_n}$, and by the definition of N, we conclude that $y \in N$. Since E is sequential, the set N is closed, and so $x \in N = \overline{M}$. □

The converse implication fails in general. Corollary 9.8.7 provides a large class of topological groups X such that $(X, \sigma(X, X^\wedge))$ has countable tightness and X is not sequential. Yosinaga [644] proved that every Silva space (equivalently, the strong dual of a Fréchet–Schwartz space) is sequential. Webb [638] extended this result to all Montel (DF)-spaces (equivalently, strong duals of Fréchet–Montel spaces), and proved also that only finite-dimensional Fréchet–Montel spaces are Fréchet–Urysohn.

We know already that proper (LB)-spaces are not Fréchet–Urysohn spaces; see Corollary 15.1.8. In particular, the non-metrizable space φ (which is a proper (LB)-space) provides an example of a space in the class $\mathfrak{G}$ that is not Fréchet–Urysohn. Nyikos asked if the direct sum of countably many (complete) metrizable groups

endowed with the box product topology is sequential. Theorems 15.2.5 and 15.2.6 provide examples to the Nyikos question.

Recall again that a lcs E is ℓ^∞-quasibarrelled if every $\beta(E', E)$-bounded sequence in E' is equicontinuous, and E is a dual metric space if E is ℓ^∞-quasibarrelled and admits a fundamental sequence of bounded sets.

We will need a couple of lemmas.

Lemma 15.2.2 *Let E be either a dual metric or an (LM)-space. The following assertions are equivalent for E:*

(i) Every bounded set in E is relatively sequentially compact.
(ii) Every bounded set in E is relatively countably compact.
(iii) Every bounded set in E is relatively compact, i.e. E is semi-Montel.
(iv) E is a Montel space, i.e. E is barrelled and semi-Montel.

Proof By Theorems 10.1.3 and 10.2.2, the space E is angelic. Now the conditions (i), (ii), and (iii) are equivalent by the well-known fact stating that in angelic spaces, the (relatively) sequential compact = (relatively) countable compact = (relatively) compact; see [240].

Clearly (iv) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv): If (iii) holds, E is quasi-complete (means that every bounded closed set in E is complete); hence, E is sequentially complete. If E is an (LM)-space, then (as quasibarrelled) E is barrelled; see also [374, Proposition 27.1 (1)]. Hence (iv) holds for (LM)-spaces. If E is a dual-metric space, every bounded subset of E is precompact (by the assumption), and we apply Theorem 10.2.2 to deduce that every bounded set in E is metrizable. Hence, E is separable (as the countable union of metrizable bounded subsets). As every separable ℓ^∞-quasibarrelled space is quasibarrelled [491, Theorem 8.2.20], the sequentially complete space E is barrelled. Hence, E is a Montel space. □

We know that every sequential topological space has countable tightness by Proposition 15.2.1. Note also that every sequential ℓ^∞-quasibarrelled space E is a Mackey space, i.e. the original topology of E equals the Mackey topology of E.

Proposition 15.2.3 *If E is an ℓ^∞-quasibarrelled space with countable tightness, E is a Mackey space. Hence, every sequential ℓ^∞-quasibarrelled space is a Mackey space.*

Proof Let ξ be the original topology of E. Assume that ξ is not the Mackey topology $\mu := \mu(E, E')$. Then there exists a μ-closed set B which is not ξ-closed. Fix $x \in \overline{B} \setminus B$, where the closure is taken in ξ. Choose a μ-continuous seminorm such that $p(x - y) \geq 1$ for all $y \in B$. Fix an arbitrary countable set $\{x_n : n \in \mathbb{N}\}$ in B. Using the Hahn–Banach theorem, we select a sequence $(f_n)_n$ in E' such that

$$f_n(x - x_n) = 1, \quad |f_n(z)| \leq p(z)$$

for all $n \in \mathbb{N}$ and all $z \in E$. This implies that the sequence $(f_n)_n$ is μ-equicontinuous. Hence, there exists an absolutely convex $\sigma(E', E)$-compact subset

containing the elements of the sequence $(f_n)_n$. Thus $(f_n)_n$ is $\beta(E', E)$-bounded. Since (E, ξ) is ℓ^∞-quasibarrelled, the sequence $(f_n)_n$ is ξ-equicontinuous. Hence, its polar set U is a ξ-neighbourhood of zero. Consequently, the set

$$V := x + 2^{-1}U$$

is a ξ-neighbourhood of x but $x_n \notin V$ for all $n \in \mathbb{N}$. This shows that (E, ξ) does not have countable tightness, a contradiction. □

Next Lemma 15.2.4 provides much stronger variant of Proposition 15.2.3 and extends Webb's [638, Theorem 5.5(i)]; see [346].

Lemma 15.2.4 *If E is a sequential ℓ^∞-quasibarrelled lcs, the space E is either b-Baire-like or barrelled whose every bounded set is relatively sequentially compact.*

Proof First note that E is quasibarrelled. Indeed, let ξ be the original topology of E. Let ξ_0 be the locally convex topology on E of the uniform convergence on $\beta(E', E)$-bounded sets. Clearly $\xi \leq \xi_0$ and ξ_0 is quasibarrelled. We show that $\xi = \xi_0$ by proving that every ξ_0-closed set in ξ-closed. Note that every ξ-null sequence is a ξ_0-null sequence. Assume that there exists a sequence $(x_n)_n$ in E such that $x_n \to 0$ in ξ but $x_n \nrightarrow 0$ in ξ_0. This means that there exist a $\beta(E', E)$-bounded set $B \subset E'$, $\epsilon > 0$, and a subsequence $(x_{n_k})_k$ of $(x_n)_n$, as well as a sequence $(f_k)_k$ in B such that $|f_k(x_{n_k})| > \epsilon$ for all $k \in \mathbb{N}$. We know that

$$\epsilon^{-1}\{f_k : k \in \mathbb{N}\} \subset \epsilon^{-1}B$$

is a $\beta(E', E)$-bounded set, so the polar $\epsilon\{f_k : k \in \mathbb{N}\}^\circ$ is a neighbourhood of zero in ξ (since ξ is ℓ^∞-quasibarrelled) such that

$$x_{n_k} \notin \epsilon\{f_k : k \in \mathbb{N}\}^\circ$$

for each $k \in \mathbb{N}$. Hence, we obtain the contradiction $x_n \nrightarrow 0$ in ξ. Since E is sequential and both topologies ξ and ξ_0 have the same convergent sequences, we have that each ξ-closed subset of E is also ξ_0-closed. Hence $\xi = \xi_0$ as claimed.

Now assume that E is not b-Baire-like. We need to show that E is barrelled and every bounded set in E is relatively sequentially compact. Assume by way of contradiction that E contains a bounded sequence $(z_n)_n$ which does not have a convergent subsequence. Since E is quasibarrelled and not b-Baire-like, there exists an increasing bornivorous sequence $(B_n)_n$ of closed absolutely convex sets covering E such that for each set B_n, there exists a bounded set not absorbed by B_n. Taking a subsequence, if necessary, we may assume that $n^{-1}z_k \in B_n$ for each $k \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists a bounded sequence $(v_{k,n})_k$ that misses n^2B_n. If $u_{k,n} := n^{-1}v_{k,n}$, then for the null sequence $(u_{k,n})_k$, we have:

$$u_{k,n} \notin nB_n, \ u_{k,n} \neq -n^{-1}z_k$$

for each $k \in \mathbb{N}$. Hence

$$0 \notin A_n := \{n^{-1}z_k + u_{k,n} : k \in \mathbb{N}\}.$$

We reach a contradiction by showing that the set $A := \bigcup_n A_n$ is sequentially closed but not closed.

Claim 1. The set A is not closed in E.
Indeed, take a neighbourhood of zero U in E. Then there exists $n \in \mathbb{N}$ such that $n^{-1}z_k \subset U$ for all $k \in \mathbb{N}$. There exists $k_0 \in \mathbb{N}$ such that $u_{k,n} \in U$ for all $k \geq k_0$. Finally,

$$n^{-1}z_k + u_{k,n} \in A \cap (U + U) \neq \emptyset$$

for all $k \geq k_0$. This shows that $0 \in \overline{A} \setminus A$. The claim is proved.

Claim 2. The set A is sequentially closed.
In fact, let $(x_j)_j$ be a sequence in A converging to $x \in E$. Note that only finitely many distinct elements of x_j are included in any A_n. Indeed, otherwise, we can find a subsequence of the sequence $(n^{-1}z_k + u_{k,n})_k$ which would converge to element x, and then the corresponding subsequence of $(n^{-1}z_k)_k$ would converge to $x - lim_k\, u_{k,n} = x$. This provides a contradiction, since then a subsequence of $(z_k)_k$ would converge. Something more is even true: Only finitely many A_n contain one element of the sequence $(x_j)_j$. Indeed, assume that the sequence $(x_j)_j$ has elements of the form $n^{-1}z_n + u_{k,n}$ for arbitrary large $n \in \mathbb{N}$. Then, since the sequence $(u_{k,n})_k$ is unbounded (because $u_{n,k} \notin nB_n$ for $k, n \in \mathbb{N}$ and $(B_n)_n$ is bornivorous) and $(z_k)_k$ is bounded, so the corresponding sequence $(x_j)_j$ of sums is an unbounded converging sequence, a contradiction. This implies that only finitely many of the sets A_n may contain elements of the sequence $(x_j)_j$ and, as we proved above, each such set A_n contains only finitely many distinct elements of $(x_j)_j$. This yields the conclusion that the sequence $(x_j)_j$ has only finitely many distinct elements. Therefore, there exists $m \in \mathbb{N}$ such that $x = \lim_k x_j = x_m \in A$. We proved the claim.

Since, as we proved that, if E is not b-Baire-like, the space E is sequentially complete, we deduce that the quasibarrelled space E is sequentially complete. Hence E is barrelled. □

Now we are ready to prove the following.

Theorem 15.2.5 (Kąkol–Saxon) *The following conditions are equivalent for a dual metric space E:*

(i) E is sequential.
(ii) E is normable, or E is a Montel (DF)-space.

Proof By $(S_n)_n$ we denote a fundamental sequence of bounded sets in E consisting of closed absolutely convex subsets of E:

(i) $\Rightarrow$ (ii): If E is non-normable, E does not admit a bounded neighbourhood of zero. Then E cannot be b-Baire-like. By Lemma 15.2.4 it follows that E is a

barrelled space whose each bounded set is relatively sequentially compact. Now Lemma 15.2.2 applies to deduce that E is a Montel (DF)-space.

(ii) $\Rightarrow$ (i): Assume E is a Montel space. It suffices to show that if $A \subset E$ is a sequentially closed set such that $0 \notin A$, then $0 \notin \overline{A}$. Let $(S_n)_n$ be a fundamental sequence of bounded absolutely convex subsets of E. There exists $\epsilon_1 > 0$ such that

$$A \cap \epsilon_1 S_1 = \emptyset,$$

since A does not contain sequences converging to zero. Since $\epsilon_1 S_1$ is sequentially compact by Lemma 15.2.2, the set $A - \epsilon_1 S_1$ is sequentially closed and not containing 0. Hence there exists $\epsilon_2 > 0$ such that $(A - \epsilon_1 S_1) \cap \epsilon_2 S_2 = \emptyset$, and the compact set $\epsilon_1 S_1 + \epsilon_2 S_2$ is sequentially compact. Continuing this procedure we obtain a sequence $(\epsilon_n)_n$ of positive numbers such that $K_n \cap A = \emptyset$ for each $n \in \mathbb{N}$, where $K_n := \sum_{k \leq n} \epsilon_k S_k$. On the other hand, the set $K := \bigcup_n K_n$ is absolutely convex and absorbs bounded sets of E. As any Montel (DF)-space is an (LB)-space, the space E is bornological. Hence K is a neighbourhood of zero. As $K \cap A = \emptyset$, we have $0 \notin \overline{A}$. □

For (LM)-spaces we note the following.

Theorem 15.2.6 (Kąkol–Saxon) *The following assertions are equivalent for an (LM)-space E:*

(i) E is sequential.
(ii) E is metrizable or is a Montel (DF)-space.

Proof (ii) $\Rightarrow$ (i): Apply (ii) $\Rightarrow$ (i) in the proof of Theorem 15.2.5.

(i) $\Rightarrow$ (ii): Since E is an (LM)-space, E is bornological, and consequently E is ℓ^∞-quasibarrelled. Assume E is non-metrizable. We prove that E is a Montel (DF)-space. Exactly as it was proved in Saxon–Narayanaswami's paper [534] for (LF)-spaces, one obtains that E is not b-Baire–like. Then Lemmas 15.2.4 and 15.2.2 imply that E is sequentially complete and Montel, so E is an (LF)-space. Now it is enough to show that E admits a fundamental sequence of bounded sets, since then E, as barrelled will be a (DF)-space. By Corollary 2.4.7 we deduce that E contains an isomorphic copy of φ. Let ξ be the original topology of E. Let $(E_n, \xi_n)_n$ be a defining sequence for E of Fréchet spaces. We claim that for each $n \in \mathbb{N}$, there exists an absolutely convex neighbourhood of zero W_n in (E_n, ξ_n) which is bounded in E. Assume the claim fails. Then, there exists (E_m, ξ_m) such that no ξ_m-neighbourhood of zero is a bounded set in E. Let $(U_n)_n$ be a decreasing basis of absolutely convex neighbourhoods of zero in the space (E_m, ξ_m). As each set U_n is unbounded, for each $n \in \mathbb{N}$, there exists $f_n \in E'$ such that $f_n(U_n) = \mathbb{K}$, where $\mathbb{K}$ denotes the scalar field of E. By $(e_n)_n$ we denote a Hamel basis for the space $\varphi \subset E$. For

$$y = a_1 e_1 + \cdots + a_k e_k \in \varphi$$

with $a_k \neq 0$, set:

$$l(y) = k, \quad v(y) = a_k, \quad \|y\| := \sup\{|a_1|, |a_2|, \dots, |a_k|\}.$$

Set:

$$A := \{x + y : 0 \neq y \in \varphi, \|y\| \leq 1, x \in \overline{U}_{l(y)}, f_{l(y)}(x + y) = [v(y)]^{-1}\},$$

where the closure is taken in E. We show that A is sequentially closed and not closed in E. This contradiction will prove the claim.

A is not closed in E. Indeed, clearly $0 \notin A$, since each member of A has a non-zero image under a suitable linear functional. We show that $0 \in \overline{A} \setminus A$. Let U and V be neighbourhoods of zero in E such that $V + V \subset U$. There exists $k \in \mathbb{N}$ such that $U_k \subset V$. It is easy to select a non-zero element $y \in V \cap \varphi$ such that $l(y) = k$ and $\|y\| \leq 1$. Now choose a point $x \in U_k$ such that $f_k(x) = [v(y)]^{-1} - f_k(y)$. Then

$$x + y \in A \cap (V + V) \subset A \cap U.$$

This means that $0 \in \overline{A} \setminus A$.

A is sequentially closed in E. Indeed, let $(z_n)_n$ be a sequence in A that converges to $z \in E$. Then, applying the definition of A, we note that there exists non-zero $y_n \in \varphi$ such that

$$\|y_n\| \leq 1, \quad x_n := z_n - y_n \in \overline{U}_{l(y_n)}, \quad f_{l(y_n)}(z_n) = [v(y_n)]^{-1},$$

for all $n \in \mathbb{N}$. Note that $m := \sup\{l(y_n) : n \in \mathbb{N}\}$ is finite. Indeed, if this is not true, there exists a subsequence $l(y_{n_k})_k$ of $l(y_n)_n$ tending to infinity. Since for every closed neighbourhood of zero U in E there exists $m \in \mathbb{N}$ such that $\overline{U}_{l(y_k)} \subset U$ for all $k \geq m$, we have $x_{n_k} \to 0$. Hence $y_{n_k} \to z$, so the sequence $(y_{n_k})_k$ defines an infinite-dimensional bounded subset in the space φ. This is impossible as all bounded sets in φ are finite-dimensional. Set:

$$L := \operatorname{span}\{e_1, e_2, \dots, e_m\},$$

and let γ be a topology generated by the norm $\|y\|$. Then $\xi|L = \gamma|L$. This shows that the sequence $(y_n)_n$ is ξ-bounded in L. The classical Bolzano–Weierstrass theorem [322] applies to select a convergent subsequence

$$(y_{m_k})_k \to y \in \varphi$$

of a subsequence of $(y_n)_n$ such that there exists $p \leq m$ with $l(y_{m_k}) = p$ for each $k \in \mathbb{N}$. Then $\|y\| \leq 1$ and

$$f_p(z) = \lim_k f_p(z_{m_k}) = \lim_k [v(y_{m_k})]^{-1};$$

therefore $\lim_k v(y_{m_k}) > 0$ and then, the proceeding equalities and the continuity of each coefficient functional imply that

$$f_p(z) = [\lim_k v(y_{m_k})]^{-1} = [v(y)]^{-1}$$

and $p = l(y)$. Finally, since $x_{m_k} \subset \overline{U}_{l(y)}$, we note:

$$x = z - y = \lim_k (z_{m_k} - y_{m_k}) \in \overline{U}_{l(y)}.$$

Hence

$$z = x + y \in A,$$

which proves that A is sequentially closed. This contradiction with the assumption (i) shows that the claim holds. Using the claim we obtain an increasing sequence $(S_n)_n$ of absolutely convex bounded sets in ξ covering E such that each S_n is a ξ_n-closed neighbourhood of zero in ξ_n.

Note that $(\overline{S_n})_n$ is a fundamental sequence of ξ-bounded sets, where the closure is taken in ξ. Indeed, assume that there exists a bounded set B in (E, ξ) which is not absorbed by any $\overline{S_n}$. Then, for each $n \in \mathbb{N}$, there exists $x_n \in n^{-1}B$ such that $x_n \notin \overline{S_n}$. For each $n \in \mathbb{N}$, let V_n be a ξ_n-neighbourhood of zero such that $x_n \notin S_n + V_n$. Clearly the sequence $(V_n)_n$ can be taken as decreasing. Then $S_k \cap V_k \subset S_n + V_n$ for each $n, k \in \mathbb{N}$. Consequently, $V \subset S_n + V_n$, where

$$V =: \mathrm{ac}\{\bigcup_k (S_k \cap V_k)\}$$

is a ξ-neighbourhood of zero in E. This proves $x_n \notin V$ for each $n \in \mathbb{N}$. We showed that the neighbourhood of zero V in ξ misses all elements of the null-sequence $(x_n)_n$, a contradiction. The proof is completed. □

Theorem 15.2.6 applies to provide an example of a Montel (LF)-space that is not a (DF)-space.

Example 15.2.7 The space $E := \mathbb{R}^{\mathbb{N}} \times \varphi$ is a Montel (LF)-space that is not a (DF)-space; hence, E is not sequential.

Proof Since $\mathbb{R}^{\mathbb{N}}$ does not have a fundamental sequence of bounded sets, otherwise as metrizable, would be normable by using the Baire category theorem, Theorem 15.2.6 shows that E is not sequential. □

Example 15.2.7 provides another approach to produce examples of non-sequential lcs with countable tightness (recall that any (LF)-space has countable tightness; see Theorem 12.2.1).

We complete this section with the following converse to Webb's result [638, Theorem 5.5 (1)].

Proposition 15.2.8 (Kąkol–Saxon) *Every sequential proper (LB)-space is a Montel space.*

Proof Since E is a proper (LB)-space, E contains φ; see Corollary 2.4.6. As every bounded set in φ is finite-dimensional, we can select in E a linearly independent sequence $(y_n)_n$ whose linear span does not admit an infinite-dimensional bounded set. By Lemma 15.2.2 it is enough to show that every bounded sequence $(z_n)_n$ in E contains a convergent subsequence. Assume this is not true, and let $(z_n)_n$ be a bounded sequence that does not contain a convergent subsequence. Set:

$$A := \{n^{-1}z_k + k^{-1}y_n : k, n \in \mathbb{N}\}.$$

The set A is not closed, since $0 \in \overline{A} \setminus A$. Indeed, choose an arbitrary neighbourhood of zero V in E. Let U be a neighbourhood of zero in E such that $U + U \subset V$. The sequence $(z_k)_k$ is bounded, so there exists $n \in \mathbb{N}$ such that $n^{-1}z_k \subset U$ for each $k \in \mathbb{N}$. For this $n \in \mathbb{N}$, choose $k \in \mathbb{N}$ such that $k^{-1}y_n \subset U$. Then there exist $k, n \in \mathbb{N}$ such that

$$n^{-1}z_k + k^{-1}y_n \in U + U \subset V.$$

This proves that $0 \in \overline{A} \setminus A$. Now we prove that A is sequentially closed by showing that every convergent sequence $(n_p^{-1}z_{k_p} + k_p^{-1}y_{n_p})_p$ in A has only finitely many distinct members. If the set $\bigcup_p \{n_p\}$ were infinite, the set $\bigcup_p \{k_p^{-1}y_{n_p}\}$ would be an infinite-dimensional bounded set, a contradiction. Then the set $\bigcup_p \{n_p\}$ is finite and, without loss of generality, we may assume that $n_p = m$ for some fixed $m \in \mathbb{N}$. Then, if $\bigcup_p \{k_p\}$ were infinite, then $\lim_p k_p^{-1}y_m = 0$, implying that the sequence $(m^{-1}z_{k_p})_p$, and then $(z_{k_p})_p$ also, would be convergent, which provides another contradiction. This argument proves that A is sequentially closed and not closed. Hence, E is not sequentially closed giving a contradiction. We proved that E is Montel. □

A tvs E is said to have the property C_4 if for each sequence $(x_n)_n$ in E there exists a sequence $(t_n)_n$ of positive scalars such that $0 \in \overline{\{t_n x_n : n \in \mathbb{N}\}}$. The following lemma motivates the next proposition.

Lemma 15.2.9 *For a tvs E, the property C_3 implies C_4. There exists a set Γ such that the space $\ell^\infty(\Gamma)$ endowed with the topology of pointwise convergence has the property C_4 but not C_3.*

Proof Let $(x_n)_n$ be a sequence in E. Choose a non-zero vector $a \in E \setminus \{k^{-1}nx_n : k, n \in \mathbb{N}\}$. Set:

$$H := \{n^{-1}a - k^{-1}x_n : k, n \in \mathbb{N}\},$$

and denote by H^- the sequential closure of H. Then $n^{-1}a \in H^-$ for all $n \in \mathbb{N}$, and 0 belongs to the sequential closure H^{--} of H^-. If E is assumed to have the

property C_3, we have $0 \in H^-$. Therefore, there are sequences $(n_p)_p$ and $(k_p)_p$ in $\mathbb{N}$ such that

$$n_p^{-1} a - k_p^{-1} x_{n_p} \to 0, \quad p \to \infty.$$

Note that $(n_p)_p$ is unbounded. Indeed, otherwise, taking a subsequence, we may assume that there exists $m \in \mathbb{N}$ such that $m^{-1}a - k_p^{-1} x_m \to 0$, and if $(k_p)_p$ is unbounded, we have $a = 0$, and if $(k_p)_p$ is bounded, there exists $r \in \mathbb{N}$ such that $m^{-1}a - r x_m = 0$. These two cases provide a contradiction. We may assume that $(n_p)_p$ is strictly increasing. Clearly $(k_p^{-1} x_{n_p}) \to 0$. Define:

$$t_{n_p} := k_p^{-1}, \ \ p \in \mathbb{N}, \ \ t_n = 1, \ \ n \in \mathbb{N} \setminus \{n_p : p \in \mathbb{N}\}.$$

Then, the sequence $(t_n)_n$ is as required. Now we prove the other part of lemma. Let Σ be the set of sequences of different pairs $(i, j) \in \mathbb{N} \times \mathbb{N}$ such that for every $\sigma \in \Sigma$ and every $j \in \mathbb{N}$, there exists at most one $i \in \mathbb{N}$ with $(i, j) \in \sigma$. Set $\Gamma = \Sigma \cup \mathbb{N}$ and let $\ell^\infty(\Gamma)$ be the space of bounded functions on Γ with the topology of the pointwise convergence. Set $x_{(i,j)} = (x^\gamma_{(i,j)})_{\gamma \in \Gamma}$, where

$$x^\gamma_{(i,j)} = \begin{cases} x^\gamma_{(i,j)} = 0, \ \text{if} \ \ \gamma \in \Sigma, \ \ (i, j) \notin \gamma \ \text{or if} \ \ \gamma \in \mathbb{N}, \ \ j \neq \gamma, \\ x^\gamma_{(i,j)} = 1, \ \ \text{if} \ \ \gamma \in \Sigma, \ \ (i, j) \in \gamma, \ \ \text{or if} \ \ \gamma \in \mathbb{N}, \ \ j = \gamma. \end{cases}$$

Then, for each $j \in \mathbb{N}$, one has $x_{(i,j)} \to e_j, \ \ i \to \infty$, where $e_j := (e^\gamma_j)_{\gamma \in \Gamma}$ is defined by

$$e^\gamma_j = \begin{cases} 1 & \text{if} \ \ \gamma \in \mathbb{N}, \ \ j = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Also $e_j \to 0$ if $j \to \infty$. Set:

$$B := \{x_{(i,j)} : (i, j) \in \mathbb{N} \times \mathbb{N}\}.$$

Then 0 does not belong to the sequential closure of B. Indeed, otherwise we can find in B a sequence $x_{(i_n, j_n)} \to 0$, i.e. $x^\gamma_{(i_n, j_n)} \to 0$, for each $\gamma \in \Gamma$. But then either there exists $j_0 \in \mathbb{N}$ such that the set $A := \{(i_n, j_n)\}$ contains infinite many of pairs (i, j_0) or there exists in A a subsequence $(i_{n_k}, j_{n_k})_k$ in Σ. The first case for $\gamma := j_0$ and the other one for $\gamma := (i_{n_k}, j_{n_k})_k$ yields a sequence $x^\gamma_{(i_n, j_n)}$ which does not converge to zero, since it contains a constant subsequence convergent to 1, a contradiction.

We showed that $\ell^\infty(\Gamma)$ does not have the property C_3 although satisfies the property C_4. □

Proposition 15.2.10 (Kąkol–Saxon) *For a (DF)-space E, the following assertions are equivalent:*

(i) E does not contain φ.

(ii) E admits a finer normed topology.
(iii) E has the property C_4.

Proof (i) $\Rightarrow$ (ii): Let $(S_n)_n$ be a fundamental sequence of absolutely convex bounded closed sets. It is enough to prove that there exists $m \in \mathbb{N}$ such that S_m is absorbing in E. Then the Minkowski functional norm

$$\|x\| := \inf\{t > 0 : t^{-1}x \in S_m\}$$

generated by the set S_m will provide a norm as required. By a contradiction assume that none of S_n is absorbing in E. Taking a subsequence of $(S_n)_n$,if necessary, we select a sequence $(x_n)_n$ such that

$$x_n \in S_{n+1} \setminus \operatorname{span}(S_n), \quad n \in \mathbb{N}. \tag{15.1}$$

Set S be the linear hull of $\{x_n : n \in \mathbb{N}\}$. Clearly $x_n, n \in \mathbb{N}$, are linearly independent.

We prove that the induced topology on S is the finest locally convex topology. Let p be a seminorm on S. It is enough to show that p is continuous. Since (15.1) holds, we follow the proof of Theorem 2.4.3 (see also [528, Lemma]) to get a sequence $f_n \in E' \cap S_n^\circ$, $n \in \mathbb{N}$, such that

$$\max_{r \le n} |f_r(x)| \ge (1 + 2^{-n})p(x)$$

for $x = \sum_{1 \le i \le n} a_i x_i$. Set $q(x) := \sup_n |f_n(x)|$. Then

$$\{x \in E : q(x) \le 1\} \cap S_n = \bigcap_r \{x \in E : |f_r(x)| \le 1\} \cap S_n$$

$$= \bigcap_{1 \le r < n} \{x \in E : |f_r(x)| \le 1\} \cap S_n.$$

This holds, since $|f_r(x)| \le 1$ for $x \in S_n$ and $r \ge n$. We showed that $\{x \in E : q(x) \le 1\} \cap S_n$ is a neighbourhood of zero in S_n for each $n \in \mathbb{N}$. Since E is a (DF)-space, $\{x \in E : q(x) \le 1\}$ is a neighbourhood of zero in E; see [374, 29.3(2)], or [491, Corollary 8.3.3]. Consequently, q is continuous on E, and then p is continuous on S. We proved that E contains φ, which provides a contradiction with (i).

(ii) $\Rightarrow$ (i): Obvious.

(ii) $\Rightarrow$ (iii): Clear since a normed topology has the property C_4.

(iii) $\Rightarrow$ (ii): Again it is enough to show that there exists $m \in \mathbb{N}$ such that S_m is absorbing. Assume this is not true. Then for each $n \in \mathbb{N}$, there exists x_n in E which is not absorbed by S_n. By (iii) there exists a sequence $(t_n)_n$ of positive scalars such that 0 is in the closure $\overline{\{t_n x_n : n \in \mathbb{N}\}}$. Clearly $t_n x_n \notin S_n$ for all $n \in \mathbb{N}$. Since each set S_n is closed, for each $n \in \mathbb{N}$, there exists an absolutely convex neighbourhood of zero U_n in E such that

$$t_n x_n \notin \overline{S_n + U_n}.$$

Set $U := \bigcap_n \overline{S_n + U_n}$. It is known that U is a neighbourhood of zero in E; see [491, Proposition 8.3.5]. This yields a contradiction, since then $t_m x_m \in U$ for some $m \in \mathbb{N}$, and the proof is completed. □

15.3 (LF)-Spaces with the Property C_3^-

First observe that in the class of quasibarrelled (DF)-spaces E, the property C_3 implies the metrizability of E; see [346]; see also the proof of [374, Theorem 29.1].

Proposition 15.3.1 *A quasibarrelled (DF)-space E has the property C_3 if and only if E is normable.*

Proof Let $(S_n)_n$ be a fundamental (increasing) sequence of closed absolutely convex bounded sets in E. Assume E is not normable. Since E is quasibarrelled, for each S_n there exists in E a bounded absolutely convex set A_n such that S_n does not absorb A_n. Hence, for each pair $(n, k) \in \mathbb{N} \times \mathbb{N}$, there exists

$$z_{n,k} \in k^{-1} A_n \setminus (k+1) S_n.$$

Choose a sequence $(x_n)_n$ of non-zero elements of A_1 such that $x_n \to 0$. Clearly $A_1 \subset S_{m_1}$ for some $m_1 \in \mathbb{N}$ and $z_{n,k} \to 0$ if $k \to \infty$, $n \in \mathbb{N}$. Set:

$$M := \{x_n + z_{n,k} : n, k \in \mathbb{N}\}.$$

Note that each x_n belongs to the sequential closure M^- of the set M. On the other hand, $0 \notin M^-$. Indeed, otherwise there exist sequences $(n_p)_p$ and $(k_p)_p$ in $\mathbb{N}$ such that $n_p \to \infty$ and

$$x_{n_p} + z_{n_p,k_p} \to 0.$$

Hence there exists $r > m_1$ such that

$$x_{n_p} + z_{n_p,k_p} \in S_r,$$

for all $n_p, k_p \in \mathbb{N}$. This implies that

$$z_{n_p,k_p} \in S_r + S_{m_1} \subset S_r + S_r = 2S_r.$$

Hence $n_p \leq r$, a contradiction. The converse implication is obvious. □

This shows that the space φ contains a subset whose sequential closure is not sequentially closed in φ.

In [92] Bonet and Defant proved that if E is an infinite-dimensional nuclear (DF)-space different from φ, the space E contains a subspace whose sequential closure is not a sequentially closed set. This motivates us to distinguish a class of tvs having the following property: a tvs E is said to have the property C_3^- if the sequential closure of any linear subspace of E is sequentially closed; see [346].

We will see that the only infinite-dimensional Montel (DF)-space with the property C_3^- is the space φ. An (LF)-space E has the property C_3^- if and only if E is isomorphic to some metrizable (LF)-space M, to φ, or to the product space $M \times \varphi$; see [346, Theorem 6.13]. A corresponding characterization for (LB)-spaces with the property C_3^- is also provided in [346, Corollary 6.12], and will be presented in Theorem 15.3.8.

A tvs E is said to be *docile* if every infinite-dimensional subspace of E admits an infinite-dimensional bounded subset. The next proposition uses a deep result of Josefson–Nissenzweig [324, 469].

Theorem 15.3.2 *A Banach space E is finite-dimensional if and only if every sequence in E' which $\sigma(E', E)$-converges to zero converges to zero in the norm topology of the dual E'.*

A natural extension of this result to Fréchet spaces is due to Bonet, Lindström, Schlumprecht, and Valdivia; see [94] for detail.

A simple relation between the docility and the property C_3 shows the following.

Proposition 15.3.3 *Every tvs with the property C_3 is docile. The weak* dual of any infinite-dimensional Banach space E is docile without the property C_3.*

Proof Assume E is not docile. Then, there exists a linearly independent sequence $(y_n)_n$ in E whose linear span does not contain any infinite-dimensional bounded set. Set:

$$A := \{n^{-1}y_1 + k^{-1}y_n : k, n \in \mathbb{N}\}.$$

Then, the sequential closure A^- of A equals the set $A \cup \{n^{-1}y_1 : n \in \mathbb{N}\}$. Since $0 \notin A^-$, so A^- is not sequentially closed. Therefore E does not have the property C_3.

Now assume E is an infinite-dimensional Banach space. Clearly the space $(E', \sigma(E', E))$ is docile. Let B be the closed unit ball in E'. By Theorem 15.3.2 it follows that B contains a linearly independent sequence $(f_n)_n$ which converges to zero in $\sigma(E', E)$. Now we proceed as before: set

$$S := \{f_n + n^{-1}f_m : n, m \in \mathbb{N}\}.$$

Then the sequential closure S^- equals the set $S \cup \{f_n : n \in \mathbb{N}\}$. Since $0 \notin S^-$, the sequential closure of S is not sequentially closed. □

To prove the main result of this section, we need a few additional lemmas.

Lemma 15.3.4 *Let E be an infinite-dimensional docile tvs. Then E contains a sequentially dense subspace F such that* $\dim(E) = \dim(E/F)$.

Proof Case 1. Assume $\dim(E) = \aleph_0$. Then there exists a subspace $G \subset E$ such that $\dim(G) = \dim(E/G) = \aleph_0$. By the assumption there exists in G an infinite-dimensional bounded set B. Hence, we can choose a double indexed set:

$$S := \{z_{n,k} : k, n \in \mathbb{N}\} \subset B$$

of linearly independent elements.
Choose a Hamel basis $\{x_n : n \in \mathbb{N}\}$ for an algebraic complement of span(S) in E. Set:

$$F := \text{span}\{x_n + k^{-1} z_{n,k} : n, k \in \mathbb{N}\}.$$

Then $x_n \in F^-$ (the sequential closure) for each $n \in \mathbb{N}$, since $x_n + k^{-1} z_{n,k} \to x_n$ if $k \to \infty$ for each $n \in \mathbb{N}$. Then

$$z_{n,k} = (z_{n,k} + x_n) - x_n \in F^-,$$

and hence $E \subset F^-$. This proves the case 1.

Case 2. $\dim(E) = \aleph_0 \cdot \dim(E)$. Then there exists a family $\{E_\alpha : \alpha \in A\}$ of vector subspaces of E such that $|A| = \dim(E)$, $\dim(E_\alpha) = \aleph_0$ for each $\alpha \in A$, and $E = \bigoplus_\alpha E_\alpha$. The first case provides a family of sequentially dense proper subspaces $F_\alpha \subset E_\alpha$. Then $F := \bigoplus_\alpha F_\alpha$ is sequentially dense in E and $\dim(E/F) = |A| = \dim(E)$. The proof is completed. □

Lemma 15.3.5 *Let E be a lcs with an increasing bornivorous sequence $(S_n)_n$ of subsets of E. Assume that E contains docile infinite-dimensional subspaces $G_n \subset \text{span}(S_n)$ such that $G_{n+1} \cap \text{span}(S_n) = \{0\}$, $n \in \mathbb{N}$. Then E does not have the property C_3^-.*

Proof Applying Lemma 15.3.4 we obtain a sequentially dense subspace $F \subset G_1$ and $x \in G_1 \setminus F$. Let $(y_n)_n$ be a sequence in F with $y_n \to x$. We may assume that the set $\{x\} \cup \{y_n : n \in \mathbb{N}\}$ is linearly independent, passing to a subsequence if necessary.

We claim that exist sequences $(f_n)_n \subset E'$ and $(z_{r,s})_s \subset E$ such that:

(i) $f_n(x) \neq 0$ for all $n \in \mathbb{N}$.
(ii) $f_n(y_i) = 0$ for all $i < n$.
(iii) The sequence $(z_{n,s})_s$ is linearly independent in G_{n+1} for each $n \in \mathbb{N}$.
(iv) $f_n(z_{r,s}) = 0$ for all $r, s \in \mathbb{N}$.

Indeed, by the Hahn-Banach theorem, there exist $f_1 \in E'$ such that $f_1(x) \neq 0$. Clearly there exists a non-zero $z_{1,1} \in G_2 \cap f_1^\perp$, where $f^\perp := \{z \in E : f_1(z) = 0\}$. Fix arbitrary $k \in \mathbb{N}$ and assume that for any $1 \leq n, r, s \leq k$, we have already obtained f_n and $z_{r,s}$ such that the conditions (i), (ii), and (iv) hold and $\{z_{n,1}, z_{n,2}, \ldots, z_{n,k}\}$ is linearly independent in G_{n+1}. Since $S := \{x, y_1, \ldots, y_k\}$ is linearly independent in G_1, also $T := S \cup \{z_{r,s} : r, s \leq k\}$ is a linearly independent

finite set. Again, using the Hahn–Banach theorem, we obtain an element $f_{k+1} \in E'$ such that

$$f_{k+1}(x) \neq 0, \quad f_{k+1}(T \setminus \{x\}) = \{0\}.$$

Next, choose linearly independent:

$$z_{k+1,1}, z_{k+1,2}, \ldots, z_{k+1,k+1} \in G_{k+2} \cap f_1^{\perp} \cap f_2^{\perp} \cap \cdots \cap f_{k+1}^{\perp},$$

and for each $n \leq k$, we select:

$$z_{n,k+1} \in [G_{n+1} \cap f_1^{\perp} \cap f_2^{\perp} \cap \cdots \cap f_{k+1}^{\perp}] \setminus \operatorname{span}\{z_{n,s} : s \leq k\}.$$

The claim is proved.

We may assume that each sequence $(z_{n,s})_s$ is bounded, since the space $\operatorname{span}\{z_{n,s} : s \in \mathbb{N}\}$ contains an infinite-dimensional bounded set (by the docility). Set:

$$H := \operatorname{span}\{y_n + s^{-1} z_{n,s} : n, s \in \mathbb{N}\}.$$

Then, as before, $y_n \in H^-$ for each $n \in \mathbb{N}$, since $y_n + s^{-1} z_{n,s} \to y_n$ for $s \to \infty$. Hence $x \in H^{--}$, since $y_n \to x$ if $n \to \infty$.

We show that $x \notin H^-$, and this will show that E does not have the property C_3^-. Assume that there exists a sequence $(x_n)_n \subset H$ such that $x_n \to x$. Then there exists $m \in \mathbb{N}$ such that S_m absorbs all the set $\{x_n : n \in \mathbb{N}\}$. Note

$$\operatorname{span}(S_m) \cap H = \operatorname{span}\{y_n + s^{-1} z_{n,s} : n < m, s \in \mathbb{N}\}.$$

This implies that $f_m(x_n) = 0$ for all $n \in \mathbb{N}$. From (i) it follows that $f_m(x) \neq 0$, so the conclusion holds. □

Now we are ready to prove the following main result; see [346].

Proposition 15.3.6 (Kąkol–Saxon) *The only infinite-dimensional Montel (DF)-space E with the property C_3^- is the space φ.*

Proof Let $(S_n)_n$ be a fundamental (increasing) sequence of closed bounded absolutely convex sets in E. Since E is Montel, we may assume that S_n are compact.

Case 1. For each $n \in \mathbb{N}$, the space $\operatorname{span}(S_n)$ is infinite-codimensional in $\operatorname{span}(S_{n+1})$. We may assume that $G_1 := \operatorname{span}(S_1)$ is infinite-dimensional. By induction we obtain a sequence $(G_n)_n$ of subspaces of E such that $G_{n+1} \subset \operatorname{span}(S_{n+1})$ is infinite-codimensional for each $n \in \mathbb{N}$ and $G_{n+1} \cap \operatorname{span}(S_n) = \{0\}$. Since each G_n admits a stronger normed topology generated by the Minkowski functional norm associated with the set S_n, each G_n is docile. By Lemma 15.3.5 we conclude that E does not have the property C_3^-.

Case 2. There exists $m \in \mathbb{N}$ such that $\operatorname{span}(S_m)$ is finite-codimensional in $\operatorname{span}(S_{m+1})$. Since $(S_n)_n$ is increasing, we may assume that each $\operatorname{span}(S_n)$

has finite codimension in $\mathrm{span}(S_{n+1})$. Hence $H_n := \mathrm{span}(S_n)$ is countable-codimensional in E for each $n \in \mathbb{N}$. Since every countable-codimensional subspace of a barrelled space is barrelled [491], each H_n is a barrelled subspace of E having a compact neighbourhood of zero S_n. Therefore, each H_n is finite-dimensional, consequently yielding the countable dimensionality of the whole space E. So E is a barrelled space of dimension $\aleph_0$. Hence, E is isomorphic to the space φ; apply Proposition 2.4.14. □

Proposition 15.3.7 *Let M be a metrizable lcs. Then the product $M \times \varphi$ has the property C_3^- and does not have the property C_3.*

Proof Let $\{e_n : n \in \mathbb{N}\}$ be a Hamel basis in φ. Set:

$$S_n := \{(x, y) : x \in M,\ y \in n\,\mathrm{ac}\{e_1, e_2, \ldots, e_n\}\}$$

for each $n \in \mathbb{N}$. Note that each S_n is a closed subset of $M \times \varphi$, $(S_n)_n$ covers the whole space $M \times \varphi$, and $(S_n)_n$ is increasing bornivorous in $M \times \varphi$.

Let $F_n := \mathrm{span}(S_n)$ for each $n \in \mathbb{N}$. Clearly each F_n is metrizable and has codimension 1 in the space F_{n+1}, and $S_m \cap F_n$ is closed for each $m > n$. We need to show that $E := M \times \varphi$ has the property C_3^-: fix a linear subspace H of E. We show that $H^- = H^{--}$; this will show that E has the property C_3^-. Fix arbitrary $x \in H^{--}$. Then there exist sequences $(y_n)_n \subset H^-$ with $y_n \to x$, $(z_{n,k})_k \subset H$ with $z_{n,k} \to y_n$ for each $n \in \mathbb{N}$. There exists $p \in \mathbb{N}$ such that $\{y_n : n \in \mathbb{N}\} \subset pS_p$ (since $(S_n)_n$ is increasing bornivorous), and for each $n \in \mathbb{N}$, there is $m_n > p$ with

$$\{z_{n,k} : k \in \mathbb{N}\} \subset m_n S_{m_n}.$$

Note that there exists a sequence $(w_{n,k})_k$ such that

$$w_{n,k} \in 2\,\mathrm{ac}\{z_{n,k} : k \in \mathbb{N}\} \cap F_p$$

and such that $w_{n,k} \to y_n$ for each $n \in \mathbb{N}$. Indeed, fix $n \in \mathbb{N}$. Note that F_p is finite-codimensional in F_{m_n} and $S_{m_n} \cap F_p$ is closed. Let $F_{m_n} = F_p \oplus L_p$ algebraically, where L_p is an algebraic finite-dimensional complement of F_p in F_{m_n}. For each $k \in \mathbb{N}$, there exist $u_{n,k} \in F_p$ and $v_{n,k} \in L_p$ such that

$$z_{n,k} = u_{n,k} + v_{n,k}.$$

Since $n \in \mathbb{N}$ is fixed, to simplify the notations, set:

$$z_k := z_{n,k},\quad u_k := u_{n,k},\quad v_k := v_{n,k},\quad y := y_n.$$

If all $v_k = 0$, the claim is trivial. Therefore, we may assume that the sequence $(v_k)_k$ contains a maximal linearly independent subset $\{e_1, e_2, \ldots, e_r\}$. Define a norm on the linear span $\mathrm{span}\{e_1, e_2, \ldots, e_r\}$ by the formula:

$$\|\sum_{j\le r}\lambda_j e_j\| := \sum_{j\le r}|\lambda_j|.$$

Let t_j be the first z_k for which $v_k = e_j$. Clearly $t_j - e_j \in F_p$ and for each $k \in \mathbb{N}$, there exist unique scalars $\lambda_{k,j}$ such that $v_k = \sum_{j\le r}\lambda_{k,j}e_j$. Note that from $u_k = z_k - v_k \in F_p$ it follows that

$$w_k := z_k - \sum_{j\le r}\lambda_{k,j}t_j = z_k - v_k + \sum_{j\le r}\lambda_{k,j}(e_j - t_j) \in F_p.$$

Since $v_k \to 0$, so $\|\sum_{j\le r}\lambda_{k,j}e_j\| \to 0$. Hence $\sum_{j\le r}\lambda_{k,j}t_j \to 0$, and since $\sum_{j\le r}|\lambda_{k,j}| \le 1$ for almost all $k \in \mathbb{N}$ (by the definition of the norm $\|.\|$), we deduce that

$$w_k \in 2\,\mathrm{ac}\{z_k : k \in \mathbb{N}\}$$

for almost all $k \in \mathbb{N}$. Therefore

$$w_k \in F_p \cap 2\,\mathrm{ac}\{z_k : k \in \mathbb{N}\}$$

for almost all $k \in \mathbb{N}$. Finally

$$\lim_k w_k = \lim(z_k - \sum_{j\le r}\lambda_{k,j}t_j) = \lim z_k - 0 = y.$$

The claim is proved. In fact we showed that $\{w_{n,k} : k \in \mathbb{N}\} \subset H \cap F_p$. Since $x = \lim_n y_n \in pS_p \subset F_p$ and F_p has the property C_3^- as a metrizable space, we note $x \in H^-$. This shows that $H^{--} = H^-$, so $E = M \times \varphi$ has the property C_3^-. The space φ does not have the property C_3 (see Proposition 15.3.1), so the proof is completed. □

For (LB)-spaces we have the following.

Theorem 15.3.8 *An (LB)-space E with its defining sequence $(E_n)_n$ of Banach spaces E_n has the property C_3^- if and only if E is isomorphic to some E_m, to φ, or to the product $E_m \times \varphi$.*

Proof Assume E has the property C_3^-. For each $n \in \mathbb{N}$, let S_n be the unit ball in the Banach space E_n. Let B_n be the closure of S_n in E. Clearly $E = \bigcup_n \mathrm{span}(B_n)$ and each set B_n is bounded in E. Moreover, $(B_n)_n$ is a fundamental sequence of bounded sets in E; see the proof of the last part of Theorem 15.2.6. If each $\mathrm{span}(B_n)$ is infinite-dimensional in $\mathrm{span}(B_{n+1})$, then (as in the proof of Case 1 in Proposition 15.3.6), we deduce that E does not have the property C_3^-, a contradiction. Therefore, there exists $m \in \mathbb{N}$ such that $N := \mathrm{span}(B_m)$ is countable-codimensional in E. Let P be an algebraic complement of N in E. Then $E = N \oplus P$ topologically. Indeed, since any countable-codimensional subspace of a barrelled

space is barrelled (see [491, Theorem 4.3.6]), the space N is barrelled and P is a topological complement endowed with the strongest locally convex topology; see also [533]. Applying the closed graph theorem [491, Theorem 4.1.10], we deduce that N is isomorphic to the Banach space E_m. If P is isomorphic to φ, then E is isomorphic to $E_M \times \varphi$. If E_m is finite-dimensional, E is isomorphic to φ. Now assume that P is finite-dimensional. Then, there exists $n > m$ such that $P \subset E_n$. Consequently, $E = E_n$, and the closed graph theorem applies again to show that this equality is topological. For the converse implication, it is enough to apply Proposition 15.3.7. □

Chapter 16
Tightness and Distinguished Fréchet Spaces

Abstract In this chapter, we apply the concept of tightness to study distinguished Fréchet spaces. We show that a Fréchet space is distinguished if and only if its strong dual has countable tightness. This approach to studying distinguished Fréchet spaces leads to a rich supply of (DF)-spaces whose *weak** duals are quasi-Suslin but not K-analytic. The small cardinals b and d will be used to improve the analysis of Köthe's echelon non-distinguished Fréchet space $\lambda_1(A)$.

16.1 A Characterization of Distinguished Spaces

Let E be a metrizable lcs with a decreasing basis $(U_n)_n$ of absolutely convex neighbourhoods of zero.

Apart typical dual topologies on E' like as the strong topology $\beta(E', E)$ or the Mackey topology $\mu(E', E)$, there is a natural way to topologize the space E' by the inductive limit topology generated by a defining sequence $(\mathrm{span}(U_n^\circ))_n$.

Indeed, let $E'_n := \mathrm{span}(U_n^\circ)$ be endowed with the Minkowski functional norm topology. Then each E'_n is a Banach space, the sequence $(E'_n)_n$ is increasing, and $E' = \bigcup_n E'_n$.

By $(E', i(E', E))$, we denote the space E' endowed with the inductive limit topology $i(E', E)$ generated by the above defining sequence $(E'_n)_n$. It is known (Grothendieck) that $i(E', E)$ is the bornological topology associated with the strong topology $\beta(E', E)$, i.e. $\beta(E', E)$ is bornological if and only if $i(E', E) = \beta(E', E)$; see [322, Theorem 13.4.2].

Dieudonné and Schwartz called a Fréchet space E distinguished if the strong dual $(E', \beta(E', E))$ is barrelled. Grothendieck observed that E is distinguished if and only if $(E', \beta(E', E))$ is bornological.

In fact it is known that for a metrizable lcs E, the dual $(E', \beta(E', E))$ is quasibarrelled if and only if $(E', \beta(E', E))$ is barrelled if and only if $(E', \beta(E', E))$ is bornological; see [439, Proposition 25.12], [374, 29.4.(3)], [491, 8.3.44].

We call a metrizable lcs E *distinguished* if $(E', \beta(E', E))$ is quasibarrelled. Therefore, E is distinguished if and only if $(E', \beta(E', E)$ is bornological, i.e. $\beta(E, E) = i(E', E)$.

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A classical result of Grothendieck states that every (DF)-space for which every bounded set is metrizable is quasibarrelled; see [374, 29.3.12 (b)]. Hence:

(*) A metrizable lcs E whose strong dual $(E', \beta(E', E))$ has all bounded sets metrizable is distinguished.

The first example of a non-distinguished Fréchet space was described by Grothendieck and Köthe; it was the Köthe echelon space $\lambda_1(A)$ for the Köthe matrix $A = (a_n)_n$ defined on the set $\mathbb{N} \times \mathbb{N}$ by the formula $a_n(i, j) := j$ for $i \leq n$ and $a_n(i, j) := 1$ otherwise. For more examples of non-distinguished Fréchet spaces, we refer to [93].

The distinguished Köthe echelon spaces were intensively studied by many specialists; see, for example, [73, 80, 82, 83, 85, 86] and [216, 220, 221].

Taskinen [586] provided a concrete Fréchet space $C(\mathbb{R}) \cap L^1(\mathbb{R})$ (endowed with the intersection topology). A simple argument to this effect was presented in [95].

The argument below, due to Bierstedt and Bonet [84, Theorem 1], is also valid for the space $C^\infty(\Omega)$ of infinitely differentiable functions on an open subset $\Omega \subset \mathbb{R}^{\mathbb{N}}$, endowed the compact-open topology for the functions and each of their derivatives.

Proposition 16.1.1 *The intersection space $E := C^\infty(\Omega) \cap L^1(\Omega)$ is a non-distinguished Fréchet space.*

Proof Let $p_0(f) := \int_\Omega |f| d\mu$ for $f \in E$. Choose an increasing sequence $(K_n)_n$ of compact sets covering Ω such that every compact subset of Ω is contained in some K_n and the interior of $K_{n+1} \setminus K_n$ is non-void for each $n \in \mathbb{N}$. The topology of E is defined by the increasing sequence $(p_n)_n$ of seminorms:

$$p_n(f) := p_0(f) + \max_{|\alpha| \leq n} \max_{x \in K_n} |f^\alpha(x)|$$

for each $f \in E$.

We prove that $(E', \beta(E', E))$ is not bornological. It is enough to show that $i(E', E) \neq \beta(E', E)$. We show that for each bounded set $B \subset E$, there exists $u \in B^\circ$ such that for each $n \in \mathbb{N}$ there exists $f_n \in E$ with $p_n(f_n) \leq 1$ and $u(f_n) = 2$. This is enough, since then

$$V := \bigcup_n \{v \in E' : |v(f)| \leq p_n(f), f \in E\}$$

will be a neighbourhood of zero in $i(E', E)$ and not in $\beta(E', E)$.

So, fix a bounded set B in E. By the boundedness for each $n \in \mathbb{N}$, there exists $M_n > 0$ such that $p_n(f) \leq M_n$ for all $f \in B$. For each $n \in \mathbb{N}$, choose a compact set I_n with non-empty interior in $K_{n+1} \setminus K_n$ such that the Lebesque measure $\mu(I_n) < 2^{-n-1} M_{n+1}^{-1}$. Set $u(f) := 2 \sum_n \int_{I_n} f d\mu$ for all $f \in E$. Since $|u(f)| \leq 2p_0(f)$, we have $u \in E'$. On the other hand,

$$|u(f)| \leq 2 \sum_n \mu(I_n) \max_{x \in K_{n+1}} |f(x)| \leq 2 \sum_n \mu(I_n) M_{n+1} \leq 1.$$

Hence $u \in B^\circ$. For each $n \in \mathbb{N}$, choose a non-negative test function f_n on I_n such that $\int_{I_n} f_n d\mu = 1$. Then $f_n \in E$ for each $n \in \mathbb{N}$. Each f_n vanishes on an open neighbourhood of K_n. Then $p_n(f_n) = p_0(f_n) = 1$ and $u(f_n) = 2$. □

If E is a non-distinguished metrizable lcs, the topologies $i(E', E)$ and $\beta(E', E)$ are different. Even in that case, one may happen that $(E', i(E', E))' = (E', \beta(E', E))'$. Indeed, there exists a non-distinguished Fréchet space E for such that $(E', i(E', E))' = (E', \beta(E', E))'$. The first example of this type was provided by Kōmura; see [73] or [611].

On the other hand, Grothendieck [287] showed that for the non-distinguished Köthe echelon space $E := \lambda_1(A)$, there exists a discontinuous linear functional on $(E', \beta(E', E))$ which is bounded, i.e. transforms bounded sets to bounded sets (that means $(E', i(E', E))' \neq (E', \beta(E', E))'$).

Valdivia [611] proved that, if E is a separable Fréchet space not containing a copy of the space ℓ^1, then $(E', i(E', E))' = (E', \beta(E', E))'$. This surely motivates a question if every separable Fréchet space not containing a copy of ℓ^1 is necessarily distinguished. This problem has been answered in the negative by Diaz [161].

In order to prove Theorem 16.1.2, we need a couple of additional facts about the vector-valued Fréchet space $\ell^1(E)$, where E is a Fréchet space. The strong dual

$$\ell^1(E)'_b := (\ell^1(E)', \beta(\ell^1(E)', \ell^1(E))$$

is isomorphic to the space $(\ell_1 \hat{\otimes}_\pi E)'_b$. The latter space is isomorphic to the space $\ell^\infty(E'_b)$ of all bounded sequences in E'_b endowed with the topology of uniform convergence. The duality is defined by the map $u(x) := \sum_i < x(i), u(i) >$, where $x = (x(i))_i \in \ell^1(E)$ and $u = u(i)_i \in \ell^\infty(E'_b)$; see [494, Theorem 1.5.8]. The proof of (i) ⇔ (iii) in Theorem 16.1.2 is adopted from [84, Theorem 10].

Theorem 16.1.2 (Bierstedt–Bonet) *For a Fréchet space E, the following conditions are equivalent:*

(i) E satisfies the density condition.
(ii) Every bounded set in $(E', \beta(E', E))$ is metrizable.
(iii) The space $\ell^1(E)$ is distinguished.

Proof Note that the equivalence (i) ⇔ (ii) was already proved in Proposition 6.7.4. Let $(U_n)_n$ be a decreasing basis of absolutely convex neighbourhoods of zero in E. Then, the polars $B_n := U_n^\circ$ form a fundamental sequence of bounded sets in the strong dual $E'_b := (E', \beta(E', E))$. Consequently, the sets

$$D_n := \{u \in \ell^\infty(E'_b) : u(i) \in B_n, i \in \mathbb{N}\}$$

form a fundamental sequence of bounded sets in the (DF)-space $\ell^\infty(E'_b)$.

(i) ⇒ (iii): Since $(\ell^1(E))'_b$ is isomorphic to $\ell^\infty(E'_b)$, it is enough to show that $\ell^\infty(E'_b)$ is bornological. Let D be an absolutely convex set in $\ell^\infty(E'_b)$ which absorbs bounded sets in $\ell^\infty(E'_b)$. Then there exists a sequence $(\lambda_j)_j$ of positive numbers such that

$$H := \bigcup_n \sum_{j=1}^{n} \lambda_j D_j \subset D.$$

Since $\ell^\infty(E'_b)$ is a (DF)-space, to show that H is a neighbourhood of zero, it is enough to prove that $H \cap D_n$ is a neighbourhood of zero in D_n for each $n \in \mathbb{N}$, [374, 29.3.(2)]. Then, we will conclude D is a neighbourhood of zero. E satisfies the density condition (see Proposition 6.7.4). For the sequence $(\lambda_j)_j$ (chosen above) and each $n \in \mathbb{N}$, by using the bipolar theorem, we find $m > n$ and a bounded set $B \subset E$ such that

$$B_n \cap B^\circ \subset \sum_{1 \le j \le m} \lambda_j B_j.$$

Set:

$$V := \{u \in \ell^\infty(E'_b) : u_i \in B^\circ, i \in \mathbb{N}\}.$$

Then V is a neighbourhood of zero in $\ell^\infty(E'_b)$. Also

$$D_n \cap V \subset \sum_{1 \le j \le m} \lambda_j B_j \subset H.$$

The last inclusion yields the conclusion.

(iii) $\Rightarrow$ (i): Assume E does not satisfy the density condition and $\ell^\infty(E'_b)$ is bornological, i.e. $\ell^1(E)$ is distinguished. By the bipolar theorem, there exist a sequence $(\lambda_j)_j$ of positive numbers and $n \in \mathbb{N}$ such that for each m and each bounded set $B \subset E$, the set $B_n \cap B^\circ$ is not contained in $D_m := \mathrm{ac}(\bigcup_{1 \le j \le m} \lambda_j B_j)$. For each $m \in \mathbb{N}$, define:

$$A_m := \{u \in \ell^\infty(E'_b) : u(i) \in D_m, i \in \mathbb{N}\}.$$

Note that $A := \bigcup_m A_m$ is absolutely convex and absorbs bounded sets in $\ell^\infty(E'_b)$. Therefore, A is a neighbourhood of zero in the bornological space $\ell^\infty(E'_b)$. Hence, there exists a bounded set $B \subset E$ such that

$$T := \{u \in \ell^\infty(E'_b) : u(i) \in B^\circ, i \in \mathbb{N}\} \subset A.$$

On the other hand, for B, and for each $m \in \mathbb{N}$, there exists $u(m) \in (B_n \cap B^\circ) \setminus D_m$. Also $u = (u(m))_m \in T \subset A$. Hence there exists $k \in \mathbb{N}$ such that $u \in A_k$, so by the definition of the set A_k, we have that $u(k) \in D_k$, a contradiction. □

By (*) any condition from Theorem 16.1.2 implies that E is distinguished. The converse fails in general, as the following observation shows. It is known that every separable (DF)-space is quasibarrelled; see [491, Proposition 8.3.13].

There exist reflexive Fréchet spaces whose strong dual is separable (such spaces are distinguished) and which do not satisfy the density condition [82].

It turns out that for the Köthe echelon spaces λ_1, the density condition characterizes the distinguished property of λ_1; see [82].

Theorem 16.1.3 *The Köthe echelon space λ_1 is distinguished if and only if it satisfies the density condition.*

Proof Assume λ_1 satisfies the density condition. Applying Theorem 16.1.2 we deduce that $\ell^1(\lambda_1)$ is distinguished. Since $\ell^1(\lambda_1)$ is isomorphic to the space $\ell_1 \hat{\otimes}_\pi \lambda_1$, the space λ_1 is distinguished; see [82, Proposition 3].

Conversely, assume that λ_1 is distinguished. In order to prove that λ_1 satisfies the density condition, it is enough to show that $\ell^1(\lambda_1)$ is distinguished and apply Theorem 16.1.2. To see that $\ell_1 \hat{\otimes}_\pi \lambda_1$ is distinguished, we refer the reader again to the article [82, Proposition 3]. □

Let $V := (v_n)_n$ be the associated decreasing sequence on I, i.e. $v_n := a_n^{-1}$ for all $n \in \mathbb{N}$, where $A := (a_n)_n$ is a strictly positive Köthe matrix on I, i.e. an increasing sequence of strictly positive functions a_n on I. Proposition 16.1.4 provides the condition (D), due to of Bierstedt and Meise [85], which characterizes the density condition for λ_p, $1 \leq p < \infty$, or $p = 0$; see [83, Theorem 3].

Proposition 16.1.4 *An echelon space $\lambda_p := \lambda_p(I, A)$, $1 \leq p < \infty$, or $p = 0$, satisfies the density condition if and only if $A = (a_n)_n$ satisfies the condition (D) (independent of p), i.e. there exists an increasing sequence $(I_n)_n$ of subsets of I such that:*

(i) for each $m \in \mathbb{N}$, there is $n(m) \in \mathbb{N}$ with $\inf_{i \in I_m} a_{n(m)}(i)(a_k(i))^{-1} > 0$ for $k \geq n(m)$, while

(ii) for each $n \in \mathbb{N}$ and each $J \subset I$ with $J \cap (I \setminus I_m)$ non-empty for all $m \in \mathbb{N}$, there is $k = k(n, J) > n$ with $\inf_{i \in J} a_k(i)(a_n(i))^{-1} = 0$.

This can be used to define a sufficient condition (ND) for the non-distinguishedness of λ_1; see [84] for many discussions concerning these conditions and consequences.

We provide another characterization for distinguished spaces. First we prove two additional propositions. The first one is motivated by the proof of Proposition 15.2.3.

Proposition 16.1.5 *Let τ and ϑ be two locally convex topologies on a vector space E such that $\tau \leq \vartheta$ and each countable ϑ-equicontinuous set is equicontinuous in τ. If A is a ϑ-closed set and has τ-countable tightness, then $\tau|A = \vartheta|A$.*

Proof Assume, by contradiction, that $\tau|A$ is different from $\vartheta|A$. Then there exists a $\vartheta|A$-closed (hence ϑ-closed) subset B of A which is not $\tau|A$-closed. Let $\overline{B}$ denote the $\tau|A$-closure of B. There exists $x \in A$ such that $x \in \overline{B} \setminus B$. Hence there exists a ϑ-continuous seminorm p on E such that $p(x - y) \geq 1$ for each $y \in B$. We show that $x \notin \overline{\{x_n : n \in \mathbb{N}\}}$ (the closure in τ restricted to A) for each sequence $(x_n)_n$ in B, contradicting the τ-countable tightness of A. Let $(x_n)_n$ be a sequence in B. By F denote the linear hull of the set $\{x - x_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, choose a linear functional g_n on F such that

$$g_n(x - x_n) = p(x - x_n) \geq 1, \quad |g_n(y)| \leq p(y)$$

for all $y \in F$. For each $n \in \mathbb{N}$, let h_n be a linear extension of g_n to the whole E such that $|h_n(z)| \leq p(z)$ for each $z \in E$. Define:

$$f_n(z) := h_n(z)(p(x - x_n))^{-1}$$

for each $n \in \mathbb{N}$. Clearly each f_n is ϑ-continuous and $f_n(x - x_n) = 1$, $|f_n(z)| \leq p(z)$ for each $z \in E$. If $\epsilon > 0$ and $U = \{z \in E : p(z) < \epsilon\}$, the set U is a ϑ-neighbourhood of zero in E such that $|f_n(z)| \leq p(z) < \epsilon$ for all $z \in U$ and $n \in \mathbb{N}$. Hence, $(f_n)_n$ is a ϑ-equicontinuous sequence; consequently it must be τ-equicontinuous (by the assumption on E). Therefore, there exists an absolutely convex τ-neighbourhood of zero V such that $|f_n(z)| < 2^{-1}$ for all $z \in V$, $n \in \mathbb{N}$, and

$$(x + V) \cap \{x_n : n \in \mathbb{N}\} = \emptyset.$$

This provides a contradiction. □

The next proposition extends Grothendieck's result [374, 29.3.12(b)].

Proposition 16.1.6 *Let $E = (E, \tau)$ be a (DF)-space:*

(i) E is quasibarrelled if and only if every bounded set in E has τ-countable tightness.

(ii) If the Mackey space $(E, \mu(E, E'))$ is quasibarrelled, then $(E, \sigma(E, E'))$ has countable tightness. If every bounded set in E has $\sigma(E, E')$-countable tightness, then $(E, \mu(E, E'))$ is quasibarrelled.

Proof Let $(S_n)_n$ be a fundamental sequence of absolutely convex closed bounded sets in E:

(i): If E is quasibarrelled, by Theorem 12.2.1 the space E has countable tightness, and the conclusion follows.
To prove the converse, assume that every bounded set in E has countable tightness. Since every linear functional on E that is continuous on each S_n is continuous on E, we apply (the proof of) Theorem 12.1.4 (i) $\Rightarrow$ (ii) and Proposition 12.1.3, to deduce the

Claim 1. The *weak** dual $(E', \sigma(E', E))$ is realcompact.

Now we show that $(E, \mu(E, E'))$ is quasibarrelled. Indeed, for every sequence $\alpha := (n_k) \in \mathbb{N}^{\mathbb{N}}$, set $B_\alpha := \bigcap_k n_k S_k^o$. Since E is a (DF)-space, every sequence in any B_α is equicontinuous. Hence B_α is relatively countably compact in $(E', \sigma(E', E))$. Since $(E', \sigma(E', E))$ is realcompact, every B_α is relatively compact, so $\mu(E, E')$-equicontinuous. As every $\beta(E', E)$-bounded set is contained in some B_α, each $\beta(E', E)$-bounded set is $\mu(E, E')$-equicontinuous. This proves that $(E, \mu(E, E'))$ is quasibarrelled.

Claim 2. We have the equality $\tau = \mu(E, E')$.
Indeed, since (E, τ) is a (DF)-space, the assumption of Proposition 16.1.5 for $\vartheta := \mu(E, E')$ is satisfied. Then

$$\tau|A = \mu(E, E')|A \tag{16.1}$$

for every bounded set A of E. Since (E, τ) is a (DF)-space, the topology τ is the finest locally convex topology on E satisfying (16.1); see [374, 29.3.2]. This implies the conclusion.

(ii): If $(E, \mu(E, E'))$ is quasibarrelled, then $(E, \sigma(E, E'))$ has countable tightness. Indeed, the sets B_α defined above compose a $\mathfrak{G}$-representation for $(E, \mu(E, E'))$. Now it is enough to apply Theorem 12.2.1. Note that the same argument as above shows the remaining implication. □

For the next result, we refer to [216].

Theorem 16.1.7 (Ferrando–Kąkol–López-Pellicer) *A metrizable lcs E is distinguished if and only if every bounded set in the strong dual of E has countable tightness.*

Proof The proof follows from Proposition 16.1.6, since the strong dual of a metrizable lcs is a (DF)-space. □

On the other hand, there are a lot of non-quasibarrelled spaces with countable tightness whose every bounded set is metrizable; the weak topology of every infinite-dimensional Fréchet–Montel space has this property. Proposition 16.1.6 for (DF)-spaces $C_c(X)$ can be read as follows.

Proposition 16.1.8 *The following conditions are equivalent for a (DF)-space $C_c(X)$:*

(i) The compact-open topology τ_c of $C_c(X)$ is equivalent to the Banach topology generated by the unit ball $[X, 1] := \{f \in C_c(X) : \sup_{x\in X} |f(x)| \leq 1\}$.
(ii) Every bounded set of $C_c(X)$ has countable tightness.
(iii) Every bounded set of $C_c(X)$ has countable tightness in the weak topology.

Proof Recall that X is pseudocompact; see Proposition 2.6.10 or Theorem 2.6.11 (vii). It is known that $[X, 1]$ generates on $C_c(X)$ a finer Banach topology ϑ, $\tau_c \leq \mu(C_c(X), C_c(X)') \leq \vartheta$, and the weak dual of $(C_c(X), \tau_c)$ is locally complete, Theorem 2.6.11(vi). Clearly (i)⇒ (ii). If (ii) holds, Proposition 16.1.6 applies to show that τ_c is barrelled (since a quasibarrelled space E is barrelled if and only if $(E', \sigma(E', E))$ is locally complete, [322, Theorem 11.2.5(b)]). Then, the closed graph theorem applied to the identity map $I : (C_c(X), \tau_c) \to (C_c(X), \vartheta)$ yields $\vartheta = \tau_c$; see [491, Theorem 4.1.10]. This proves (i).

The same argument applies to show (i) $\Leftrightarrow$ (iii). Indeed, if (i) holds, the weak topology of $C_c(X)$ has countable tightness by Theorem 12.1.4. Conversely, by Proposition 16.1.6 the Mackey topology $\mu(C_c(X), C_c(X)')$ is barrelled. Again the closed graph theorem applies to deduce that $\vartheta = \mu(C_c(X), C_c(X)')$. □

16.2 $\mathfrak{G}$-Bases and Tightness

Recall that the *character* of a lcs E, denoted by $\chi(E)$, is the smallest infinite cardinality for a basis of neighbourhoods of zero. A lcs is metrizable if and only if its character is countable.

Given an infinite cardinal number $\mathfrak{m}$, denote by $\mathfrak{G}_\mathfrak{m}$ the class of those locally convex spaces E for which $\chi(E) \leq \mathfrak{m}$. Note that $\mathfrak{G}_\mathfrak{m}$ is stable by taking subspaces, quotients by closed subspaces, completions, and products of no more than $\mathfrak{m}$ spaces; see [118]. We need the following result from [118].

Proposition 16.2.1 *Let X and Y be topological spaces and let $\psi : X \to 2^Y$ be an (usco) compact-valued map such that $Y = \bigcup\{\psi(x) : x \in X\}$. If the weight $\omega(X)$ is infinite, we have the following: (i) The Lindelöf number $\ell(Y^n) \leq \omega(X)$ for every $n \in \omega$; (ii) If Y is moreover metric, then* $\operatorname{dens}(Y) \leq \omega(X)$.

Proof To prove (i) observe that for every $n \in \mathbb{N}$, the multi-valued map $\psi^n : X^n \to 2^{Y^n}$ defined by

$$\psi^n(x_1, x_2, \ldots, x_n) := \psi(x_1) \times \psi(x_2) \times \cdots \times \psi(x_n)$$

is (usco) compact-valued, and

$$Y^n = \bigcup\{\psi^n(x_1, x_2, \ldots, x_n) : (x_1, x_2, \ldots, x_n) \in X^n\}.$$

Since $\omega(X)$ is infinite, we note $\omega(X^n) = \omega(X)$. Hence, we only need to prove our case for $n = 1$. Take $(G_i)_{i\in I}$ any open cover of Y. For each $x \in X$, the compact set $\psi(x)$ is covered by the family $(G_i)_{i\in I}$. Therefore, there exists a finite subset $I(x)$ of I such that $\psi(x) \subset \bigcup_{i\in I(x)} G_i$. By the upper semicontinuity, for each $x \in X$ there exists an open set $O_x \subset X$ such that $x \in O_x$ and $\psi(O_x) \subset \bigcup_{i\in I(x)} G_i$. The family $(O_x)_{x\in X}$ is an open cover of X; therefore, there is a set $F \subset X$ such that $|F| \leq \omega(X)$ and $X = \bigcup_{x\in F} O_x$; see, for example, [195, Theorem 1.1.14]. Then

$$Y = \psi(X) = \bigcup_{x\in F} \psi(O_x) = \bigcup_{x\in F} \bigcup_{i\in I(x)} G_i.$$

Hence $(G_i)_{i\in I}$ has a subcover of at most $w(X)$ elements.

To get (ii) assume Y is a metric space, and for every $n \in \mathbb{N}$, choose $F_n \subset Y$ a maximal set of points the distance between any two of which is at least n^{-1}. Then

F_n is closed, each $x \in X$ has a neighbourhood U such that $\psi(U) \cap F_n$ is finite, and therefore $|F_n| \leq \omega(X)$. Then $F = \bigcup_{n=1}^{\infty} F_n$ is dense in Y, and thus we obtain $\operatorname{dens}(Y) \leq \omega(X)$ which finishes the proof. □

Corollary 16.2.2 *Let X and Y be topological spaces. Let $\psi : X \to 2^Y$ be an (usco) compact-valued map such that $Y = \bigcup\{\psi(x) : x \in X\}$. If $\omega(X)$ is infinite and if $Y_0 \subset Y$ is a closed subspace, $\ell(Y_0^n) \leq \omega(X)$ for every $n \in \omega$.*

Proof Since Y_0^n is closed in Y^n, we have $\ell(Y_0^n) \leq \ell(Y^n)$, and then we apply Proposition 16.2.1. □

We are ready to prove the following result [118].

Theorem 16.2.3 (Cascales–Kąkol–Saxon) *Let $\{E_s : s \in S\}$ be a family of lcs in the class $\mathfrak{G}_{\mathfrak{m}}$. Let $f_s : E_s \to E$ be linear maps for $s \in S$, and let $E = \sum_{s\in S} f_s(E_s)$ be the locally convex hull of $\{f_s(E_s) : s \in S\}$. Then $\mathrm{t}(E) \leq \mathfrak{m}$ and $\mathrm{t}(E, \sigma(E, E')) \leq \mathfrak{m}$ if $|S| \leq \mathfrak{m}$.*

Proof For every $s \in S$, fix a basis $\mathfrak{B}_s$ of absolutely convex neighbourhoods of zero in E_s such that $|\mathfrak{B}_s| \leq \mathfrak{m}$. Observe that $\mathrm{t}(E, \tau) \leq \mathfrak{m}$. It is enough to show that, if $A \subset E$ and $0 \in \overline{A}$ (the closure in E), there exists $B \subset A$ with $|B| \leq \mathfrak{m}$ such that $0 \in \overline{B}$. The family

$$\mathfrak{B} := \{\mathrm{ac}(\bigcup_{s\in S} f_s(U_s)) : U_s \in \mathfrak{B}_s, s \in S\}$$

is a basis of zero in E, and the family

$$\mathfrak{B}_0 := \{\mathrm{ac}(\bigcup_{s\in S'} f_s(U_s)) : U_s \in \mathfrak{B}_s,\ s \in S,\ S' \text{ finite subset of } S\}$$

has at most $\mathfrak{m}$ elements. For $A \subset E$ and $0 \in \overline{A}$, set

$$B := \{x_{U_0} : x_{U_0} \text{ is a chosen point in } U_0 \cap A \text{ if } U_0 \cap A \neq \emptyset, U_0 \in \mathfrak{B}_0\}.$$

Clearly $B \subset A$, $|B| \leq \mathfrak{m}$, and $0 \in \overline{B}$. Indeed, if $U \in \mathfrak{B}$, we have $A \cap U \neq \emptyset$. Hence there exists $U_0 \in \mathfrak{B}_0$ such that $U_0 \subset U$ and $U_0 \cap A \neq \emptyset$. Consequently, $x_{U_0} \in B \cap U$ and $0 \in \overline{B}$.

Now we prove that $\mathrm{t}(E, \sigma(E, E')) \leq \mathfrak{m}$. Since $(E, \sigma(E, E'))$ is a subset of $C_p(E', \sigma(E', E))$, it suffices to show that

$$\ell(E', \sigma(E', E))^n \leq \mathfrak{m} \tag{16.2}$$

for each $n \in \mathbb{N}$. Indeed, then $\mathrm{t}(C_p(E', \sigma(E', E))) \leq \mathfrak{m}$, by Theorem 9.4.1, and this implies $\mathrm{t}(E, \sigma(E, E')) \leq \mathfrak{m}$. Since E is topologically isomorphic to the quotient space $\widehat{E} = (\bigoplus_{s\in S} E_s)/H$, then $(E', \sigma(E', E))$ is linearly homeomorphic to a closed subspace of $\prod_{s\in S}(E_s', \sigma(E_s', E_s))$.

To prove (16.2) it is enough to show that $\prod_{s\in S}(E'_s, \sigma(E'_s, E_s))$ is an (usco) compact-valued image of a space of the weight at most $\mathfrak{m}$ and then apply Corollary 16.2.2. For $s \in S$ consider $\mathfrak{B}_s$ as a discrete space. The map

$$\psi_s : \mathfrak{B}_s \to 2^{(E'_s,\sigma(E'_s,E_s))}$$

defined by

$$\psi_s(U) := U^\circ \text{ for every } U \in \mathfrak{B}_s$$

is (usco) compact-valued and $E'_s = \bigcup\{\psi_s(U) : U \in \mathfrak{B}_s\}$. The map

$$\psi : \prod_s \mathfrak{B}_s \to 2^{\prod_s(E'_s,\sigma(E'_s,E_s))}$$

defined by

$$\psi((U_s)_s) := \prod_s \psi_s(U_s)$$

for $(U_s)_s \in \prod_s \mathfrak{B}_s$ is compact-valued and (usco) (see also [184, Proposition 3.6]) and satisfies:

$$\prod_s E_s = \bigcup\{\psi((U_s)_s) : (U_s)_s \in \prod_s \mathfrak{B}_s\}.$$

Finally, we get $\omega(\prod_s \mathfrak{B}_s) \leq \mathfrak{m}$ by applying [195, Theorem 2.3.13]. Hence $\mathrm{t}(E, \sigma(E, E')) \leq \mathfrak{m}$. □

Theorem 16.2.3 yields the following classical result due to Kaplansky.

Corollary 16.2.4 *If E is a lcs, then* $\mathrm{t}(E) \leq \chi(E)$, $\mathrm{t}(E, \sigma(E, E')) \leq \chi(E)$.

A lcs E is said to have a $\mathfrak{G}$-*base* [220] if the condition (iii) in Lemma 15.1.2 is satisfied, i.e. there exists a family $\{U_\alpha \in \mathbb{N}^{\mathbb{N}}\}$ of neighbourhoods of zero in E such that $U_\alpha \subset U_\beta$ whenever $\beta \leq \alpha$ in $\mathbb{N}^{\mathbb{N}}$, and such that each neighbourhood of zero in E contains some U_α. Every metrizable lcs admits a $\mathfrak{G}$-base with each U_α determined by the first coordinate of α. Also, if E has a $\mathfrak{G}$-base, every linear subspace of E has a $\mathfrak{G}$-base. If F is a closed linear subspace of E, the quotient space E/F has also a $\mathfrak{G}$-base. Note also that the completion of E has a $\mathfrak{G}$-base.

Proposition 16.2.5 *(i) If $(E_n)_n$ is a sequence of lcs, each of them has a $\mathfrak{G}$-base, the space $E = \prod_{n=1}^{\infty} E_n$ has a $\mathfrak{G}$-base.*

(ii) If $(E_n)_n$ is a defining sequence for a lcs E, each of them has a $\mathfrak{G}$-base, the inductive limit space E has a $\mathfrak{G}$-base.

(iii) If $(E_n)_n$ is a family of lcs each of them has an $\mathfrak{G}$-base, the locally convex direct sum $E = \bigoplus_{n=1}^{\infty} E_n$ has a $\mathfrak{G}$-base.

Proof (i): Let $\{V_\gamma^k) : \gamma \in \mathbb{N}^\mathbb{N}\}$ be a $\mathfrak{G}$-base in E_k for each $k \in \mathbb{N}$. Set:

$$V_\alpha = \prod_{k=1}^{\alpha_1} V_\alpha^k \times \prod_{k=\alpha_1+1}^{\infty} E_k$$

for each $\alpha = (\alpha_n) \in \mathbb{N}^\mathbb{N}$. Clearly, the family $\{V_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is decreasing. Fix $m \in \mathbb{N}$. Let V_k be a neighbourhood of zero in E_k, and let $\beta_k = (\beta_{k,n}) \in \mathbb{N}^\mathbb{N}$ be such that $V_{\beta_k}^k \subset V_k$ for $1 \leq k \leq m$. Set:

$$\alpha_n = \max\{m, \beta_{1,n}, \ldots, \beta_{m,n}\}$$

for $n \in \mathbb{N}$. Then $\alpha = (\alpha_n) \in \mathbb{N}^\mathbb{N}$ for $\alpha_1 \geq m$ and $\alpha \geq \beta_k$ for $1 \leq k \leq m$. Hence

$$V_\alpha \subset \prod_{k=1}^{m} V_{\beta_k}^k \times \prod_{k=m+1}^{\infty} E_k \subset \prod_{k=1}^{m} V_k \times \prod_{k=m+1}^{\infty} E_k.$$

It follows that $\{V_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-base in E.

(ii): Let $(N_k)_k$ be a partition of $\mathbb{N}$ into infinite subsets. Let $\psi_k : \mathbb{N} \to \mathbb{N}$ be a strictly increasing map such that $\psi_k(\mathbb{N}) = N_k$ for $k \in \mathbb{N}$. If $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ for $\alpha \leq \beta$, we have $\alpha \circ \psi_k \leq \beta \circ \psi_k$ for all $k \in \mathbb{N}$. Note that the map $\varphi : \mathbb{N}^\mathbb{N} \to (\mathbb{N}^\mathbb{N})^\mathbb{N}$ defined by $\varphi(\alpha) = (\alpha \circ \psi_k)$ is injective. Moreover, φ is a surjective map. Indeed, for $\beta = (\beta_k) \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$, we set:

$$\alpha(n) = \beta_k(\psi_k^{-1}(n))$$

for $n \in N_k$ and $k \in \mathbb{N}$. Then $\alpha : \mathbb{N} \to \mathbb{N}$ for $n \mapsto \alpha(n)$, and $\varphi(\alpha) = \beta$. Let $\{V_\gamma^k : \gamma \in \mathbb{N}^\mathbb{N}\}$ be a $\mathfrak{G}$-base of absolutely convex neighbourhoods of zero in E_k for each $k \in \mathbb{N}$. For $\alpha \in \mathbb{N}^\mathbb{N}$, set

$$V_\alpha = \sum_{k=1}^{\infty} V_{\alpha \circ \psi_k}^k,$$

where $\sum_{k=1}^{\infty} V_k := \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} V_k$. Then $\{V_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-base of neighbourhoods of zero in E. Indeed, let V be a neighbourhood of zero in E. Then for every $k \in \mathbb{N}$, there exists a neighbourhood of zero V_k in E_k such that $\sum_{k=1}^{\infty} V_k \subset V$. Let $\beta_k \in \mathbb{N}^\mathbb{N}$ be such that $V_{\beta_k}^k \subset V_k$ with $k \in \mathbb{N}$. Then $\beta = (\beta_k) \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$, so $\beta = \varphi(\alpha)$ for some $\alpha \in \mathbb{N}^\mathbb{N}$. Hence

$$V_\alpha = \sum_{k=1}^{\infty} V_{\beta_k}^k \subset \sum_{k=1}^{\infty} V_k \subset V.$$

Thus $\{V_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base of neighbourhoods of zero in E. Part (iii) follows from (ii). □

Uncountable products of spaces with a $\mathfrak{G}$-base need not admit a $\mathfrak{G}$-base, since every lcs with a $\mathfrak{G}$-base belongs to the class $\mathfrak{G}$ and uncountable products are not in the class $\mathfrak{G}$ by Corollary 11.1.2. There are many spaces of this type. By Lemma 15.1.2 every quasibarrelled space in the class $\mathfrak{G}$ has a $\mathfrak{G}$-base. On the other hand, many spaces in $\mathfrak{G}$ do not have a $\mathfrak{G}$-base. Note that there exists a large class of quasibarrelled spaces (not in the class $\mathfrak{G}$) with countable tightness that do not admit a $\mathfrak{G}$-base. Indeed, let X be an uncountable metrizable compact space. The space $C_p(X)$ is quasibarrelled (see [322, Theorem 11.7.3]), and by Proposition 12.2.2 $C_p(X)$ is not in the class $\mathfrak{G}$, so $C_p(X)$ does not have a $\mathfrak{G}$-base. On the other hand, since X is compact, X^n is Lindelöf for any $n \in \mathbb{N}$. Hence $C_p(X)$ has countable tightness by Proposition 9.4.1.

Some (DF)-spaces admit a $\mathfrak{G}$-base; some do not. The next two examples provide $\mathfrak{G}$-bases for some (DF)-spaces:

(i) Every strict (LB)-space is a (DF)-space which, by Proposition 16.2.5 has a $\mathfrak{G}$-base.
(ii) The strong dual E of a metrizable lcs F is a (DF)-space having a $\mathfrak{G}$-base. Indeed, let $(V_n)_n$ be a decreasing basis of absolutely convex neighbourhoods of zero for F. For every $\alpha \in \mathbb{N}^{\mathbb{N}}$, set $U_\alpha = (\bigcap_k n_k V_k)^\circ$. Then $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base in E.

16.3 $\mathfrak{G}$-Bases, Bounding, Dominating Cardinals, and Tightness

The first part of this section deals with two results due to Saxon and Sanchez-Ruiz [536] (see also [109]), showing that:

(a) The *bounding* cardinal $\mathfrak{b}$ is the smallest infinite dimensionality for a metrizable barrelled space.
(b) Every metrizable lcs of dimension less than $\mathfrak{b}$ is spanned by a bounded set, and for any non-normable metrizable lcs E, the minimal size for a fundamental family of bounded sets in E is the *dominating* cardinal $\mathfrak{d}$.

It is worth noticing here that, according to Mazur's result (see [491]), $2^{\aleph_0}$ is the smallest infinite dimensionality for a Fréchet space.

The second part of this section deals with the bounding and dominating cardinals to study spaces in the class $\mathfrak{G}$ with a $\mathfrak{G}$-base.

Given $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ with $\alpha = (a_k)_k$ and $\beta = (b_k)_k$, we write $\alpha \leq^* \beta$ to mean that $a_k \leq b_k$ for almost all $k \in \mathbb{N}$. Thus, $\alpha \leq \beta$ implies $\alpha \leq^* \beta$, but not conversely. It is easy to see that every countable set in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ has an upper bound; this fails for $(\mathbb{N}^{\mathbb{N}}, \leq)$.

The *bounding cardinal* $\mathfrak{b}$ and the *dominating cardinal* $\mathfrak{d}$ are defined as the least cardinality for unbounded, respectively, cofinal subsets of the quasi-ordered space $(\mathbb{N}^{\mathbb{N}}, \leq^*)$; see [525].

A subset C of $\mathbb{N}^{\mathbb{N}}$ is called *cofinal* if for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ there exists $\beta \in C$ such that $\alpha \leq^* \beta$. A subset of $\mathbb{N}^{\mathbb{N}}$ is called *unbounded* (*dominating*) if it is unbounded (cofinal) in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$.

It is clear that in any ZFC-consistent system, one has $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq 2^{\aleph_0}$. The Continuum Hypothesis CH requires all four of these cardinals to coincide. Yet it is ZFC-consistent to assume that any of the three inequalities is strict. Note also that scales, i.e. well-ordered cofinal subsets of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, exist, if and only if $\mathfrak{b} = \mathfrak{d}$, [535, Remark, p.144].

Define the relation equivalence $=^*$ on $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ so that $\alpha =^* \beta$ if and only if $a_k = b_k$ for almost all $k \in \mathbb{N}$. Thus $\alpha =^* \beta$ if and only if $\alpha \leq^* \beta$ and $\beta \leq^* \alpha$.

Let $\hat{a}$ denote the equivalence class represented by α, and note that each $\hat{a}$ is countable. For any metrizable lcs E, define $\mathfrak{d}_b(E)$ as the least cardinality for fundamental systems of bounded sets in E. It is known that $\mathfrak{d}_b(E) \leq 2^{\aleph_0}$; see [606]. It is also known that for any non-normable metrizable lcs E, one has $\aleph_1 \leq \mathfrak{d}_b(E) \leq 2^{\aleph_0}$. We show the following result from [535].

Proposition 16.3.1 *If E is a non-normable metrizable lcs, then $\mathfrak{d}_b(E) = \mathfrak{d}$.*

Proof Let $(U_n)_n$ be a (decreasing) basis of absolutely convex neighbourhoods of zero in E. Let D be a $\mathfrak{d}$-sized dominating subset of $\mathbb{N}^{\mathbb{N}}$. For each $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$, there exists $\beta = (b_n) \in D$ such that $a_n \leq b_n$ for almost all $n \in \mathbb{N}$.

Set $m\alpha := (ma_n)_n$ and

$$\mathbb{D} := \{m\alpha : m \in \mathbb{N}, \alpha \in D\}.$$

Then $|\mathbb{D}| = \mathfrak{d} = \aleph_0\mathfrak{d}$. Set:

$$\mathcal{B} := \{\bigcap_n a_n U_n; \alpha = (a_n) \in \mathbb{D}\}.$$

Note that $\mathcal{B}$ is a fundamental family of bounded sets in E. Indeed, if B is a bounded set in E, for each $n \in \mathbb{N}$, there exists $a_n \in \mathbb{N}$ such that $B \subset a_n U_n$. There exists $\beta = (b_n) \in \mathbb{D}$ such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Hence

$$B \subset \bigcap_n a_n U_n \subset \bigcap_n b_n U_n \in \mathcal{B}.$$

Hence $\mathcal{B}$ is a fundamental family of bounded sets in E and

$$\mathfrak{d}_b(E) \leq |\mathcal{B}| \leq |\mathbb{D}| = \mathfrak{d}.$$

To prove the converse, let $\mathcal{B}$ be a family of bounded sets in E. The space E is assumed to be non-normed, so there exists a sequence $(f_n)_n$ in E' such that each

$f_n(U_n)$ is unbounded. For any $B \in \mathcal{B}$ and $n \in \mathbb{N}$, let $g_B(n)$ be the smallest natural number such that

$$\sup_{x \in B} |f_n(x)| < g_B(n).$$

If $|\mathcal{B}| < \mathfrak{d}$, then $|\{g_B : B \in \mathcal{B}\}| < \mathfrak{d}$. Hence there exists $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$ such that for any $B \in \mathcal{B}$ we have $g_B(n) < a_n$ for infinitely many $n \in \omega$. As we noticed, the set $f_n(U_n)$ is unbounded, so there exists $x_n \in U_n$ such that $|f_n(x_n)| > a_n$. The set $C := \{x_n : n \in \mathbb{N}\}$ is bounded and for $B \in \mathcal{B}$ there exists $n \in \mathbb{N}$ with $a_n > g_B(n)$ such that

$$\sup_{x \in C} |f_n(x)| \geq |f_n(x_n)| > a_n > g_B(n) > \sup_{x \in B} |f_n(x)|.$$

This shows that C is not in $\mathcal{B}$, so $\mathcal{B}$ is not a fundamental family of bounded sets if $|\mathcal{B}| < \mathfrak{d}$. Hence, we have $\mathfrak{d} \leq \mathfrak{d}_b(E)$. □

This easily yields the following well-known fact; see [322].

Corollary 16.3.2 *If E is a metrizable lcs and has a fundamental sequence of bounded sets, the space E is normable.*

Corollary 16.3.3 *The minimal size for a basis of neighbourhoods of zero for the strong dual of a non-normable metrizable lcs is $\mathfrak{d}$.*

We show that the cardinal $\mathfrak{b}$ is the smallest infinite dimensionality for metrizable barrelled spaces; see Theorem 16.3.9. We need the following three technical lemmas from [535] and [536].

Lemma 16.3.4 *There exists a dense barrelled subspace $\psi_{\mathfrak{b}}$ of the space $\mathbb{R}^{\mathbb{N}}$ such that* $\dim(\psi_{\mathfrak{b}}) = \mathfrak{b}$.

Proof Recall that for the ordinals α and β, one has $\alpha < \beta$ if and only if $\alpha \in \beta$. By u_i we denote the i-th coordinate functional defined on $\mathbb{R}^{\mathbb{N}}$. First we show that for $\{r_\beta : \beta \in \mathfrak{b}\} \subset \mathbb{N}^{\mathbb{N}}$, there exists

$$\{g_\beta : \beta \in \mathfrak{b}\} \subset \mathbb{N}^{\mathbb{N}}$$

such that, if $\{s_\beta : \beta \in \mathfrak{b}\} \subset \mathbb{R}^{\mathbb{N}}$ satisfies $|s_\beta(n)| \leq r_\beta(n)$ for all $\beta \in \mathfrak{b}$ and $n \in \mathbb{N}$, and if for all $k \in \mathbb{N}$ one has $v_k = \sum_{i=1}^{n_k} t_{ki} u_i$ with $t_{kn_k} \neq 0$ and $n_{k+1} > n_k$ in $\mathbb{N}$, then there exists $\alpha_0 \in \mathfrak{b}$ such that

$$\{v_k(x) + u_k(y) : k \in \mathbb{N}\}$$

is unbounded if $x, y \in \mathbb{R}^{\mathbb{N}}$ are of the form

$$x = g_\beta + \sum_{\alpha \in \sigma} a_\alpha g_\alpha, \quad y = s_\beta + \sum_{\alpha \in \sigma} a_\alpha s_\alpha$$

for some finite subset σ of β with $\beta > \alpha_0$. Without loss of generality, we may assume that each r_β is an increasing function. Fix $\{f_\beta : \beta \in \mathfrak{b}\}$ an unbounded subset of $\mathbb{N}^{\mathbb{N}}$ with each f_β increasing.

Assume that for $\beta \in \mathfrak{b}$, we have already defined $\{g_\alpha : \alpha < \beta\}$. Since $\mathfrak{b}$ is minimal, let f be an upper bound in $\mathbb{N}^{\mathbb{N}}$ for

$$\{f_\gamma[r_\beta + \sum_{\alpha\in\sigma} p_\alpha r_\alpha] + \sum_{\alpha\in\sigma} p_\alpha g_\alpha : \gamma \in \beta,\ \sigma \subset \beta \text{ is finite},\ \ p_\alpha \in \mathbb{N}\}.$$

Without loss of generality, one can assume that f is strictly increasing. For each $n \in \mathbb{N}$, set:

$$g_\beta(1) := f(1),\ \ g_\beta(n+1) := f(n+1) + 2g_\beta(n)f(n+1).$$

Note that the scalar sequence $(v_k(x) + u_k(y))_k$ is unbounded. Indeed, choose $\{s_\beta : \beta \in \mathfrak{b}\}$ and $(v_k)_k$ as required. There exists $h \in \mathbb{N}^{\mathbb{N}}$ such that

$$h(j)|t_{kn_k}| - \sum_{i=1}^{n_k-1} |t_{ki}| > j$$

for each $k \in \mathbb{N}$ and $n_{k-1} < j \leq n_k$, where $n_0 = 0$.

Fix $\alpha_0 \in \mathfrak{b}$ such that $f_{\alpha_0} \not\leq^* h$. We need a couple of inequalities which will be used to determine the unboundedness of the sequence $(v_k(x)+u_k(y))_k$. If $\beta > \alpha_0$ and x and y are of the form previously given, choose $p_\alpha \in \mathbb{N}$ such that $|a_\alpha| \leq p_\alpha$ for each $\alpha \in \sigma$. Clearly there exists $M > 0$ such that

$$g_\beta(n) \geq f(n) \geq (f_{\alpha_0}[r_\beta + \sum_{\alpha\in\sigma} p_\alpha r_\alpha] + \sum_{\alpha\in\sigma} p_\alpha g_\alpha)(n)$$

for each $n \geq M$. This yields:

$$|x(n+1)[x(n)]^{-1}| = |[g_\beta(n+1) + \sum_{\alpha\in\sigma} a_\alpha g_\alpha(n+1)][g_\beta(n) + \sum_{\alpha\in\sigma} a_\alpha g_\alpha(n)]^{-1}|$$

$$\geq [g_\beta(n+1) - \sum_{\alpha\in\sigma} p_\alpha g_\alpha(n+1)][g_\beta(n) + \sum_{\alpha\in\sigma} a_\alpha p_\alpha(n)]^{-1}$$

$$\geq [g_\beta(n+1) - f(n+1)][2g_\beta(n)]^{-1} = 2g_\beta(n)f(n+1)[2g_\beta(n)]^{-1} = f(n+1)$$

for all $n \geq M$.

Note that $|x(n)| \leq 2g_\beta(n)$. As $f(n+1) \geq n+1$ for all $n \in \mathbb{N}$, the last inequalities show that there exists $L > M$ such that $|x(n)| \geq \max_{1\leq i\leq n} |x(i)|$ for all $n > L$. Since $f_{\alpha_0} \leq^* f$ and $f_{\alpha_0} \not\leq^* h$, there exists $j > L$ such that

$$f(j) \geq f_{\alpha_0}(j) > h(j)$$

and $n_{k-1} < j \leq n_k$ for some $k \in \mathbb{N}$. Then, since f is strictly increasing, $f(n_k) > h(j)$ and

$$f_{\alpha_0}(n_k)|t_{kn_k}| > h(j)|t_{kn_k}| > 1.$$

Now we are ready to get the final conclusion:

$$|v_k(x) + u_k(y)| \geq |v_k(x)| - |u_k(y)| = |\sum_{i=1}^{n_k} t_{ki}x(i)| - |y(k)| \geq$$

$$|t_{kn_k}||(g_\beta + \sum_{\alpha\in\sigma} a_\alpha g_\alpha)(n_k)| - \sum_{i=1}^{n_k-1} |t_{ki}x(i)| - |y(k)| \geq$$

$$|t_{kn_k}|(g_\beta - \sum_{\alpha\in\sigma} p_\alpha g_\alpha)(n_k) - |x(n_k-1)| \sum_{i=1}^{n_k-1} |t_{ki}| - (r_\beta + \sum_{\alpha\in\sigma} p_\alpha r_\alpha)(k) \geq$$

$$|t_{kn_k}|[f(n_k) + 2g_\beta(n_k-1)f(n_k) - \sum_{\alpha\in\sigma} p_\alpha g_\alpha(n_k)] - 2g_\beta(n_k-1) \sum_{i=1}^{n_k-1} |t_{ki}| -$$

$$(r_\beta + \sum_{\alpha\in\sigma} p_\alpha r_\alpha(n_k) \geq |t_{kn_k}|(f - (f_\alpha[r_\beta + \sum_{\alpha\in\sigma} p_\alpha r_\alpha] +$$

$$\sum_{\alpha\in\sigma} p_\alpha g_\alpha))(n_k) + 2g_\beta(n_k-1)[f(n_k)|t_{kn_k}| - \sum_{i=1}^{n_k-1} |t_{ki}|] \geq$$

$$2[f(n_k)|t_{kn_k}| - \sum_{i=1}^{n_k-1} |t_{ki}|] \geq 2[h(j)|t_{kn_k}| - \sum_{i=1}^{n_k-1} |t_{ki}|] > 2j.$$

This shows that the sequence $(v_k(x) + u_k(y))_k$ is unbounded as claimed. Now let

$$\psi_{\mathfrak{b}} := F + \text{span}\{e_i : i \in \mathbb{N}\},$$

where F is the linear span of $\{g_\beta : \beta \in \mathfrak{b}\}$ and $e_i, i \in \mathbb{N}$ are the unit vectors. Clearly $\psi_{\mathfrak{b}}$ is dense in $\mathbb{R}^{\mathbb{N}}$. The dual of $\psi_{\mathfrak{b}}$ is $\text{span}\{u_i : i \in \mathbb{N}\}$.

We prove that $\psi_{\mathfrak{b}}$ is barrelled (in fact Baire-like, since $\psi_{\mathfrak{b}}$ is metrizable; see Corollary 2.4.4). Indeed, for $\{s_\beta : \beta \in \mathfrak{b}\} = \{0\}$ and an infinite-dimensional set $\{v_k : k \in \mathbb{N}\}$ in E', we apply the first part of the proof to get $\alpha_0 \in \mathfrak{b}$ such that $\{v_k(x) : k \in \mathbb{N}\}$ is unbounded for $x \in \text{span}\{g_\beta : \beta \in \mathfrak{b}\}$ having a component beyond α_0. This shows that every bounded set T in the *weak** dual of $\psi_{\mathfrak{b}}$ is finite-dimensional; hence, T is equicontinuous. This proves that $\psi_{\mathfrak{b}}$ is barrelled. □

Lemma 16.3.5 *Let $\mathcal{B}$ be a family of bounded sets in a metrizable lcs E such that $|\mathcal{B}| < \mathfrak{b}$. Then there exist scalars t_B such that $A := \bigcup\{t_B B : B \in \mathcal{B}\}$ is bounded in E.*

Proof Fix a basis $(U_n)_n$ of absolutely convex neighbourhoods of zero in E. If $B \in \mathcal{B}$ and $n \in \mathbb{N}$, there exists $t_B(n) \in \mathbb{N}$ such that $B \subset t_B(n)U_n$. Since $|\mathcal{B}| < \mathfrak{b}$, there exists $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$ such that $a_n \geq t_B(n)$ for each $B \in \mathcal{B}$ and almost all $n \in \mathbb{N}$. Hence, for each $B \in \mathcal{B}$, we can find $0 < t_B \leq 1$ such that

$$t_B B \subset \bigcap_n t_B(n)U_n \subset \bigcap_n a_n U_n.$$

Since $\bigcap_n a_n U_n$ is bounded, the set $A := \bigcup\{t_B B : B \in \mathcal{B}\}$ is bounded. □

Lemma 16.3.6 *Let A be a set in a lcs E with $|A| < \mathfrak{b}$. Let $(f_n)_n \subset E'$ such that $f_n(x) \to 0$ for each $x \in A$. Then there exists an increasing sequence $(s_i)_i$ in $\mathbb{N}$ with $1 \leq s_i \leq i+1$ for each $i \in \mathbb{N}$ such that $s_i \to \infty$ and $(s_i f_i(x))_i$ is bounded for each $x \in A$.*

Proof For $x \in A$, set:

$$a_x(n) := \min\{k \in \mathbb{N} : |f_i(x)| \leq (n+1)^{-1} \text{ for all } i \geq k\}.$$

Clearly $\alpha_x = (a_x(n)) \in \mathbb{N}^{\mathbb{N}}$. By the definition of $\mathfrak{b}$, we have $\beta = (b_n) \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha_x \leq^* \beta$ for all $x \in A$. Without loss of generality, one can assume that $b_1 = 1$ and $(b_n)_n$ is strictly increasing. The definition of α_x ensures that for each $x \in A$ one has $|f_i(x)| \leq (n+1)^{-1}$ for all $i \geq a_x(n)$ and $n \in \mathbb{N}$. Then $(n+1)|f_i(x)| \leq 1$ if $b_n \leq i < b_{n+1}$ for almost all $n \in \mathbb{N}$. Consequently, the set

$$\{(n+1)f_i(x) : b_n \leq i < b_{n+1}, n \in \mathbb{N}\}$$

is bounded for each $x \in A$. It is enough to set $s_i := n+1$ for $b_n \leq i < b_{n+1}$ and $n \in \mathbb{N}$. □

Lemma 16.3.6 provides the following.

Corollary 16.3.7 (Saxon–Sanchez-Ruiz) *Let E be an infinite-dimensional normed barrelled space. The dimension of E is at least $\mathfrak{b}$.*

Proof Assume that E has dimension less than $\mathfrak{b}$. Then, using Josefson–Nissenzweig's theorem 15.3.2 for the completion F of E, we obtain a sequence $(f_n)_n$ in $E' = F'$ such that $f_n(x) \to 0$ for all $x \in E$ and $\|f_n\| = 1$ for all $n \in \mathbb{N}$. Lemma 16.3.6 applies to find a sequence $(s_n)_n$ in $\mathbb{N}$ such that $(s_n f_n)_n$ is $\sigma(E', E)$-bounded and $\|s_n f_n\| \to \infty$, a contradiction, since E is barrelled, so any $\sigma(E', E)$-bounded set must be equicontinuous. □

Proposition 16.3.8 (Saxon–Sanchez-Ruiz) *Any metrizable lcs of dimension less than $\mathfrak{b}$ is spanned by a bounded set.*

Proof The conclusion follows from Lemma 16.3.5. □

We note the following stronger result than Corollary 16.3.7. The proof follows from Lemma 16.3.5 and Corollary 16.3.7.

Theorem 16.3.9 (Saxon–Sanchez-Ruiz) *The least infinite dimensionality of a metrizable barrelled space E is $\mathfrak{b}$.*

Proof If E is normed, the conclusion follows from Corollary 16.3.7. If the non-normable infinite-dimensional metrizable barrelled space E has dimension less than $\mathfrak{b}$, then E contains a bounded barrel (Lemma 16.3.5). Hence E is a normed space, a contradiction. □

In particular, note that $\psi_{\mathfrak{b}}$ is not spanned by a bounded set. Indeed, if $\psi_{\mathfrak{b}}$ is spanned by a bounded set B, it must be normable, since the closed absolutely convex hull L of B is a barrel in $\psi_{\mathfrak{b}}$, so L is a neighbourhood of zero (which yields that $\psi_{\mathfrak{b}}$ is normed).

The second part of this section deals with spaces in the class $\mathfrak{G}$ and cardinals $\mathfrak{b}$ and $\mathfrak{d}$. First observe that non-metrizable lcs with $\mathfrak{G}$-bases have precisely limited characters. We prove the following proposition [220].

Proposition 16.3.10 *The character $\chi(E)$ of a non-metrizable lcs E having a $\mathfrak{G}$-base satisfies the condition $\mathfrak{b} \leq \chi(E) \leq \mathfrak{d}$.*

Proof Let $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-base for E. By the definition of $\mathfrak{d}$, there exists a cofinal set D in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ with $|D| = \mathfrak{d}$. Then the set $\hat{D} := \bigcup\{\hat{\alpha} : \alpha \in D\}$ satisfies $\left|\hat{D}\right| = \aleph_0 \cdot \mathfrak{d} = \mathfrak{d}$ and is cofinal in $(\mathbb{N}^{\mathbb{N}}, \leq)$ so that $\{U_\beta : \beta \in \hat{D}\}$ is a basis of neighbourhoods of zero of cardinality $\mathfrak{d}$, where $\hat{\beta}$ is the countable equivalence class defined by β; see the part before Proposition 16.3.1. This implies that $\chi(E) \leq \mathfrak{d}$. Since the cardinals are well-ordered, there exists a subset A of $\mathbb{N}^{\mathbb{N}}$ with $|A| = \chi(E)$ such that $\{U_\alpha : \alpha \in A\}$ is a basis of neighbourhoods of zero in E. Observe that $|A| \geq \mathfrak{b}$. Suppose that $|A| < \mathfrak{b}$. Then, by the definition of $\mathfrak{b}$, there is some $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha \leq^* \beta$ for every $\alpha \in A$. Hence, for every $\alpha \in A$, there exists $\gamma \in \hat{\beta}$ such that $\alpha \leq \gamma$, being $\hat{\beta}$ the countable equivalence class defined by β; see the part before Proposition 16.3.1. It follows that $\{U_\gamma : \gamma \in \hat{\beta}\}$ is a countable basis of neighbourhoods of zero, a contradiction. □

This and Corollary 16.2.4 yield the following.

Corollary 16.3.11 *If a lcs space E has a $\mathfrak{G}$-base, the tightness of E and $(E, \sigma(E, E'))$ is at most $\mathfrak{d}$.*

For the convenience of the reader, recall again the example of a non-distinguished Fréchet space attributed to Grothendieck and Köthe, [374, 31.7]. This is the vector space $E := \lambda_1$ of all numerical double sequences $x = (x_{ij})$ such that for each $n \in \mathbb{N}$, we define:

$$p_n(x) = \sum_{i,j} \left|a_{ij}^{(n)} x_{ij}\right| < \infty,$$

where $a_{ij}^{(n)} = j$ for $i \leq n$ and all j, $a_{i,j}^{(n)} = 1$ for $i > n$ and all j. The seminorms p_n for $n \in \mathbb{N}$ generate a locally convex topology under which E is a Fréchet space. The dual E' is identified with the space of double sequences $u = \left(u_{ij}\right)$ such that $\left|u_{ij}\right| \leq ca_{ij}^{(n)}$ for all i, j and suitable $c > 0$, $n \in \mathbb{N}$.

We show (using the concept of tightness) that the Köthe echelon space is indeed non-distinguished; see [220].

Theorem 16.3.12 (Ferrando–Kąkol–López-Pellicer–Saxon) *The tightness of the strong dual $(E', \beta(E', E))$ of the Köthe echelon space $E := \lambda_1$ is $\mathfrak{d}$, the dominating cardinal. Moreover, the tightness of $\left(E', \sigma\left(E', E''\right)\right)$ is between $\mathfrak{b}$ and $\mathfrak{d}$.*

Proof By $\mathrm{t}(E')$ we denote the tightness of $(E', \beta(E', E))$. For each $f : \mathbb{N} \to \mathbb{N}$, define the double sequence $v^f := \left(v_{ij}^f\right) \in E'$ so that, for all $i, j \in \mathbb{N}$,

$$v_{ij}^f = \begin{cases} 0 & \text{if } j \leq f(i), \\ 1 & \text{if } j > f(i). \end{cases}$$

Thus, if i is fixed, the single sequence $\left(v_{ij}^f\right)_j$ consists of zeros for the first $f(i)$ coordinates, and ones thereafter. Set $A = \left\{v^f : f \in \mathbb{N}^{\mathbb{N}}\right\}$. We prove the theorem in three steps.

Claim 1. The origin 0 belongs to the $\beta\left(E', E\right)$-closure of A. Indeed, let B be an arbitrary bounded set in E, choose $g \in \mathbb{N}^{\mathbb{N}}$ such that $g(i)$ is an upper bound for $p_i(B)$ for each $i \in \mathbb{N}$, and set $f(i) := 2^i \cdot g(i)$, thus determining $v^f \in A$.

Let $x := \left(x_{ij}\right)$ be an arbitrary member of B. Then

$$\begin{aligned} \left|\left\langle x, v^f \right\rangle\right| &\leq \sum_{i,j} \left|x_{ij} v_{ij}^f\right| = \sum_{i \geq 1} \sum_{j > f(i)} \left|x_{ij}\right| \\ &\leq \sum_{i \geq 1} \sum_{j > f(i)} \frac{j}{f(i)} \left|x_{ij}\right| \\ &\leq \sum_{i \geq 1} \frac{1}{f(i)} \sum_{j \geq 1} j \left|x_{ij}\right| \leq \sum_{i \geq 1} \frac{1}{f(i)} p_i(x) \\ &\leq \sum_{i \geq 1} \frac{1}{2^i} = 1. \end{aligned}$$

This proves that $v \in A \bigcap B^\circ$. Thus A meets every $\beta\left(E', E\right)$-neighbourhood of zero. The claim 1 is proved.

Claim 2. $\mathrm{t}\left(E'\right) \geq \mathfrak{d}$. Indeed, it is enough to show that 0 is not in the closure of any subset of A having fewer than $\mathfrak{d}$ elements. Let $C := \left\{v^f : f \in \mathcal{F}\right\}$, where $\mathcal{F}$ is a subset of $\mathbb{N}^{\mathbb{N}}$ with $|\mathcal{F}| < \mathfrak{d}$. By the definition of $\mathfrak{d}$, the set $\mathcal{F}$ is not cofinal in $\left(\mathbb{N}^{\mathbb{N}}, \leq^*\right)$. Hence there exists $h \in \mathbb{N}^{\mathbb{N}}$ such that $h \leq^* f$ does not hold for every

$f \in \mathcal{F}$. For each $r \in \mathbb{N}$, define $x^r = \left(x_{ij}^r\right)_{ij} \in E$ by the formula:

$$x_{ij}^r = \begin{cases} 2 & \text{if } (i,j) = (r, h(r)), \\ 0 & \text{if } (i,j) \neq (r, h(r)). \end{cases}$$

Note that $D := \{x^r : r \in \mathbb{N}\}$ is bounded in E since, for a given n, we have $p_n(x^r) = 2$ for all $r > n$, implying that $p_n(D)$ is a finite, hence bounded set. Let f be an arbitrary member of $\mathcal{F}$. Because $h \leq^* f$ fails, there exists some $r \in \mathbb{N}$ with $h(r) > f(r)$. Therefore,

$$\left\langle x^r, v^f \right\rangle = \sum_{i,j} x_{ij}^r v_{ij}^f = 2 \cdot v_{r,h(r)}^f = 2.$$

Finally we note that $v^f \notin D^\circ$ and D° is a $\beta(E', E)$-neighbourhood of 0 in E' which misses C.

Claim 3. $\mathfrak{t}\left(E'\right) \leq \mathfrak{d}$. This follows from Corollary 16.3.11 and (ii) by the end of Sect. 16.2.

Now we prove the second part of theorem.

Step I. Since the strong dual E' of any metrizable lcs E has a $\mathfrak{G}$-base, by Corollary 16.3.11 we have that $t\left(E', \sigma\left(E', E''\right)\right) \leqslant \mathfrak{d}$.

Step II. We prove that $t\left(E', \sigma\left(E', E''\right)\right) \geqslant \mathfrak{b}$. Define the set A as in the proof of the previous part. The $\beta\left(E', E\right)$-closure of A contains 0, so does its closure in the coarser topology $\sigma\left(E', E''\right)$. It is enough to prove that 0 is not in the $\sigma\left(E', E''\right)$-closure of any subset $C := \left\{v^f : f \in \mathcal{F}\right\}$ of A with $|\mathcal{F}| < \mathfrak{b}$. By definition of $\mathfrak{b}$, the set $\mathcal{F}$ is bounded in $\left(\mathbb{N}^\mathbb{N}, \leqslant^*\right)$. Hence $g \in \mathbb{N}^\mathbb{N}$ such that $f \leqslant^* g$ for every $f \in \mathcal{F}$. Thus for each $f \in \mathcal{F}$ there exists $m(f) \in \mathbb{N}$ such that $f(n) \leqslant g(n)$ for all $n > m(f)$.

If we define $h \in \mathbb{N}^\mathbb{N}$ so that each $h(n) = g(n) + 1$, then, for every $f \in \mathcal{F}$,

$$h(n) > f(n) \text{ for all } n > m(f).$$

Just as before, we define x^r in terms of this h and note that $D := \{x^r : r \in \mathbb{N}\}$ is bounded in E, so that its polar D° is a neighbourhood of 0 in E'. Thus D, viewed canonically as a subset of E'', is equicontinuous on E'. The Alaoglu-Bourbaki theorem provides $z \in E''$ such that $x^r \in z + V$ for infinitely many $r \in \mathbb{N}$ whenever V is a $\sigma\left(E'', E'\right)$-neighbourhood of 0. For an arbitrary $f \in \mathcal{F}$, set $V = \left\{u \in E'' : \left|u\left(v^f\right)\right| < 1\right\}$ and choose $r > m(f)$ such that $x^r - z \in V$. Then

$$2 - \left|z\left(v^f\right)\right| \leqslant \left|2 - z\left(v^f\right)\right| = \left|\left\langle x^r, v^f\right\rangle - z\left(v^f\right)\right| = \left|\left(x^r - z\right)\left(v^f\right)\right| < 1,$$

which implies that $\left|z\left(v^f\right)\right| > 1$. Hence $\{z\}^\circ$ is a $\sigma\left(E', E''\right)$-neighbourhood of 0 which excludes each $v^f \in C$. □

This theorem combined with Theorem 12.2.1 shows that the Köthe echelon space $E = \lambda_1$ is non-distinguished.

In [611, (24), p. 66], Valdivia showed that for a Fréchet space E the second dual $(E'', \sigma(E'', E'))$ is a K-analytic space if and only if $(E', \mu(E', E''))$ is barrelled. Consequently, $(E'', \sigma(E'', E'))$ is K-analytic for any distinguished Fréchet space E. Here is another argument: if a Fréchet space E is distinguished, then $(E', \beta(E', E))$ is a quasibarrelled (DF)-space. Then by Theorem 12.2.1 the space $(E', \sigma(E', E''))$ has countable tightness. Consequently, using Theorem 12.1.4, we deduce that $(E'', \sigma(E'', E'))$ is K-analytic. On the other hand, it is easy to deduce from [374, 29.4.(3)] that for a Fréchet space E, every locally bounded (i.e. bounded on bounded sets) linear functional on $(E', \beta(E', E))$ is continuous if and only if $(E', \mu(E', E''))$ is bornological.

In [370, 4] Kōmura constructed a non-distinguished Fréchet space E such that every locally bounded linear functional on $(E', \beta(E', E))$ is continuous. For non-distinguished Köthe echelon spaces λ_1, the situation is different. Proposition 16.1.6 applies to provide another (simpler) proof of the following deep result of Bastin and Bonet [73, Theorem 2].

Corollary 16.3.13 *If λ_1 is a non-distinguished Köthe echelon space, there exists on $(\lambda_1', \beta(\lambda_1', \lambda_1))$ a locally bounded discontinuous linear functional.*

Proof Since, by Theorem 16.3.12, the space $(\lambda_1', \sigma(\lambda_1', \lambda_1''))$ does not have countable tightness, we apply Proposition 16.1.6 to deduce that the Mackey space $\mu(\lambda_1', \lambda_1'')$ is not quasibarrelled. □

Valdivia [611] invented a non-distinguished Fréchet space whose *weak** bidual is quasi-Suslin and not K-analytic. We prove that Köthe's original non-distinguished Fréchet space provides the same effect.

Example 16.4.6 deals with certain spaces $C_c(X)$; see [220]. In fact we will work with spaces $C_c(\kappa) := C_c([0, \kappa))$, where κ is an infinite ordinal. The space $C_c(\omega_1)$ was studied by Morris and Wulbert [457], where ω_1 is the first uncountable ordinal. The cardinal of $[0, \omega_1)$ is the first uncountable cardinal $\aleph_1$.

The set $[0, \kappa)$ of all ordinals less than κ is endowed with its usual interval topology. For each ordinal α, the closed interval $[0, \alpha]$ is compact. However, for κ an infinite ordinal, $[0, \kappa)$ is not compact, and has a fundamental system of compact sets consisting of the sets $[0, \alpha]$ as the ordinal α ranges over a cofinal subset A of $[0, \kappa)$. Clearly the compact-open topology for $C_c(\kappa)$ has a basis of neighbourhoods of zero described by sets of the form

$$U_{n,\alpha} := \{f \in C(\kappa) : |f(\gamma)| \leq n^{-1}, \gamma \in [0, \alpha]\},$$

where $n \in \mathbb{N}$ and $\alpha \in A$. Since

$$|\{U_{n,\alpha} : n \in \mathbb{N}, \alpha \in A\}| = \aleph_0 \cdot |A| = |A|,$$

we obtain that the character $\chi(C_c(\kappa))$ equals the cofinality $\mathrm{cf}(\kappa)$.

The *cofinality* $\mathrm{cf}(\kappa)$ of an infinite ordinal κ is the smallest cardinality for cofinal subsets of $[0, \kappa)$, where $\mathcal{S} \subset [0, \kappa)$ is *cofinal* if for each ordinal $\alpha < \kappa$, i.e. for each $\alpha \in [0, \kappa)$, there exists $\beta \in \mathcal{S}$ such that $\alpha \leq \beta$.

The space $C_c(\kappa)$ is a Fréchet space or a (DF)-space provided $\mathrm{cf}(\kappa) = \aleph_0$, or $\mathrm{cf}(\kappa) > \aleph_0$, respectively. Indeed, it is easy to see that $C_c(\kappa)$ is sequentially complete, and hence it is a Fréchet space if $\mathrm{cf}(\kappa) = \aleph_0$. Warner [636] proved that $C_c(X)$ is a (DF)-space if and only if every countable union of compact sets in X is relatively compact. A countable union of compact sets in $[0, \kappa)$ is contained in a countable union of closed intervals, and their right-end points have supremum $\beta < \kappa$ if $\mathrm{cf}(\kappa)$ is uncountable. Therefore, the countable union of compact sets is contained in a compact interval $[0, \beta]$, so it is relatively compact.

We prove the following.

Proposition 16.3.14 $t\left(C_c\left(\kappa\right)\right) = t\left(\sigma\left(C_c\left(\kappa\right), C_c\left(\kappa\right)'\right)\right) = \chi\left(C_c\left(\kappa\right)\right) = \mathrm{cf}(\kappa)$.

Proof The set C of characteristic functions of the open intervals (α, κ) is a subset of $C_c\left(\kappa\right)$ whose closure contains 0. If B is any subset of C of size less than $\mathrm{cf}(\kappa)$, the collection of left endpoints has the supremum $\beta < \kappa$, so that all members of B are identically one on the open interval (β, κ). Choose $\gamma \in (\beta, \kappa)$. The evaluation functional δ_γ is in the dual $C_c\left(\kappa\right)'$ and bounds B away from 0. Hence zero is not in the closure of B in $\sigma\left(C_c\left(\kappa\right), C_c\left(\kappa\right)'\right)$. This shows that the tightness of both the original and weak topologies for $C_c\left(\kappa\right)$ is at least $\mathrm{cf}(\kappa)$. Now we apply Corollary 16.2.4; we deduce that the tightness of the original and weak topologies for $C_c\left(\kappa\right)$ is at most $\mathrm{cf}(\kappa)$. □

We collect a couple of examples providing more spaces with (and without) $\mathfrak{G}$-bases. Consider the Banach space $\ell^p(\Lambda)$, where p is fixed with $1 \leq p < \infty$ and Λ is an uncountable indexing set. Let D be the closed unit ball in $\ell^p(\Lambda)$. For each $S \subset \Lambda$, we define:

$$E_S := \{u \in \ell^p(\Lambda) : u(x) = 0, x \notin S\},$$

and for each countable $T \subset \Lambda$ and each $n \in \mathbb{N}$, define also:

$$[n, T] := (n^{-1}D) + E_{\Lambda \setminus T}.$$

By E we denote the space $\ell^p(\Lambda)$ endowed with the locally convex topology ξ having as a basis of neighbourhoods of zero all sets of the form $[n, T]$. Note that, for each countable T, the subspaces E_T and $E_{\Lambda \setminus T}$ are topologically complementary in E, and E_T inherits the same Banach topology from E as it does from the Banach space $\ell^p(\Lambda)$, and the dual of E is the same as that of $\ell^p(\Lambda)$.

Observe that $E := (E, \xi)$ is a sequentially complete non-quasibarrelled (DF)-space which does not have countable tightness and whose *weak** dual is K-analytic. Indeed, note that since each sequence in E is contained in a Banach subspace of E, also the sequential completeness is clear. Since ξ is compatible with the Banach topology, the *weak** dual of E is K-analytic. Also $(nD)_n$ forms a fundamental

sequence of bounded sets. Then, to prove that E is a (DF)-space, we need to check that E is $\aleph_0$-quasibarrelled. Indeed, let $(U_n)_n$ be a sequence of absolutely convex closed neighbourhoods of zero whose intersection U is bornivorous in E. For each $n \in \mathbb{N}$, there has 0 in its closure and is not in the closure of any countable subset of B). It exists a countable $T_n \subset \Lambda$ and $k_n \in \mathbb{N}$ such that $[k_n, T_n] \subset U_n$. Therefore, $E_{\Lambda\setminus T_n} \subset U_n$, and $E_{\Lambda\setminus T} \subset U$, where $T := \bigcup_n T_n$. Since E_T is a Banach space, it is $\aleph_0$-barrelled, and $U \cap E_T$ is a relative neighbourhood of zero. Since U intersects both summands E_T and $E_{\Lambda\setminus T}$ in neighbourhoods of zero, it is a neighbourhood of zero in E. This insures that E is a (DF)-space, and it is clear that E is not Mackey, since it is not the Banach space $\ell^p(\Lambda)$. Therefore, E cannot be quasibarrelled, nor, can it have countable tightness by Proposition 16.1.6.

(An alternative proof: The set B of all characteristic functions of the singleton subsets of Λ has 0 in its closure but not in the closure of any countable subset of B.) Note that $E \in \mathfrak{G}$ for every choice of uncountable Λ, but we will prove that E admits a $\mathfrak{G}$-base only when Λ is restricted under an axiomatic assumption milder than CH.

Example 16.3.15 If $\aleph_1 = \mathfrak{b} = |\Lambda|$, then E has a $\mathfrak{G}$-base.

Proof By the definition of $\mathfrak{b}$, there is an injective map φ from $[0, \mathfrak{b})$ onto a set A unbounded in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, and by the assumption there is an injective map ψ from Λ onto $[0, \mathfrak{b})$.

For arbitrary $\sigma = (a_1, a_2, \dots) \in \mathbb{N}^{\mathbb{N}}$, let $\beta(\sigma)$ be the first member of $[0, \mathfrak{b})$ such that $\varphi(\beta(\sigma)) \not\leq^* (a_2, a_3, \dots)$, and define the corresponding neighbourhood U_σ of zero by

$$U_\sigma := [a_1, \psi^{-1}([0, \beta(\sigma)])].$$

Note that $\beta(\sigma) < \mathfrak{b} = \aleph_1$ implies the set $[0, \beta(\sigma)]$ is countable. Thus $\psi^{-1}([0, \beta(\sigma)])$ is a countable subset of Λ; hence, U_σ is a neighbourhood of zero in E. Clearly $\sigma \leq \tau$ implies $\beta(\sigma) \leq \beta(\tau)$ and $U_\tau \subset U_\sigma$. To justify this inclusion, we need to compare the first coordinates. Finally $\{U_\sigma : \sigma \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base because for given $n \in \mathbb{N}$ and a countable $T \subset \Lambda$, set $\alpha = \sup \psi(T)$ and note that $\alpha < \mathfrak{b}$, since $\mathfrak{b}$ has uncountable cofinality and $\psi(T)$ is countable. Since $\varphi([0, \alpha])$ has less than $\mathfrak{b}$ elements, it is bounded in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ by some $(a_2, a_3, \dots)$. Putting $a_1 = n$ and $\sigma = (a_1, a_2, a_3, \dots)$ we have that $\beta(\sigma) \notin [0, \alpha]$, so that $[0, a] \subset [0, \beta(\sigma)]$. It follows that

$$T \subset \psi^{-1}([0, \alpha]) \subset \psi^{-1}([0, \beta(\sigma)]$$

and finally that $U_\alpha \subset [n, T]$. □

Example 16.3.16 If $\aleph_1 < \mathfrak{b}$, then E does not admit a $\mathfrak{G}$-base.

Proof If E has a $\mathfrak{G}$-base, then E_S has a $\mathfrak{G}$-base, where S is a subset of Λ of size $\aleph_1$. Since the character of E_S is $\aleph_1$, we reach a contradiction with Proposition 16.3.10 and the assumption that $\aleph_1 < \mathfrak{b}$. □

Example 16.3.17 If $|\Lambda| < \mathfrak{b}$ or $|\Lambda| > \mathfrak{d}$, then E does not admit a $\mathfrak{G}$-base.

Proof Always $\aleph_1 \leq |\Lambda|$, so the inequality $|\Lambda| < \mathfrak{b}$ would imply $\aleph_1 < \mathfrak{b}$, and the conclusion follows from the previous example. Now suppose that $|\Lambda| > \mathfrak{d}$ and that $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-base for E. Let $\mathcal{D}$ be a cofinal subset of $(\mathbb{N}^\mathbb{N}, \leq^*)$ of size $\mathfrak{d}$. Then $\hat{\mathcal{D}} := \bigcup\{\hat{\alpha} : \alpha \in \mathcal{D}\}$ is cofinal in $(\mathbb{N}^\mathbb{N}, \leq)$ and still of size $\aleph_0 \cdot \mathfrak{d} = \mathfrak{d}$. The cofinality of $\hat{\mathcal{D}}$ in $(\mathbb{N}^\mathbb{N}, \leq)$ ensures that $\mathcal{U} := \{U_\alpha : \alpha \in \hat{\mathcal{D}}\}$ is a base of neighbourhoods of zero in E. We may choose a fixed

$$t \in \Lambda \setminus \bigcup\{S(\alpha) : \alpha \in \hat{\mathcal{D}}\},$$

since this last union has size $\aleph_0 \cdot \mathfrak{d} = \mathfrak{d}$ due to the countability of the sets $S(\alpha)$ used in the definition of U_α and $\mathfrak{d} < |\Lambda|$. Thus

$$\chi_t \in E_{\Lambda \setminus S(\alpha)} \subset U_\alpha$$

for each $\alpha \in \hat{\mathcal{D}}$, contradicting the fact that $\mathcal{U}$ is a basis of neighbourhoods of zero in the Hausdorff space E. □

16.4 More about the Morris–Wulbert Space $C_c(\omega_1)$

It is interesting to know when precisely the tightness of the space $C_c(X)$ is countable; the same problem for $C_p(X)$ has been discussed and solved in previous chapters. Lemma 16.4.1 will be used to show that the space $C_c(\omega_1)$ does not have countable tightness.

We shall say that an open cover Σ of X is *compact-open* if every compact subset of X is contained in some member of Σ. The following fact is due to McCoy.

Lemma 16.4.1 *For any completely regular Hausdorff topological space X, the space $C_c(X)$ has countable tightness if and only if every compact-open cover of X has a compact-open countable subcover.*

Proof Assume that every compact-open cover of X has a countable compact-open subcover and assume $f \in \overline{A}$, where the closure is taken in $C_c(X)$. Let $\mathcal{K}(X)$ be the family of all compact subsets of X. For every compact $K \in \mathcal{K}(X)$ and every $n \in \mathbb{N}$, choose:

$$f_{K,n} \in A \cap \{g \in C_c(X) : |f(x) - g(x)| < n^{-1}, x \in K\}.$$

Set:

$$\Sigma(K, n) := \{x \in X : |f_{K,n}(x) - f(x)| < n^{-1}\}.$$

Then, for every $n \in \mathbb{N}$, the family $\Sigma(K, n)$ is a compact-open cover of X. By the assumption for every $n \in \mathbb{N}$ there exists a sequence $(K_{(n,j)})_j$ in $\mathcal{K}(X)$ such that $\Sigma(K_{(n,j)}, n)$ is a compact-open cover of X. Define

$$B := \{f_{K_{(n,j)}n} : n, j \in \mathbb{N}\}.$$

Then $f \in \overline{B}$. Conversely, assume that $C_c(X)$ has countable tightness and let Σ be a compact-open cover of X. For every compact K in X, choose $U_K \in \Sigma$ such that $K \subset U_K$. For every $n \in \mathbb{N}$ and every compact $K \subset X$, choose $f_{K,n} \in C_c(X)$ such that $n^{-1} \leq f_{K,n} \leq n$, and

$$f_{K,n}(K) = \{n^{-1}\}, \quad f_{K,n}(X \setminus U_K) = \{n\}.$$

Define:

$$A := \{f_{K,n} : n \in \mathbb{N}, A \in \mathcal{K}(X)\}.$$

Since the closure of A contains the zero function 0, there are sequences $(n_j)_j$ in $\mathbb{N}$ and $(K_j)_j$ in $\mathcal{K}(X)$ such that

$$0 \in \overline{\{f_{K_j,n_j} : n, j \in \mathbb{N}\}}.$$

It is easy to see that $\{U_{K_j} : j \in \mathbb{N}\}$ is a compact-open subcover of Σ. □

We apply Lemma 16.4.1 to get the following.

Proposition 16.4.2 *The Morris–Wulbert space $C_c(\omega_1)$ does not have countable tightness.*

Proof Note that $\Sigma := \{[0, \alpha) : \alpha < \omega_1\}$ is a compact-open cover of $[0, \omega_1)$. If $K \subset [0, \omega_1)$ is compact, $\sup K < \omega_1$. Assume that there exists a countable compact-open subcover of Σ. Then there exists an injective sequence $(\alpha_n)_n$ (i.e. $\alpha_n \neq \alpha_m$ if $n \neq m$) of countable ordinals such that $\Gamma := \{[0, \alpha_n) : n \in \mathbb{N}\}$ is a compact-open cover of Ω. Clearly $\alpha := \sup_n \alpha_n < \omega_1$. If $\gamma \in (\alpha, \omega_1)$, then $(\alpha_n)_n$ is contained in the compact interval $[0, \gamma]$. By the assumption there exists $m \in \mathbb{N}$ such that $[0, \gamma] \subset [0, \alpha_m)$, a contradiction since $\alpha_m \in [0, \gamma] \setminus [0, \alpha_m)$. □

We show that the existence of a $\mathfrak{G}$-base in $C_c(\omega_1)$ depends on the ZFC-consistent axiom system. We refer to results in [220].

Proposition 16.4.3 *If $\aleph_1 < \mathfrak{b}$, then $C_c(\omega_1)$ does not admit a $\mathfrak{G}$-base.*

Proof Since the character of $C_c(\omega_1)$ is $\aleph_1$, Proposition 16.3.10 applies. □

In the next proposition, as usual, $\mathfrak{b}$ ($\mathfrak{d}$) denotes the first ordinal of cardinality $\mathfrak{b}$ ($\mathfrak{d}$).

Proposition 16.4.4 *Both spaces $C_c(\mathfrak{b})$ and $C_c(\mathfrak{d})$ have a $\mathfrak{G}$-base. If the ordinal κ has the cofinality $\aleph_0$, $\mathfrak{b}$, or $\mathfrak{d}$, then $C_c(\kappa)$ has a $\mathfrak{G}$-base.*

Proof For the space $C_c(\mathfrak{b})$, choose $A \subset \mathbb{N}^{\mathbb{N}}$ with $|A| = \mathfrak{b}$ such that A is unbounded in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. Let φ be an injective map from $[0, \mathfrak{b})$ onto A. For arbitrary $\sigma = (a_1, a_2, \dots) \in \mathbb{N}^{\mathbb{N}}$, we define the corresponding neighbourhood U_σ of zero by $U_\sigma := U_{a_1,\beta}$, where β is the first member of $[0, \mathfrak{b})$ such that $\varphi(\beta) \not\leq^* (a_2, a_3, \dots)$. Obviously,

$$\sigma \leq \tau \Rightarrow U_\tau \subset U_\sigma.$$

Note that $\{U_\sigma : \sigma \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base. Indeed, given $n \in \mathbb{N}$ and an ordinal $\alpha < \mathfrak{b}$, some $(a_2, a_3, \dots)$ bounds the set $\varphi([0, \alpha])$ in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, by definition of $\mathfrak{b}$.

Set $a_1 := n$ and $\sigma = (a_1, a_2, \dots)$. One gets a corresponding $\beta \notin [0, \alpha]$, so that $[0, \alpha] \subset [0, \beta]$ and $U_\sigma = U_{a_1,\beta} \subset U_{n,\alpha}$.

One repeats the construction for $C_c(\mathfrak{d})$ with $\mathfrak{d}$ replacing $\mathfrak{b}$, with A cofinal in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, and with β the first member of $[0, \mathfrak{d})$ such that

$$(a_2, a_3, \dots) \leq^* \varphi(\beta).$$

By definition of $\mathfrak{d}$, one concludes that the result is a $\mathfrak{G}$-base.

Now we prove the second part: if $\mathrm{cf}(\kappa) = \aleph_0$, there exists a decreasing sequence $(V_n)_n$ that is a basis of neighbourhoods of zero in $C_c(\kappa)$. A $\mathfrak{G}$-base is given by setting $U_{(a_1,a_2,\dots)} := V_{a_1}$. If $\mathrm{cf}(\kappa)$ is $\mathfrak{b}$ or $\mathfrak{d}$, let φ be an injective map from a cofinal subset M of $[0, \kappa)$ onto a subset A of $\mathbb{N}^{\mathbb{N}}$ such that $|M| = \mathfrak{b}$ or $\mathfrak{d}$ and A is unbounded or cofinal in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, respectively. Finally follow as before to construct a $\mathfrak{G}$-base for $C_c(\kappa)$. □

Combining previous results, we obtain the following full characterization for the space $C_c(\omega_1)$ to have a $\mathfrak{G}$-base.

Proposition 16.4.5 *The space $C_c(\omega_1)$ has a $\mathfrak{G}$-base if and only if $\aleph_1 = \mathfrak{b}$.*

Theorems 12.1.4, 12.2.1, and Proposition 16.4.2 imply that $C_c(\omega_1)$ is a non-quasibarrelled (DF)-space and the *weak** dual of $C_c(\omega_1)$ is not K-analytic. If we assume that $\aleph_1 = \mathfrak{b}$, we note the following example providing also restrictions on possible extensions of Theorem 6.6.3.

Example 16.4.6 Set $E := C_c(\omega_1)$. Then $F := (E', \sigma(E', E))$ is not K-analytic. If $\aleph_1 = \mathfrak{b}$, the space F has a resolution of metrizable and compact absolutely convex sets.

Proof Since every compact set in $X := [0, \omega_1)$ is metrizable, the polar of every neighbourhood of zero in E is $\sigma(E', E)$-metrizable by Proposition 6.4.1 and Lemma 6.4.7. It is also known that E is a locally complete (DF)-space and X is pseudocompact, non-compact, and also the family $\{f \in E : f(X) \leq 1\}$ generates on E a Banach topology ϑ such that $\mu(E, E') \leq \vartheta$; see [348] for more detail.

We show again (with a somewhat different argument) that $(E', \sigma(E', E))$ is not a K-analytic space. Assume that F is K-analytic. Then $\sigma(E, E')$ has countable tightness by Theorem 12.1.4 and $(E, \mu(E, E'))$ is quasibarrelled by Proposition 16.1.6. Hence the space $(E, \mu(E, E'))$ is barrelled (since E is a locally complete space). By the closed graph theorem applied to the identity map $I : (E, \mu(E, E')) \to (E, \vartheta)$, we have the equality $\mu(E, E') = \vartheta$. As X is non-compact, the topological duals of E and (E, ϑ) are different; see [213, Lemma 2]. Hence, $(E', \sigma(E', E))$ indeed is not a K-analytic space.

Now assume that $\aleph_1 = \mathfrak{b}$. Then by Proposition 16.4.5 we deduce that E has a basis of absolutely convex neighbourhoods of zero $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that $U_\alpha \subset U_\beta$ if $\beta \leq \alpha$. Clearly the polars of the sets U_α compose a resolution consisting of $\sigma(E', E)$-metrizable compact absolutely convex sets. □

From Fremlin's [522, Theorem 5.5.3] it follows that under CH there exists a non-analytic K-analytic space E such that each compact set in E is metrizable. It is known that in any ZFC-consistent system, one has $\aleph_1 \leq \mathfrak{b} \leq 2^{\aleph_0}$. If we assume Martin's axiom and the negation of the Continuum Hypothesis, then any K-analytic space in which every compact set is metrizable is analytic [522, Theorem 5.5.3] and $\aleph_1 < \mathfrak{b} = 2^{\aleph_0}$; see, e.g. [625].

We complete this section with some extra facts concerning the Morris–Wulbert space $C_c(\omega_1)$. Recall, as Morris and Wulbert noted in [457], that $C_c(\omega_1)$ is an $\aleph_0$-space (i.e. has a countable pseudobase) that is not barrelled and, in fact, is not even a Mackey space. Moreover, since $[0, \omega_1)$ is pseudocompact and not compact, it follows that the dual of the Banach space $C_u(\omega_1)$ with the sup-norm topology is strictly stronger than that of $C_c(\omega_1)$ [213, Lemma 2], and thus the Mackey topology $\mu := \mu(C_c(\omega_1, C_c(\omega_1)')$ is also not barrelled, as the closed graph theorem shows. Let us compare directly the dual of $C_c(\omega_1)$ and $C_u(\omega_1)$. First, for each $\alpha \in [0, \omega_1)$, set $(\alpha, \omega_1) := [0, \omega_1) \setminus [0, \alpha]$ and let

$$F = \{f \in C_c(\omega_1) : f((\alpha, \omega_1)) = \{0\}, \text{ for some } \alpha \in [0, \omega_1)\}.$$

Let h be a function whose value is identically 1 on $[0, \omega_1)$.

Claim 1. F is algebraically complemented in $C_c(\omega_1)$ by the span of h. Indeed, if g is a real-valued function that is non-constant on each set (α, ω_1), there are uncountably many $\beta \in [0, \omega_1)$ such that $|f(\beta) - f(\beta + 1)| > 0$, and thus there exists some $n \in \mathbb{N}$ such that $|f(\beta) - f(\beta + 1)| > n^{-1}$ for uncountably many β. Therefore, we may choose $\beta_1 < \beta_2 < \dots$ such that

$$|f(\beta_k) - f(\beta_k + 1)| > n^{-1}$$

for all $k \in \mathbb{N}$. Note also that the sequence $(\beta_k)_k$ converges in $[0, \omega_1)$, although $(f(\beta_k))_k$ and $(f(\beta_k + 1))_k$ do not converge to the same point in $\mathbb{R}$. Therefore, g is not continuous. We conclude that each $f \in C_c(\omega_1)$ is eventually constant, so that $C_c(\omega_1)$ is the algebraic direct sum of F and the span of h.

In the sup-norm, h is clearly unit distance from the subspace F, and is in the compact-open closure of F.

Claim 2. F is a closed hyperplane in the space $C_u(\omega_1)$, and it is a dense hyperplane in $C_c(\omega_1)$. Therefore, members of $C_c(\omega_1)'$ are uniquely determined by their restrictions to F.

Let F_c and F_u denote F with its relative compact-open and uniform sup-norm topology, respectively. Then F_u is a Banach space that dominates F_c. Given $\lambda \in F_u'$, there exists a point $\alpha \in [0, \omega_1)$ such that $\lambda(f) = 0$ whenever $f \in F$ with $f([0, \alpha]) = \{0\}$. Indeed, otherwise for some $n \in \mathbb{N}$, there would be uncountably many $\alpha \in [0, \omega_1)$ and $f \in F$ with

$$\|f\| = 1, \quad \lambda(f) > n^{-1}, \quad f([0, \alpha]) = \{0\}.$$

But then we can find $f_1, \ldots, f_k \in F$ having disjoint supports and unit norm, with each $\lambda(f_j) > n^{-1}$. Take $k > n\|\lambda\|$. By virtue of disjoint supports, $\|f_1 + \cdots + f_k\| = 1$, yielding the contradiction that

$$kn^{-1} < \lambda(f_1 + \cdots + f_k) \leq \|f_1 + \cdots + f_k\| = \|\lambda\|.$$

Having now established the existence of a point $\alpha \in \omega_1$ such that $\lambda(f) = 0$ whenever $f([0, \alpha]) = \{0\}$, we see that $\lambda \in F_c'$, and $F_u' = F_c'$. Let $\varphi \in C_u(\omega_1)'$ which vanishes on F and $\varphi(h) = 1$. For any member λ of $F_u' = F_c'$, there is a unique continuous linear extension λ_0 to $C_c(\omega_1)$, whereas each linear extension remains continuous on $C_u(\omega_1)$ and may be realized in the form $\lambda_0 + c\varphi$, where c is an arbitrary scalar. We proved the following.

Claim 3. $C_c(\omega_1)'$ is a one-codimensional subspace of $C_u(\omega_1)'$.

Finally we prove the following statement.

Claim 4. $C_c(\omega_1)'$ is a closed hyperplane of the dual Banach space $C_u(\omega_1)'$. Indeed, for any $\gamma \in C_c(\omega_1)'$, there is some point $\alpha \in [0, \omega_1)$ such that the characteristic function χ_α of $[0, \alpha]$ satisfies $\gamma(\chi_\alpha) = \gamma(h)$. Otherwise, there would be an $n \in \mathbb{N}$ and an uncountable set Λ of points in ω_1 such that $|\gamma(h) - \gamma(\chi_\alpha)| > n^{-1}$ for each $\alpha \in \Lambda$. Then h is in the $C_c(\omega_1)$-closure of the set $\{\chi_\alpha : \alpha \in \Lambda\}$, but $\gamma(h)$ is not in the closure of $\{\gamma(\chi_\alpha) : \alpha \in \Lambda\}$, contradicting continuity of γ. With α fixed such that $\gamma(\chi_\alpha) = \gamma(h)$, we have:

$$\|\gamma + \varphi\| \geq \|\gamma + \varphi\|\|h - \chi_\alpha\| \geq |(\gamma + \varphi)(h - \chi_\alpha)| = \varphi(h) = 1,$$

which shows that φ is unit distance from the hyperplane $C_c(\omega_1)'$ in the Banach space $C_u(\omega_1)'_\beta$, so that the hyperplane is closed.

Another approach might be the following: since $C_c(\omega_1)$ is a (DF)-space (see the part before Proposition 16.3.14), it is a (df)-space, and then its strong dual $C_c(\omega_1)'_\beta$ is a Banach space by Theorem 2.6.11. It is known that the spaces $C_c(\omega_1)$ and $C_u(\omega_1)$ have the same bounded sets; see, for example, [541, Theorem 11]. Then the topology on $C_c(\omega_1)'_\beta$ is that induced by $C_u(\omega_1)'_\beta$.

16.5 $\mathfrak{G}$-Bases for Spaces $C_c(X)$

In this section we characterize (see [210]) those Tychonoff spaces X for which the space $C_c(X)$ has a $\mathfrak{G}$-base.

A resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact sets swallowing the compact sets of X will be called a *fundamental compact resolution*.

By Tkachuk's theorem 9.4.16, the space $C_p(X)$ has a fundamental compact resolution if and only if X is countable and discrete. The case for $C_c(X)$ is different.

The following theorem generalizes a well-known result due to Arens (see [636]), stating that $C_c(X)$ *is metrizable if and only if* X *is hemicompact*.

Theorem 16.5.1 (Ferrando–Kąkol) *The space $C_c(X)$ has a $\mathfrak{G}$-base if and only if X has a fundamental compact resolution.*

Proof Let K be a compact set in X and choose $\epsilon > 0$. Then, define

$$[K, \epsilon] = \left\{ f \in C(X) : \sup_{x \in K} |f(x)| \leq \epsilon \right\}.$$

Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a compact resolution of X. Set

$$U_\alpha = [A_\alpha, \alpha(1)^{-1}]$$

for $\alpha \in \mathbb{N}^{\mathbb{N}}$ and put $\mathfrak{U} = \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

Clearly $\mathfrak{U}$ is a family of absolutely convex and absorbing sets in $C(X)$ such that $U_\beta \subseteq U_\alpha$ if $\alpha \leq \beta$ which composes a filter base. Hence there exists on $C(X)$ a locally convex topology τ whose basis of neighbourhoods consists of the sets of the family $\mathfrak{U}$.

Clearly $\tau_p \leq \tau \leq \tau_c$, so that $(C(X), \tau)$ has a $\mathfrak{G}$-base.

Now assume that $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ swallows all compact sets and let V be a neighbourhood of zero of $C_c(X)$. Let K be a compact set in X with $[K, \epsilon] \subseteq V$ for some $\epsilon > 0$. Choose $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $K \subseteq A_\gamma$ and $\gamma(1)^{-1} < \epsilon$. Then $U_\gamma \subseteq [K, \epsilon] \subseteq V$. This shows that $\tau = \tau_c$, so $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base for $C_c(X)$.

Conversely, assume that $C_c(X)$ has a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

For every set U in $C(X)$, define a corresponding set $U^\diamondsuit$ in X by the formula:

$$U^\diamondsuit = \{x \in X : |f(x)| \leq 1 \ \forall f \in U\}.$$

Clearly $U^\diamondsuit$ is closed in X and $U \subseteq V$ implies that $U^\diamondsuit \supseteq V^\diamondsuit$. If K is compact and $\epsilon > 0$, then $[K, \epsilon]^\diamondsuit \subseteq K$.

Indeed, if $x \in X \setminus K$, there is $f \in C(X)$ with $f(x) = 2$ and $f(K) = \{0\}$, so that $f \in [K, \epsilon]$ and $x \notin [K, \epsilon]^\diamondsuit$.

If K is compact and $0 < \epsilon \leq 1$, then $K \subseteq [K, \epsilon]^\diamondsuit$. Hence $[K, \epsilon]^\diamondsuit = K$.

Moreover, if U is a neighbourhood of zero in $C_c(X)$, then $U^\diamondsuit$ is compact. Indeed, if K is a compact set in X such that $[K, \epsilon] \subseteq U$ for some $\epsilon > 0$, by the

previous observations one has $U^\diamondsuit \subseteq [K, \epsilon]^\diamondsuit \subseteq K$. Hence $U^\diamondsuit$ is a closed subset of a compact set.

For any compact set K in X, there exists some $U_\alpha \subseteq [K, 1]$, which means that

$$K = [K, 1]^\diamondsuit \subseteq U_\alpha^\diamondsuit.$$

This shows that the family

$$\mathcal{A} = \{U_\alpha^\diamondsuit : \alpha \in \mathbb{N}^\mathbb{N}\}$$

swallows all compact sets, so particularly this family covers X. Since $U_\alpha^\diamondsuit \subseteq U_\beta^\diamondsuit$ for $\alpha \leq \beta$, we conclude that $\mathcal{A}$ is a compact resolution of X. This shows the theorem. □

Corollary 16.5.2 *If $C_c(X)$ has a $\mathfrak{G}$-base, then υX is K-analytic.*

Proof If $C_c(X)$ has a $\mathfrak{G}$-base, Theorem 16.5.1 provides a compact resolution for X. Hence X is a quasi-Suslin space and then υX is K-analytic. □

Corollary 16.5.3 *The ordinal space $[0, \omega_1)$ has a fundamental compact resolution if and only if $\aleph_1 = \mathfrak{b}$.*

Proof According to Proposition 16.4.5, the space $C_c([0, \omega_1))$ admits a $\mathfrak{G}$-base if and only if we assume that $\aleph_1 = \mathfrak{b}$. □

16.6 Infinite-Dimensional Compact Sets in Locally Convex Spaces with a $\mathfrak{G}$-Base

We know already that every compact set in any lcs with a $\mathfrak{G}$-base is metrizable. This statement follows from Theorem 11.2.1. This fact applies, among others, to each (LM)-space, i.e. a countable inductive limit of metrizable lcs, since (LM)-spaces have a $\mathfrak{G}$-base.

On the other hand, every Baire-like lcs with a $\mathfrak{G}$-base is metrizable. This follows from Corollary 15.1.8.

Let us recall the following facts (already mentioned above) about the space φ, i.e. an $\aleph_0$-dimensional vector space endowed with the finest locally convex topology:

(i) φ is the strong dual of the Fréchet–Schwartz space $\mathbb{R}^\mathbb{N}$.
(ii) All compact subsets in φ are finite-dimensional.
(iii) φ is a complete bornological space.

One can ask for a possible large class of lcs E for which every infinite-dimensional subspace of E contains an infinite-dimensional compact metrizable subset. Clearly every metrizable lcs trivially fulfills this request.

Below we prove the following general theorem from [60].

Theorem 16.6.1 (Banakh–Kąkol–Schütz) *Every uncountably dimensional lcs E with a $\mathfrak{G}$-base contains an infinite-dimensional metrizable compact subset.*

The assumption that E is uncountably dimensional cannot be removed. Indeed, the space φ is (countably) infinite-dimensional, has a $\mathfrak{G}$-base, and contains no infinite-dimensional compact subsets. However, φ is a unique locally convex k_R-space with this property. Recall that a topological space X is a *k_R-space* if a function $f : X \to \mathbb{R}$ is continuous whenever for every compact subset $K \subseteq X$ the restriction $f|K$ is continuous.

We prove also the following.

Theorem 16.6.2 (Banakh–Kąkol–Schütz) *A lcs E is isomorphic to the space φ if and only if E is a k_R-space containing no infinite-dimensional compact subsets.*

Theorem 16.6.2 implies that a lcs is isomorphic to φ if and only if it is homeomorphic to φ. The topological uniqueness property of φ was first proved by Banakh in [53].

We know already that $C_c(X)$ has a $\mathfrak{G}$-base if and only if X admits a fundamental compact resolution, Theorem 16.5.1. Since every Čech-complete Lindelöf space X is a continuous preimage of a Polish space under a perfect map (and the latter space admits a fundamental compact resolution), the space $C_c(X)$ has a $\mathfrak{G}$-base. Theorem 16.6.1 applies to get the following.

Corollary 16.6.3 *Let X be an infinite Čech-complete Lindelöf space. Then, every uncountable-dimensional subspace of the space $C_c(X)$ contains an infinite-dimensional metrizable compact set.*

The following concept will be used in the sequel. Recall that for a topological space X, its *free locally convex space* is a lcs $L(X)$ endowed with a continuous function $\delta : X \to L(X)$ such that for any continuous function $f : X \to E$ to a lcs E, there exists a unique linear continuous map $T : L(X) \to E$ such that $T \circ \delta = f$. Applying [513] one gets that the set X forms a Hamel basis for $L(X)$ and δ is a topological embedding. We refer the reader to articles [65] and [57] for results concerning this concept.

For a Tychonoff space X elements $\mu \in L(X)$ can be seen as finitely supported sign measures on X, which can be uniquely written as the linear combination $\mu = \sum_{x \in supp(\mu)} \mu(x)\delta_x$ with non-zero coefficients $\mu(x) \in \mathbb{R} \setminus \{0\}$. The real number $\|\mu\| = \sum_{x \in supp(x)} |\mu(x)|$ is the norm.

Recall the following result of Banakh; for the proof we refer to [55, Lemma 10.11.3].

Lemma 16.6.4 *For any Tychonoff space X and any bounded subset $K \subset L(X)$, the set $\bigcup_{\mu \in K} supp(\mu)$ is bounded in X and the set $\{\|\mu\| : \mu \in K\}$ is bounded in $\mathbb{R}$.*

Two lcs E and F are *bornologically isomorphic* if there exists a linear bijective function $f : E \to F$ such that f and f^{-1} are bounded maps. Recall again that a lcs E is called *bornological* if each bounded linear operator from E to a lcs F is continuous. A linear space E is called *κ-dimensional* if E has a Hamel basis of

cardinality κ. In this case we set $\kappa = \dim(E)$. A lcs E is *free* if it carries the finest locally convex topology. In this case E is isomorphic to the free lcs $L(\kappa)$ over the cardinal $\kappa = \dim(E)$ with the discrete topology.

We note the following easy fact which characterizes lcs containing no infinite-dimensional compact subsets.

Theorem 16.6.5 *For a lcs E, the following conditions are equivalent:*

(i) Each compact subset of E has finite topological dimension.
(ii) each bounded linearly independent set in E is finite.
(iii) E is bornologically isomorphic to a free lcs.

If E is bornological, then the conditions (i)–(iii) *are equivalent to*

(iv) E is free.

Proof (i) $\Rightarrow$ (ii): Assume that each compact subset of E has finite topological dimension. Assuming that E contains an infinite bounded linearly independent set, we get a bounded linearly independent set $\{x_n\}_{n\in\mathbb{N}}$ consisting of pairwise distinct points. Then the sequence $(2^{-n}x_n)_{n\in\mathbb{N}}$ converges to zero and

$$K = \bigcup_n \Big\{ \sum_{k=n}^{2n} t_k x_k : (t_k)_{k=n}^{2n} \in \prod_{k=n}^{2n} [0, 2^{-k}] \Big\}$$

is an infinite-dimensional compact set in E, which contradicts our assumption.

(ii) $\Rightarrow$ (iii): Let τ be the strongest locally convex topology on E. Then the identity map $(E,\tau) \to E$ is continuous and hence bounded. If each bounded linearly independent set in E is finite, then each bounded set $B \subset E$ is contained in a finite-dimensional subspace of E and hence is bounded in the topology τ. This means that the identity map $E \to (E,\tau)$ is bounded and hence E is bornologically isomorphic to the free lcs (E,τ).

(iii) $\Rightarrow$ (i): If E is bornologically isomorphic to a free lcs F, then each bounded linearly independent set in E is finite since the free lcs F has this property.

Clearly (iv) $\Rightarrow$ (iii) is trivial. If E is bornological then the implication (iii) $\Rightarrow$ (iv) follows from the continuity of bounded linear maps on bornological spaces. □

Recall here that a subset B of a topological space X is called *topologically bounded* if for any continuous real-valued function f, the set $f(B)$ is bounded.

Proposition 16.6.6 *For a Tychonoff space X, the following conditions are equivalent:*

(i) Each compact subset of the free lcs $L(X)$ has finite topological dimension;
(ii) Each bounded linearly independent set in $L(X)$ is finite;
(iii) Each topologically bounded subset of X is finite.

Proof The equivalence $(i) \Leftrightarrow (ii)$ follows from Theorem 16.6.5. The implication (iii) $\Rightarrow$ (i) follows from Lemma 16.6.4, and the implication (ii) $\Rightarrow$ (iii) follows from the fact that each topologically bounded set in a lcs is bounded. □

Let κ, λ be two cardinals. A lcs E is defined to have (κ, λ)-*tall bornology* if every subset $A \subseteq E$ of cardinality $|A| = \kappa$ contains a bounded subset $B \subseteq A$ of cardinality $|A| = \lambda$. The family of all bounded sets of E is called the *bornology* of E.

We need also the following.

Theorem 16.6.7 *Let κ be an infinite cardinal. For a lcs E, the following conditions are equivalent:*

(i) E is bornologically isomorphic to the free lcs $L(\kappa)$.
(ii) Each bounded linearly independent set in E is finite and the bornology of E is (κ^+, w)-tall but not (κ, w)-tall.

If E is bornological, then the conditions (i)–(ii) *are equivalent to*

(iii) E is topologically isomorphic to $L(\kappa)$.

Proof (i) $\Rightarrow$ (ii): Assume that E is bornologically isomorphic to $L(\kappa)$. Then, E has algebraic dimension κ and each bounded linearly independent set in E is finite (since this is true in $L(\kappa)$).

To see that the bornology of E is (κ^+, w)-tall, take any set $K \subseteq E$ of cardinality $|K| = \kappa^+$. Since E has algebraic dimension κ, there exists a cover $\{B_\alpha\}_{\alpha\in\kappa}$ of E by κ many compact sets. By the pigeonhole principle, there exists $\alpha \in \kappa$ such that $|K \cap B_\alpha| = \kappa^+$. This means that the bornology of E is (κ^+, κ^+)-tall and hence (κ^+, w)-tall.

Note that the bornology of E is not (κ, w)-tall. Indeed, observe that the Hamel basis κ of $L(\kappa)$ has the property that no infinite subset of κ is bounded in $L(\kappa)$. Since E is bornologically isomorphic to $L(\kappa)$, the image of κ in E is a subset of cardinality κ containing no bounded infinite subsets and showing that E is not (κ, w)-tall.

(ii) $\Rightarrow$ (i): Assume that each bounded linearly independent set in E is finite and the bornology of E is (κ^+, w)-tall but not (κ, w)-tall. Fix a Hamel basis B of E. We show that $|B| = \kappa$. Assume that $|B| > \kappa$. Then we conclude that E is not (κ^+, w)-tall, a contradiction. Assume that $|B| < \kappa$. Then E is the union of $< \kappa$ many bounded sets and hence is (κ, κ)-tall by the pigeonhole principle. But this contradicts our assumption. Therefore $|B| = \kappa$.

Let $h : \kappa \to B$ be any bijective function and $\bar{h} : L(\kappa) \to E$ be the unique extension of h to a linear continuous operator. Since B is a Hamel basis for E, the operator $\bar{h}$ is bijective. Since each bounded set in E is contained in a finite-dimensional linear subspace, the operator $\bar{h}^{-1} : E \to L(\kappa)$ is bounded and hence $\bar{h} : L(\kappa) \to E$ is a bornological isomorphism. If the space E is bornological, then the equivalence (i) $\Leftrightarrow$ (iii) follows from the bornological property of E and $L(\kappa)$. □

We need the following concepts; see [60].

Definition 16.6.8 Let κ, λ be cardinals. A topological space X is:

- $(\kappa, \lambda)_p$-*equiconvergent* at a point $x \in X$ if for any indexed family $\{x_\alpha\}_{\alpha\in\kappa} \subseteq \{s \in X^{\mathbb{N}} : \lim_{n\to\infty} s(n) = x\}$, there exists a subset $\Lambda \subseteq \kappa$ of cardinality $|\Lambda| = \lambda$

such that for every neighbourhood $O_x \subseteq X$ of x, there exists $n \in \mathbb{N}$ such that the set $\{\alpha \in \Lambda : x_\alpha(n) \notin O_x\}$ is finite;

- $(\kappa, \lambda)_k$*-equiconvergent at a point* $x \in X$ if for any indexed family $\{x_\alpha\}_{\alpha\in\kappa} \subseteq \{s \in X^\omega : \lim_{n\to\infty} s(n) = x\}$, there exists a subset $\Lambda \subseteq \kappa$ of cardinality $|\Lambda| = \lambda$ such that for every neighbourhood $O_x \subseteq X$ of x, there exists $n \in \mathbb{N}$ such that for every $m \geq n$ and $\alpha \in \Lambda$ we have $x_\alpha(m) \in O_x$;
- $(\kappa, \lambda)_p$*-equiconvergent* if X is $(\kappa, \lambda)_p$-equiconvergent for all $x \in X$;
- $(\kappa, \lambda)_k$*-equiconvergent* if X is $(\kappa, \lambda)_k$-equiconvergent for all $x \in X$.

Clearly every $(\kappa, \lambda)_k$-equiconvergent space is $(\kappa, \lambda)_p$-equiconvergent. We will use the following fact below.

Proposition 16.6.9 *If a lcs E is $(\kappa, \lambda)_p$-equiconvergent, then its bornology is (κ, λ)-tall.*

Proof Let $K \subseteq E$ be of cardinality $|K| = \kappa$. For every $\alpha \in K$, consider the convergent sequence $x_\alpha \in X^{\mathbb{N}}$ defined by $x_\alpha(n) = 2^{-n}\alpha$. Assume that E is $(\kappa, \lambda)_p$-equiconvergent. We find a subset $L \subset K$ of cardinality $|L| = \lambda$ such that for every neighbourhood of zero $U \subseteq E$, there exists $n \in \mathbb{N}$ such that the set

$$\{\alpha \in L : 2^{-n}\alpha \notin U\}$$

is finite.

Note that L is bounded. Indeed, for every neighbourhood $U \subseteq E$ of zero, we find a neighbourhood $V \subseteq E$ of zero such that $[0, 1] \cdot V \subseteq U$. By our assumption, there exists $n \in \mathbb{N}$ such that the set $F = \{\alpha \in K : 2^{-n}\alpha \notin V\}$ is finite. Take $m \geq n$ such that $2^{-m}\alpha \in U$ for every $\alpha \in F$. Then

$$2^{-m}L \subset 2^{-m}(L \setminus F) \cup 2^{-m}F \subset ([0, 1] \cdot V) \cup U = U.$$

This shows that L is bounded. □

We need also to recall some definitions and concepts. By a *poset* we mean a non-empty set endowed with a reflexive and transfinite binary relation $\leq$.

Again recall that a subset A of a poset set P we call *cofinal* in P if for each $x \in P$, there exists $a \in A$ such that $x \leq a$.

A set $\subset P$ is bounded in P if there exists $x \in P$ such that $a \leq x$ for all $a \in A$. The *cofinality* $cof(P)$ of a poset P is the smallest cardinality of a cofinal subsets $D \subset P$.

For each $x \in P$, set $\uparrow x = \{p \in P : x \leq p\}$. We will use also the following result due to Banakh [57, Lemma 2.3.5].

Lemma 16.6.10 (Banakh) *For every monotone function $f : \omega^\omega \to P$ into a poset P with $cof(P) \leq \omega$ and every $\alpha \in \omega^\omega$, there exists $k \in \omega$ such that $f[\uparrow (\alpha(n))] = \{f(\beta) : \beta \in \uparrow (\alpha(n))\}$ is bounded in P.*

Now we prove the following.

Theorem 16.6.11 *If a topological space X admits a $\mathfrak{G}$-base at a point $x \in X$, then X is $(\omega_1, \omega)_k$-equiconvergent at the point x.*

Proof Let $(U_f)_{f\in\mathbb{N}^{\mathbb{N}}}$ be a $\mathfrak{G}$-base at x. To show that X is $(\omega_1, \omega)_k$-equiconvergent to x, fix an indexed family

$$\{x_\alpha\}_{\alpha\in \mathrm{w}_1} \subseteq \{s \in X^{\mathbb{N}} : \lim_{n\to\infty} s(n) = x\}$$

of sequences that converge to x.

For every $\alpha \in \omega_1$, consider the function $\mu_\alpha : \omega^\omega \to \omega$ assigning to each $f \in \omega^\omega$ the smallest number $n \in \omega$ such that $\{x_\alpha(m)\}_{m\geq n} \subseteq U_f$. Note that the function $\mu_\alpha : \omega^\omega \to \omega$ is monotone.

For every $n \in \omega$ and finite function $t \in \omega^n$, set $\omega^\omega_t = \{f \in \omega^\omega : f|n = t\}$. By Lemma 16.6.10 we deduce that for every $f \in \omega^\omega$, there exists $n \in \omega$ such that $\mu_\alpha[\omega^\omega_{f|n}]$ is finite.

Let T_α be the set of all (finite) functions:

$$t \in \omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$$

such that $\mu_\alpha[\omega^\omega_t]$ is finite but for any $\tau \in \omega^{<\omega}$ with $\tau \subset t$, the set $\mu_\alpha[\omega^\omega_\tau]$ is infinite. From [57, Lemma 2.3.5] we derive that for every $f \in \omega^\omega$ there exists a unique $t_f \in T_\alpha$ such that $t_f \subset f$.

Define

$$\delta_\alpha(f) = \max \mu_\alpha[\omega^\omega_{t_f}] \geq \mu_\alpha(f).$$

It is clear that the function $\delta_\alpha : \omega^\omega \to \omega$ is continuous and hence δ_α is an element of the space $C_p(\omega^\omega, \omega)$ of continuous functions from ω^ω to ω endow ω^ω with the product topology. We use Michael's [445, Proposition 10.4] deduce that $C_p(\omega^\omega, \omega)$ has a countable network.

Next, consider the function

$$\delta : \omega_1 \to C_p(\omega^\omega, \omega), \quad \delta : \alpha \mapsto \delta_\alpha,$$

and note that $\delta_\alpha(f) \geq \mu_\alpha(f)$ for any $\alpha \in \omega_1$ and $f \in \omega^\omega$.

Since $C_p(\omega^\omega, \omega)$ has countable network, there exists a sequence $\{\alpha_n\}_{n\in\omega} \subset A$ of pairwise distinct ordinals such that the sequence $(\delta_{\alpha_n})_{n\in\omega}$ converges to δ_{α_0} in the function space $C_p(\omega^\omega, \omega)$.

We show that the sequence $(x_{\alpha_n})_{n\in\omega}$ witnesses that X is $(\omega_1, \omega)_k$-equiconvergent at x. Indeed, given any open neighbourhood $O_x \subseteq X$ of x, we find $f \in \omega^\omega$ such that $U_f \subseteq O_x$. Since the sequence $(x_{\alpha_0}(n))_{n\in\omega}$ converges to x, there exists $m \in \omega$ such that

$$\{x_{\alpha_0}(n)\}_{n\geq m} \subseteq U_f.$$

As the sequence $(\delta_{\alpha_n})_{n\in\omega}$ converges to δ_{α_0} in $C_p(\omega,\omega)$, we can replace m by a larger number and assume that $\delta_{\alpha_n}(f) = \delta_{\alpha_0}(f)$ for all $n \geq m$. Choose a number $l \geq \delta_{\alpha_0}(f)$ such that for every $n < m$ and $k \geq l$, we have $x_{\alpha_n}(k) \in O_x$.

On the other hand, for every $n \geq m$ and $k \geq l$, we have

$$k \geq l \geq \delta_{\alpha_0}(f) = \delta_{\alpha_n}(f) \geq \mu_{\alpha_n}(f).$$

Hence $x_{\alpha_n}(k) \in U_f \subseteq O_x$. □

We are at the position to prove our Theorem 16.6.1.

Proof of Theorem 16.6.1 Assume that a lcs E has a $\mathfrak{G}$-base. By Theorem 16.6.11 we know that E is $(\omega_1,\omega)_k$-equiconvergent. Next, by Proposition 16.6.9, the space E has (ω_1,ω)-tall bornology, which means that each uncountable set in E contains an infinite bounded set. If E has an uncountable Hamel basis H, then H contains an infinite bounded linearly independent set, and we apply Theorem 16.6.5 to deduce that the space E contains an infinite-dimensional compact set. This completes the proof. □

We complete this section with some application of Theorem 16.6.1. We provide another proof of Theorem 9.5.3. This will be included in the following

Proposition 16.6.12 *For a Tychonoff space X, consider the following assertions:*

(i) $C_p(X)$ *admits a fundamental bounded resolution* $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.
(ii) X *is countable.*
(iii) $\mathbb{R}^X = \bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}} \overline{A_\alpha}^{\mathbb{R}^X}$ *for a fundamental bounded resolution* $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ *in* $C_p(X)$.
(iv) $L_\beta(X)$ *is a cosmic space, i.e. a continuous image of a metrizable separable space.*
(v) $C_p(X)$ *is a large subspace of* $\mathbb{R}^X$, *i.e. for every mapping* $f \in \mathbb{R}^X$, *there is a bounded set* $A \subseteq C_p(X)$ *such that* $f \in \overline{A}^{\mathbb{R}^X}$.

Then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Rightarrow (v)$ *but* $(v) \Rightarrow (ii)$ *fails even for compact spaces* X.

In order to prove this proposition, we need the following lemma from [223, page 392]. Recall that the strong dual $L_\beta(X)$ of $C_p(X)$ over a Tychonoff space X is the dual of $C_p(X)$ endowed with the topology of the uniform convergence on bounded sets of $C_p(X)$.

Lemma 16.6.13 *Let X be a Tychonoff space. Then every bounded set B in the strong dual $L_\beta(X)$ of the space $C_p(X)$ is finite-dimensional, i.e. the linear span of the set B is a finite-dimensional vector subspace of $L_\beta(X)$.*

Proof Let B be an infinite-dimensional subset of $L_\beta(X)$. We identify X with a Hamel basis of $L_\beta(X)$.

Choose sequences $(z_n)_n \subset B$ and $(u_n)_n \subset X$ of different elements such that, in the expansion of z_n, the coefficient c_n of u_n is non-zero. Let $(x_n)_n$ be a subsequence

of $(u_n)_n$ for which there is a sequence $(U_n)_n$ of disjoint open sets in X with $x_n \in U_n$, and let $(y_n)_n$ and $(b_n)_n$ be the corresponding subsequences of $(z_n)_n$ and $(c_n)_n$, respectively. For each n, choose an open set $V_n \subset U_n$ such that $x_n \in V_n$ and V_n misses the finitely many other $x \in X$ having non-zero coefficients in the expansion of y_n. Next, choose $g_n \in C(X)$ such that $g_n(x_n) = n/b_n$ and $g_n(X \setminus V_n) = \{0\}$. Note that $g_n(y_n) = g_n(b_n x_n) = n$ and, since the V_n are pairwise disjoint, $(g_n)_n$ is bounded in $C_p(X)$. It follows that $(y_n)_n$, hence B is not bounded in $L_\beta(X)$. □

Proof of Proposition 16.6.12 (i) $\Rightarrow$ (ii): If $C_p(X)$ has a fundamental bounded resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, then the sets

$$U_\alpha = \{\xi \in L_\beta(X) : \sup_{f \in A_\alpha} |\xi(f)| \leq 1\}$$

form a $\mathfrak{G}$-base in $L_\beta(X)$. By Lemma 16.6.13 we know that every bounded set in $L_\beta(X)$ is finite-dimensional. Applying Theorem 16.6.1, we conclude that the Hamel basis X of the lcs $L_\beta(X)$ is countable:

(ii) $\Rightarrow$ (i) is clear.

(ii) $\Rightarrow$ (iii)$\wedge$(v): Since $C_p(X)$ is dense in metrizable $\mathbb{R}^X$, the claims hold. (ii) $\Rightarrow$ (iv): If X is countable, then $L_\beta(X)$ has a fundamental sequence of compact sets covering $L_\beta(X)$ and [445, Proposition 7.7] applies to get that $L_\beta(X)$ is an $\aleph_0$-space, hence $L_\beta(X)$ is cosmic.

(iv) $\Rightarrow$ (ii): If $L_\beta(X)$ is cosmic, then it is separable. Hence every bounded set in $C_p(X)$ is metrizable. We will show that this latter property implies that X is countable. Indeed, note that the interval $(-1, 1)$ and the whole line $\mathbb{R}$ are homeomorphic. It is well known that $C_p(X, (-1, 1))$ and $C_p(X, \mathbb{R})$ (= $(C_p(X))$ are homeomorphic. On the other hand, the set $C_p(X, (-1, 1))$ is pointwise bounded in $C_p(X)$; hence, it is metrizable. We proved that $C_p(X)$ metrizable, so X is countable.

(v) $\not\Rightarrow$ (ii): Note that $C_p(X)$ over an Eberlein scattered compact, X is distinguished (see [213, Theorem 49]); hence, it satisfies (v). □

The following is a linear counterpart to item (iv) in Proposition 16.6.12.

Remark 16.6.14 A Tychonoff space X is finite if and only if $L_\beta(X)$ is a continuous linear image of a metrizable lcs.

Indeed, if X is finite, the claim is clear. Now assume that $L_\beta(X)$ is a continuous linear image of a metrizable lcs E (by an injective map). Since $L_\beta(X)$ has only finite-dimensional bounded sets and E fails this property, we conclude that X is finite.

We complete this section with the following.

Example 16.6.15 Let X be a Tychonoff space with a fundamental compact resolution. Then for the space $C_c(X)$, every uncountable-dimensional subspace contains a metrizable infinite-dimensional compact set.

Chapter 17
Distinguished Spaces $C_p(X)$ and Δ-Spaces X

Abstract In this chapter we present a number of recent results related to the study of spaces $C_p(X)$ of continuous real-valued functions on a Tychonoff spaces X that are distinguished. We prove, among the others, that the locally convex space $C_p(X)$ is distinguished if and only if X is a Δ-space in the sense of Reed. This theorem apparently provides (with several applications) a significant "bridge" between the still attractive problems from the set theory and related with Δ-sets, λ-sets, and $\mathbb{Q}$-sets X, and corresponding distinguished spaces $C_p(X)$.

17.1 Distinguished Spaces $C_p(X)$ over Tychonoff Spaces X. Introduction

Recall again that a lcs E is called *distinguished* if its strong dual $E'_\beta = (E', \beta(E', E))$ is a barrelled space. Clearly, each reflexive space is semi-reflexive, and each semi-reflexive space is distinguished [374, 23.3 (4)]. In fact, (an alternate definition [374, 23.7]), E is distinguished if and only if E is a large subspace of $\left(E'', \sigma\left(E'', E'\right)\right)$.

A subspace F of a lcs G is a *large subspace* of G if every bounded set in G is contained in the closure in G of a bounded set in F, [491, Definition 8.3.22].

It is well known that $C_p(X)$ is semi-reflexive if and only if $C_p(X)$ is reflexive, if and only if X is discrete. We will see that the simplest examples of distinguished $C_p(X)$ spaces which are not semi-reflexive are those with X any countable non-discrete Tychonoff space X.

In this section, we will work on the following problem posed originally in [213].

Problem 17.1.1 Characterize those Tychonoff spaces X such that $C_p(X)$ is distinguished.

For exploring the dual $L_p(X)$ of spaces $C_p(X)$, we provide some useful additional concepts.

We recall that a lcs E is *feral* if every bounded subset is finite dimensional, *flat* if $E' = E^*$, and *fit* if it has a dense (linear) subspace whose codimension equals the dimension of E (see [532, 540]). Each flat space is feral, but not conversely.

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If A is a Hamel basis for E, let B be all points in A whose corresponding coefficient functionals are discontinuous, set $C := A \setminus B$, and let E_B and E_C be the linear span of B and C, respectively.

Clearly, E splits algebraically into the direct sum of E_B and E_C. We say A is a *continuous basis* if $B = \emptyset$.

The splitting theorem [518] asserts that if E is barrelled, then E is *topologically* the direct sum of E_B and E_C and, moreover, the component E_B is fit, while E_C bears the strongest locally convex topology (see [540]). Equivalently, E_C is barrelled and flat. This applies to get the following theorem. Following arguments from [540], we provide the proofs of the equivalences (i), (iii), and (v). The remaining equivalences are clear.

Theorem 17.1.2 *The following assertions for a lcs E are equivalent:*

(i) *E carries its strongest locally convex topology.*
(ii) *E is the strong dual of the product of* dim *E-many lines.*
(iii) *Every absolutely convex absorbing set is a neighbourhood of zero in E.*
(iv) *E is barrelled and flat.*
(v) *E is barrelled and admits a continuous basis.*

Proof The equivalence between (i) and (iii) is an elementary fact from the theory of topological vector space. (i) $\Rightarrow$ (v): For the strongest locally topology ξ on a vector space E, we know that every linear functional is ξ-continuous and every basis is ξ-continuous.

(v) $\Rightarrow$ (iii): Let $H \subset E$ be a Hamel basis of E with continuous coefficient functionals f_x for each $x \in H$. Let V be an absolutely convex absorbing subset of E. For each $x \in H$, there exists $t_x > 0$ such that $x^* = t_x x \in V$. Set $A = \{x^* : x \in H\}$. Then, A is another Hamel basis of E with continuous coefficient $g_x^* = (t_x)^{-1} f_x$. The polar set A° is $\sigma(E', E)$-bounded, so the bipolar $A^{\circ\circ}$ is a neighbourhood of zero in the barrelled space E. Now, let $y = \sum_{x\in U} a_x x^* \in A^{\circ\circ}$ for some finite subset $U \subset H$ and scalars a_x. Choose $g = \sum_{x\in U} (\overline{sgna_x}) h_{x^*} \in A^\circ$. Then, $\sum_{x\in U} |a_x| = |g(y)| \leq 1$. This shows that $A^{\circ\circ} \subset V$. □

Recall that the *character* $\chi(E)$ of a lcs E is the smallest cardinality for a base of neighbourhoods of zero in E. Note the following general fact.

Theorem 17.1.3 ([532, Theorem 1]) *A lcs E is fit if $\chi(E) \leq$* dim *E.*

This fact can be used to prove the following result for spaces $C_p(X)$; see [213].

Theorem 17.1.4 *The strong dual of a metrizable lcs E is either fit or flat.*

Proof The proof splits into two parts.
Case 1. Suppose no infinite-dimensional subspace of E admits a continuous norm, i.e., E has its weak topology $\sigma\left(E, E'\right)$. Then, E is (isomorphically) a dense subspace of the product G of dim E'-many lines, and metrizability implies dim $E' \leq \aleph_0$. Thus, G is separable and metrizable, so E is a large subspace of G [491, 8.3.23

(b)] and has the same strong dual as G. Hence, the strong dual E'_β of E has its strongest locally convex topology and therefore is flat (and barrelled).

Case 2. Suppose that some infinite-dimensional subspace of E admits a continuous norm. Then, $\dim E' \geq \mathfrak{c}$ by applying the theorems of Mazur and Hahn–Banach. Then, metrizability of E and [537, Theorem 1] provide a fundamental family $\mathcal{B}$ of bounded sets in E with $|\mathcal{B}| \leq \mathfrak{d}$, with $\mathfrak{d}$ the dominating cardinal. Therefore, $\chi\left(E'_\beta\right) \leq \mathfrak{d} \leq \mathfrak{c} \leq \dim E'_\beta$, so that E'_β is fit by applying Theorem 17.1.3. □

Case 1 of the above proof applies also to get the following fact.

Theorem 17.1.5 ([211, Theorem 3.3 (a)]) *If X is a countable Tychonoff space, then $C_p(X)$ is distinguished.*

Case 1 from the proof of Theorem 17.1.4 shows that if X is countable, $L_\beta(X)$ has its strongest locally convex topology and thus admits a continuous basis.

Note that *all* $L_\beta(X)$ spaces admit continuous bases.

Theorem 17.1.6 *The homeomorphic copy of X in $L_p(X)$ is a continuous basis for the space $L_\beta(X)$.*

Proof The set A of functions g in $C(X)$ such that $|g(x)| \leq 1$ for all $x \in X$ is pointwise bounded, so the polar A^0 is a zero-neighbourhood in $L_\beta(X)$. Tychonoff extension theory ensures A^0 is just the absolutely convex hull of X.

To see this, let $y := \sum_{x\in\triangle} a_x \cdot x$ be a finite linear combination from X, choose $g \in A$ with $g(x) = sgna_x$ for each $x \in \triangle$, and note that

$$|\langle g, y\rangle| = \sum_{x\in\triangle} |a_x| \leq 1$$

if and only if $y \in A^0$. Thus, the coefficient functionals for the basis X are each numerically bounded by 1 on A^0, so are continuous on $L_\beta(X)$. □

Consequently, we are able to formulate the following theorem; the last statement has been already proved in [211].

Theorem 17.1.7 *For a Tychonoff space X, these three assertions are equivalent:*

(1) $C_p(X)$ is distinguished.
(2) $C_p(X)$ is a large subspace of $\mathbb{R}^X$.
(3) $L_\beta(X)$ has its strongest locally convex topology, [209].

Proof Theorems 17.1.2 and 17.1.6 prove (1) $\Leftrightarrow$ (3), which is just [209, Corollary 3.4]. Since $C_p(X)$ is a dense subspace of the product $\mathbb{R}^X$, we may algebraically identify $L(X)$ as their common dual by means of restriction. Then, $L_\beta(X)$ is their common *strong* dual if and only if $C_p(X)$ is large in $\mathbb{R}^X$, by the bipolar theorem. Finally, Theorem 17.1.2 ensures (2) $\Leftrightarrow$ (3). □

17.2 General Results on Distinguished Spaces $C_p(X)$

Most of the results in this section were presented in articles [211, 213], and [224]. We refer also the reader to [233] for some recent results about a new splitting theorem for the space $L_\beta(X)$ providing applications for distinguished spaces $C_p(X)$.

First notice that the space $C_p(X)$ to be distinguished must be a large subspace of $\mathbb{R}^X$ (see Theorem 17.1.7).

We need some extra notations. For each $g \in \mathbb{R}^X$, define

$$P_g := \left\{ f \in \mathbb{R}^X : |f| \leq |g| \right\},$$

a bounded set in $\mathbb{R}^X$. For each bounded set A in $\mathbb{R}^X$, define $\varphi_A \in \mathbb{R}^X$ by writing

$$\varphi_A(x) := \sup_{f \in A} |f(x)|$$

for each $x \in X$. For each bounded set B in $C_p(X)$, define another bounded set B^+ in $C_p(X)$ consisting of those h in $C(X)$ for which there exists a finite set $F \subseteq B$ such that

$$|h(x)| \leq \max_{f \in F} |f(x)|$$

for all $x \in X$.

We start with the following somewhat technical lemma which will be used in the sequel.

Lemma 17.2.1 *If a subset A of $\mathbb{R}^X$ is contained in the closure $\overline{B}$ in $\mathbb{R}^X$ of a bounded set B in $C_p(X)$, then $P_{\varphi_A} \subseteq \overline{B^+}$. Hence, $P_{\varphi_B} \subseteq \overline{B^+}$ holds.*

Proof Given $f_0 \in P_{\varphi_A}$, a finite set Δ in X, and $\epsilon > 0$, choose by definition of φ_A a finite set G in A with $|G| = |\Delta|$ such that, for every $x \in \Delta$

$$\varphi_A(x) < \max_{g \in G} |g(x)| + \epsilon.$$

Since $G \subseteq \overline{B}$, there exists $F \subseteq B$ with $|F| = |G|$ such that, for every $x \in \Delta$

$$|f_0(x)| \leq \varphi_A(x) < \max_{f \in F} |f(x)| + \epsilon.$$

For each $x \in \Delta$, choose a_x with $|a_x| \leq \max_{f \in F} |f(x)|$ and $|f_0(x) - a_x| < \epsilon$. As X is Tychonoff, there exists $h \in C(X)$ such that $h(x) = a_x$ for all $x \in \Delta$ and $|h(x)| \leq \max_{f \in F} |f(x)|$ for all $x \in X$. Therefore, h is in B^+ and within ϵ of f_0 on Δ, so $f_0 \in \overline{B^+}$. □

We denote by $\mathcal{P}_Y$ the canonical projection from $\mathbb{R}^X$ onto the subspace of functions whose support lies in Y.

Theorem 17.2.2 *Let $\mathcal{F}$ be a finite family of sets covering X.*

(i) If $C_p(X)$ is distinguished and $Y \subseteq X$, then $C_p(Y)$ is also distinguished.
(ii) The space $C_p(X)$ is distinguished if and only if for each bounded set A in $\mathbb{R}^X$ and $Y \in \mathcal{F}$ there is a bounded set B in $C_p(X)$ such that $\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(B)}^{\mathbb{R}^X}$.
(iii) Assume that $C_p(Y)$ is distinguished for each $Y \in \mathcal{F}$. If for each bounded set D in $C_p(Y)$ there is a bounded set B in $C_p(X)$ with $B|_Y = D$ (e.g. if each $C_p(Y)$ admits a continuous linear extender), then $C_p(X)$ is distinguished.

Proof If A is bounded in $\mathbb{R}^X$ and $Y \subseteq X$, then $\mathcal{P}_Y(A)$ is bounded in $\mathbb{R}^X$. If $C_p(X)$ is distinguished, we apply Theorem 17.1.7 to get that

$$\mathcal{P}_Y(A) \subseteq \overline{B}$$

for some bounded set B in $C_p(X)$.

Applying $\mathcal{P}_Y$, we get

$$\mathcal{P}_Y(A) \subseteq \mathcal{P}_Y\left(\overline{B}\right) \subseteq \overline{\mathcal{P}_Y(B)}.$$

The inclusion

$$\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(B)}$$

just means the restriction sets $A|_Y$ and $B|_Y$ satisfy $A|_Y \subseteq \overline{B|_Y}$, closure in $\mathbb{R}^Y$, where $B|_Y$ is bounded in $C_p(Y)$ and $A|_Y$ could be any bounded set in $\mathbb{R}^Y$. This proves (i) via Theorem 17.1.7 and necessity of the condition in (ii).

For the converse implication, let A be bounded in $\mathbb{R}^X$. For each $Y \in \mathcal{F}$, the condition posits a bounded set B_Y in $C_p(X)$ such that

$$\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(B_Y)}.$$

Thus, if $x \in Y \in \mathcal{F}$, then

$$\varphi_A(x) = \sup_{f \in A} |f(x)| \leq \sup_{h \in B_Y} |h(x)| \leq \sup_{h \in B} |h(x)| = \varphi_B(x),$$

where the finite union $B := \bigcup_{Y \in \mathcal{F}} B_Y$ of bounded sets is bounded in $C_p(X)$. Since $\mathcal{F}$ covers X, the inequality

$$\varphi_A(x) \leq \varphi_B(x)$$

holds for all $x \in X$. Using Lemma 17.2.1, we have

$$A \subseteq P_{\varphi_A} \subseteq P_{\varphi_B} \subseteq \overline{B^+}.$$

Thus, $C_p(X)$ is large in $\mathbb{R}^X$; the proof of (ii) is complete.

Now, (iii) follows from (ii). Indeed, if A is bounded in $\mathbb{R}^X$, so is $A|_Y$ in $\mathbb{R}^Y$. We obtain a bounded set D in $C_p(Y)$ with $A|_Y \subseteq \overline{D}^{\mathbb{R}^Y}$. By hypothesis, there is a bounded set B in $C_p(X)$ with $B|_Y = D$. Therefore

$$\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(B)}^{\mathbb{R}^X}$$

and (ii) applies. □

Let us start with some concept (which in [213] was studied under the name *scant cover*) and some results from [213].

Definition 17.2.3 A family $\{\mathcal{N}_x : x \in X\}$ of subsets of a Tychonoff space X is called a *point-finite open assignment* for X if each $\mathcal{N}_x$ is a neighbourhood of x and for each $u \in X$ the set $X_u = \{x \in X : u \in \mathcal{N}_x\}$ is finite.

Example 17.2.4 Each countable Tychonoff space $X = \{x_n : n \in \mathbb{N}\}$ admits a point-finite open assignment. *Simply set* $\mathcal{N}_{x_n} = \{x_i : i \geq n\}$ *for each* $n \in \mathbb{N}$.

Theorem 17.2.5 ([213]) *If a Tychonoff space X admits a point-finite open assignment $\{\mathcal{N}_x : x \in X\}$, then $C_p(X)$ is distinguished.*

Proof Let A be bounded in $\mathbb{R}^X$. For each $y \in X$, choose $0 \leq f_y \in C(X)$ such that f_y vanishes off $\mathcal{N}_y$; $f_y(y) = \varphi_A(y)$; and

$$f_y(x) \leq \varphi_A(y)$$

for all $x \in X$. Since x is in $\mathcal{N}_y$ for only finitely many $y \in X$, the supremum

$$\sup_{y \in X} \left| f_y(x) \right|$$

cannot exceed the maximum of finitely many numbers of the form $\varphi_A(y)$. Hence, we deduce that

$$B := \left\{ f_y : y \in X \right\}$$

is a bounded set in $C_p(X)$. Now, Lemma 17.2.1 implies that $P_{\varphi_B} \subseteq \overline{B^+}$. On the other hand, $\varphi_A \leq \varphi_B$, so

$$A \subseteq P_{\varphi_A} \subseteq P_{\varphi_B} \subseteq \overline{B^+}.$$

Therefore, $C_p(X)$ is a large subspace of $\mathbb{R}^X$. This shows that $C_p(X)$ is distinguished. □

Corollary 17.2.6 *If X has only finitely many non-isolated points, then the space $C_p(X)$ is distinguished.*

Proof Let $X = \Gamma \cup \{u_1, \ldots, u_n\}$, where all points $x \in \Gamma$ are isolated in X. Then, the family

$$\{\mathcal{N}_x : x \in X\}$$

consisting of $\mathcal{N}_x = \{x\}$ if $x \in \Gamma$ and $\mathcal{N}_{u_i} = X$ if $1 \leq i \leq n$ is a a point-finite open assignment for X. Now, it is enough to apply Theorem 17.2.5. □

Corollary 17.2.7 *If X is a discrete space and $\alpha(X)$ stands for the one-point compactification or the one-point Lindelöfication of X, then $C_p(\alpha(X))$ is distinguished.*

A family $\mathcal{F}$ of subsets of X is called *point-finite* [5] if each $x \in X$ belongs at most to finitely many members of $\mathcal{F}$. It is called *σ-point-finite* if $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$, where each $\mathcal{F}_n$ is point-finite. Note that every a point-finite open assignment of X is point-finite, but not every point-finite (even clopen) cover is a a point-finite open assignment (e.g. take X any infinite space and set each $\mathcal{N}_x = X$).

Now, we are at the position to show the following main result of the section.

Theorem 17.2.8 ([213]) *Let X be a Corson compact space. The following assertions are equivalent:*

(1) *X is scattered.*
(2) *X is a scattered Eberlein compact space.*
(3) *$C_p(X)$ is distinguished.*

Proof (1) $\Rightarrow$ (2). If X is a Corson scattered compact, then X is a scattered Eberlein compact space by Alster's theorem [5, Theorem].

(2) $\Rightarrow$ (3). We apply the proof of [77, Lemma 1.1] with X an arbitrary scattered Eberlein compactum, so for each $a \in X$ there is a clopen neighbourhood V_a of a such that the family $\{V_a : a \in X\}$ is point-finite, with V_a and V_b (clearly) distinct for distinct $a, b \in X$. This shows that $\{V_a : a \in X\}$ is a a point-finite open assignment for X. Now, Theorem 17.2.5 implies that $C_p(X)$ is distinguished.

(3) $\Rightarrow$ (1). Assume that $C_p(X)$ is distinguished but X is a non-scattered Corson compact space. According to theorem of Pełczyński and Semadeni, there is a continuous surjection f from X onto the closed interval $[0, 1]$.

We show that there exists a compact set Y in X which is metrizable and $|Y| = \mathfrak{c}$. Indeed, fix any countable dense subset Q of $[0, 1]$ and choose a countable subset P in X such that $f(P) = Q$. Let Y be the closure of P in X. It is obvious that Y is metrizable, since it is a Corson separable compact space. In addition, Y must have cardinality continuum since by the density of P in Y and the density of Q in $[0, 1]$ one has $f(Y) = [0, 1]$. Since $C_p(Y)$ is not distinguished, neither is $C_p(X)$ distinguished, by the first statement of Theorem 17.2.2. □

17.3 Δ-Spaces X and Distinguished $C_p(X)$

In this section, which is mostly covered by results from [332, 333], and [406], we show that $C_p(X)$ over a Tychonoff space X is distinguished if and only if X is a Δ-space. As an application of this theorem, we obtain the following results:

(i) If X is a Čech-complete (in particular, compact) space such that $C_p(X)$ is distinguished, then X is scattered.
(ii) For every separable compact space of the Isbell–Mrówka type X, the space $C_p(X)$ is distinguished.
(iii) If X is the compact space of ordinals $[0, \omega_1]$, then $C_p(X)$ is not distinguished.

Another characterization for $C_p(X)$ in terms of X has been obtained by Ferrando and Saxon in [232].

First, we recall the following notions.

Definition 17.3.1 A topological space X is said to be a Δ-space if for every decreasing sequence $\{D_n : n \in \mathbb{N}\}$ of subsets of X with empty intersection, there is a decreasing sequence $\{V_n : n \in \mathbb{N}\}$ consisting of open subsets of X, also with empty intersection, and such that $D_n \subset V_n$ for every $n \in \mathbb{N}$.

The class of all Δ-spaces is denoted by Δ. We should mention that Knight [369] called all topological spaces X satisfying above Definition 17.3.1 by Δ-sets. The original definition of a Δ-set of the real line $\mathbb{R}$ is due to Reed and van Douwen (see [515]).

Note also that a Q-set X is a subset of $\mathbb{R}$ such that each subset of X is G_δ. A subset X of $\mathbb{R}$ is called a λ-set, if each countable subset of X is G_δ in X. It is well known that every Δ-set is a λ-set.

It should be pointed out that Kuratowski constructed uncountable λ-sets in ZFC; see [393, p.517]. On the other hand, Hausdorff showed that the cardinality of an uncountable Q-set X has to be strictly smaller than continuum, so assuming the Continuum Hypothesis CH there are no uncountable Q-sets. More details about Q-sets and Δ-sets can be found in [237].

Martin's axiom plus the negation of the Continuum Hypothesis implies that every subset $X \subset \mathbb{R}$ of cardinality less than continuum is a Q-set. No Δ-set X can have cardinality continuum, [509]. Consequently, under Martin's axiom, every subset of reals that is a Δ-set is also a Q-set. We refer the readers to [404] for a discussion about these objects.

It is interesting to mention that Δ-sets in $\mathbb{R}$ have been used and investigated thoroughly in the study of two the most basic and central constructions in general topology: the Moore–Nemytskii plane and the Pixley–Roy topology. For example, if $M(X)$ is the subspace of the Moore–Nemytskii plane, which is obtained by using only a subset $X \subset \mathbb{R}$ of the x-axis, Reed observed that $M(X)$ is countably paracompact if and only if X is a Δ-set [515].

More details about Q-sets and Δ-sets can be found in [237]. Of course, there are plenty of non-metrizable Δ-spaces with non-G_δ subsets, in ZFC [332].

In the sequel, we will need also the following concepts:

(1) A disjoint cover $\{X_\gamma : \gamma \in \Gamma\}$ of X is called a *partition* of X.
(2) A collection of sets $\{U_\gamma : \gamma \in \Gamma\}$ is called an *expansion* of a collection of sets $\{X_\gamma : \gamma \in \Gamma\}$ in X if $X_\gamma \subseteq U_\gamma \subseteq X$ for every index $\gamma \in \Gamma$.
(3) A collection of sets $\{U_\gamma : \gamma \in \Gamma\}$ is called *point-finite* if no point belongs to infinitely many U_γ 's.

Using [211, 213, 232, 332, 518] (and Theorems 17.1.2 and 16.2.5 above), we put together several equivalent statements for distinguished spaces $C_p(X)$; see also [351].

Ferrando and Kąkol [211, Theorem 3.9] proved that always the strong dual of $C_p(X)$ is distinguished (so the strong bidual $M_\beta(X)$ is barrelled), and then, Ferrando and Saxon asked [232, Problem 11] if the Baire property of $M_\beta(X)$ implies that $C_p(X)$ is distinguished. The answer is negative (as noticed in [232, Addendum]) since $M(\omega_1)$ is Baire [232, Corollary 21] but $C_p(\omega_1)$ is not distinguished; see Theorem 17.4.15 below.

Recall here (see [355]) that a lcs E is called *primitive* if, given any increasing, covering sequence of subspaces $(E_n)_n$ each linear functional f over E such that each restriction to E_n is continuous, the functional f itself is continuous.

Theorem 17.3.2 *For a Tychonoff space X, the following are equivalent:*

(1) $C_p(X)$ *is distinguished.*
(2) The strong dual of $C_p(X)$ *carries the finest locally convex topology.*
(3) $C_p(X)$ *and* $\mathbb{R}^X$ *have the same strong duals.*
(4) The strong dual $L_\beta(X)$ *of* $C_p(X)$ *is the direct sum of* $|X|$*-many lines.*
(5) $L_\beta(X)$ *is a reflexive space.*
(6) $L_\beta(X)$ *is a bornological space.*
(7) $L_\beta(X)$ *is a quasibarrelled space.*
(8) $L_\beta(X)$ *is a primitive space.*
(9) The strong bidual $M_\beta(X)$ *of* $C_p(X)$ *is the product space* $\mathbb{R}^X$.
(10) $M(X) = \mathbb{R}^X$ *(as sets), i.e.* $L_\beta(X)' = L_\beta(X)^*$.
(11) $M_\beta(X)$ *is reflexive.*
(12) $M_\beta(X)$ *is quasi-complete.*
(13) $C_p(X)$ *is large in* $\mathbb{R}^X$.
(14) For each $f \in \mathbb{R}^X$*, there exists a bounded subset* B *of* $C_p(X)$ *such that* $f \in cl_{\mathbb{R}^X}(B)$.
(15) X *is a* Δ*-space.*
(16) Any countable disjoint collection of subsets of X *admits a point-finite open expansion in* X.
(17) Any countable partition of X *admits a point-finite open expansion in* X.

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follows from Theorems 17.1.2 and 17.1.6.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) are obvious.

(2) $\Rightarrow$ (6) is obvious. (6) $\Rightarrow$ (1) since every quasi-complete bornological lcs is a barrelled space ([90, Theorem 3.6.19]) and the space $L_\beta(X)$ is quasi-complete.

(1) $\Leftrightarrow$ (7) since every barrel in $L_\beta(X)$ is bornivorous.

(1) $\Rightarrow$ (8) since any barrelled locally convex space is primitive ([491, Proposition 4.1.6]).

(3) $\Leftrightarrow$ (13) follows by bipolar theorem.

(9) $\Rightarrow$ (11) $\Rightarrow$ (12) are obvious. (12) $\Rightarrow$ (9) $\mathbb{R}^X$ is the quasi-completion of the subspace G consisting of all functions $f \in \mathbb{R}^X$ with finite support. By Theorem 16.2.5, $G \subset M_\beta(X) \subset \mathbb{R}^X$. Thus, $M_\beta(X) = \mathbb{R}^X$.

(3) $\Rightarrow$ (9) follows by reflexivity of $\mathbb{R}^X$. (9) $\Rightarrow$ (10) is obvious.

(10) $\Leftrightarrow$ (14) Put $F = C_p(X)$ and $E = M_\beta(X)$. Then, $F'^* = L(X)^* = \mathbb{R}^X$ and

$$\sigma(F'^*, F') = \sigma(L(X)^*, L(X)) = \sigma(\mathbb{R}^X, L(X))$$

is the product topology of $\mathbb{R}^X$.

By Schmets [548, Theorem 5.4, IV], $f \in E$ if and only if there exists a bounded subset B of F such that

$$f \in \mathrm{cl}_{\sigma(F'^*, F')}(B) = \mathrm{cl}_{\mathbb{R}^X}(B).$$

(8) $\Rightarrow$ (10) Let $f \in \mathbb{R}^X$. Put $X_n = \{x \in X : |f(x)| \leq n\}$ and

$$L_n = \{\mu \in L(X) : \mathrm{supp}(\mu) \subset X_n\}$$

for $n \in \mathbb{N}$. Clearly, (L_n) is an increasing sequence of linear subspaces of $L(X)$ with $\bigcup_{n=1}^\infty L_n = L(X)$. Let $n \in \mathbb{N}$ and $f_n = f\chi_{X_n}$. The set $B_n = \{g \in C(X) : |g(x)| \leq n \text{ for all } x \in X\}$ is bounded in $C_p(X)$ and $f_n \in \mathrm{cl}_{\mathbb{R}^X}(B_n)$. By Schmets [548, Theorem 5.4, IV], $\mathrm{cl}_{\mathbb{R}^X}(B_n) \subset M(X)$, so $f_n \in M(X)$. Clearly, $f|_{L_n} = f_n|_{L_n}$, so $f|_{L_n}$ is continuous for any $n \in \mathbb{N}$. Hence, $f \in M(X)$. Thus, $\mathbb{R}^X = M(X)$.

(13) $\Rightarrow$ (14) is obvious.

(14) $\Rightarrow$ (15) Let (X_n) be a decreasing sequence of subsets of X with empty intersection. Put $X_0 = X$. Let $f \in \mathbb{R}^X$ with $f(x) = n + 1$ for all $x \in [X_{n-1} \setminus X_n]$, $n \in \mathbb{N}$. Let B be a bounded subset of $C_p(X)$ with $f \in \mathrm{cl}_{\mathbb{R}^X}(B)$. Let $x \in X$. Then, $x \in [X_{n-1} \setminus X_n]$ for some $n \in \mathbb{N}$.

The set $W = \{g \in \mathbb{R}^X : |g(x) - f(x)| < 1\}$ is a neighbourhood of f in $\mathbb{R}^X$, so $W \cap B \neq \emptyset$. Let $g_x \in W \cap B$. Then, $g_x(x) > n$, so the set $V_x = g_x^{-1}((n, \infty))$ is an open neighbourhood of x in X. Put $U_n = \bigcup_{x \in X_n} V_x$ for $n \in \mathbb{N}$. Clearly, (U_n) is a decreasing sequence of open subsets of X and $X_n \subset U_n$ for $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. For $x \in X_n$ and $z \in V_x$, we have $g_x(z) > n$, so $\sup_{g \in B} |g(z)| > n$ for $z \in U_n$. Let $y \in X$.

For some $m \in \mathbb{N}$, we have $\sup_{g \in B} |g(y)| \leq m$, since B is bounded in $C_p(X)$. It follows that $y \notin U_m$, so $y \notin \bigcap_{n=1}^\infty U_n$. Thus, $\bigcap_{n=1}^\infty U_n = \emptyset$.

(15) $\Rightarrow$ (13) Let A be a non-empty bounded subset of $\mathbb{R}^X$. Let $\psi : X \to \mathbb{R}$, $\psi(x) = \sup_{f \in A} |f(x)|$.

Put $X_n = \{x \in X : |\psi(x)| \geq n - 1\}$ for $n \in \mathbb{N}$. Then, (X_n) is a decreasing sequence of subsets of X with empty intersection. Thus, there exists a decreasing

sequence (U_n) of open subsets of X with empty intersection such that $X_n \subset U_n$ for $n \in \mathbb{N}$. For any $x \in X$, there exists $\varphi(x) \in \mathbb{N}$ with

$$x \in [U_{\varphi(x)} \setminus U_{\varphi(x)+1}].$$

We have $\psi(x) < \varphi(x)$ for every $x \in X$. Indeed, let $x \in X$ and $n \in \mathbb{N}$ with $n-1 \leq |\psi(x)| < n$. Then, $x \in X_n \subset U_n$ and $x \notin U_{\varphi(x)+1}$, so $n < \varphi(x)+1$. Thus, $\psi(x) < n \leq \varphi(x)$.

Clearly, the set $B = \{g \in C_p(X) : |g| \leq \varphi\}$ is bounded in $C_p(X)$. We shall prove that $A \subset \mathrm{cl}_{\mathbb{R}^X}(B)$. Let $f \in A$. Let W be a neighbourhood of f in $\mathbb{R}^X$. Then, there exists a finite subset K of X such that

$$\{g \in \mathbb{R}^X : g|_K = f|_K\} \subset W.$$

Let $\{V_x : x \in K\}$ be a family of pairwise disjoint open subsets of X with

$$x \in V_x \subset U_{\varphi(x)}, x \in K.$$

For every $x \in K$, there exists a continuous function

$$h_x : X \to [-\varphi(x), \varphi(x)]$$

with $h_x(x) = f(x)$ and $h_x(y) = 0$ for $y \in V_x^c$ (see [195, Theorem 3.1.7]).

The function $h : X \to \mathbb{R}, h = \sum_{x \in K} h_x$ is continuous and $h|_K = f|_K$, so $h \in W$.

We shall prove that $h \in B$. Clearly, $h(x) = 0$ for $x \in (\bigcup_{x \in K} V_x)^c$. Let $x \in K$. For $y \in [K \setminus \{x\}]$, we have $V_x \subset V_y^c$, so $h_y|_{V_x} = 0$.

Thus, $h|_{V_x} = h_x|_{V_x}$. For $t \in V_x$, we have

$$|h(t)| = |h_x(t)| \leq \varphi(x) \leq \varphi(t),$$

since $V_x \subset U_{\varphi(x)}$. Thus, $|h| \leq \varphi$, so $h \in B$.

It follows that $W \cap B \neq \emptyset$, so $f \in \mathrm{cl}_{\mathbb{R}^X}(B)$. Hence, $A \subset \mathrm{cl}_{\mathbb{R}^X}(B)$.

The equivalences (15) $\Leftrightarrow$ (16) $\Leftrightarrow$ (17) are clear. □

We provide a few applications of Theorem 17.3.2. Note that Corollary 17.3.3 was also proved in [213] by a different argument (see Theorem 17.2.2).

Corollary 17.3.3 ([213]) *Let Z be any subspace of X. If X a Δ-space, then Z also is a Δ-space.*

Proof If $\{Z_\gamma : \gamma \in \Gamma\}$ is any collection of pairwise disjoint subsets of Z and there exists a point-finite open expansion $\{U_\gamma : \gamma \in \Gamma\}$ in X, then obviously $\{U_\gamma \cap Z : \gamma \in \Gamma\}$ is a point-finite expansion consisting of the sets relatively open in Z. Theorem 17.3.2 applies. □

Last Corollary can be reversed, if $X \setminus Z$ is finite. Indeed, we have the following:

Proposition 17.3.4 *Let Z be a subspace of X such that $Y = X \setminus Z$ is finite. If Z is a Δ-space, then X is a Δ-space.*

Proof Let $\{X_n : n \in \mathbb{N}\}$ be any countable collection of pairwise disjoint subsets of X. Denote by F the set of those $n \in \mathbb{N}$ such that $X_n \bigcap Y \neq \emptyset$. There might be only finitely many X_n -s which intersect the finite set Y; hence, $F \subset \mathbb{N}$ is finite. If $n \in F$, then we simply declare that U_n is equal to X. Consider the subcollection $\{X_n : n \in \mathbb{N} \setminus F\}$. It is a countable collection of pairwise disjoint subsets of Z.

Since Z is a Δ-space, by Theorem 17.3.2, we derive that there is a point-finite open expansion $\{U_n : n \in \mathbb{N} \setminus F\}$ in Z. Observe that Z is open in X; therefore, all those sets U_n are open in X. Bringing all such U_n of both sorts together, we obtain a point-finite open expansion $\{U_n : n \in \mathbb{N}\}$ in X. Finally, X is a Δ-space by Theorem 17.3.2. □

Our Theorem 17.3.2 generalizes Theorem 17.2.5. Indeed, let $\{X_\gamma : \gamma \in \Gamma\}$ be any collection of pairwise disjoint subsets of X. Define

$$U_\gamma = \bigcup \{\mathcal{N}_x : x \in X_\gamma\}.$$

It is easily seen that $\{U_\gamma : \gamma \in \Gamma\}$ is a point-finite open expansion in X, by definition of a a point-finite open assignment. Applying Theorem 17.3.2, we conclude that $C_p(X)$ is distinguished.

17.4 Compact Δ-Spaces

We recall the following few facts which will be used in the sequel. A continuous surjection $\pi : X \to Y$ is called *irreducible* (see [552, Definition 7.1.11]) if for every closed subset F of X the condition $\pi(F) = Y$ implies $F = X$.

Theorem 17.4.1 ([552, Theorem 8.5.4]) *A compact space X is scattered if and only if there is no continuous mapping of X onto the segment* $[0, 1]$.

Proposition 17.4.2 ([552, Proposition 7.1.13]) *Let X be a compact space and let $\pi : X \to Y$ be a continuous surjection. Then, there exists a closed subset F of X such that $\pi(F) = Y$ and the restriction $\pi | F : F \to Y$ is irreducible.*

Proposition 17.4.3 ([552, Proposition 25.2.1]) *Let X be a compact space and let $\pi : X \to Y$ be a continuous surjection. Then, π is irreducible if and only if whenever $E \subset X$ and $\pi(E)$ is dense in Y, then E is dense in X.*

We will use the following well-known fact. We provide a simple proof; see [56, Theorem 1].

Proposition 17.4.4 *A Čech-complete space X is scattered if and only if every compact subset of X is scattered.*

Proof Assume that X is not scattered. Then, X contains a non-empty subset A without isolated points. Let Y be any compactification of X. Let $(U_n)_n$ be a decreasing sequence of open sets in Y with $X = \bigcap_n U_n$. Applying the inductive procedure for each $n \in \mathbb{N}$ and every binary sequence $t \in \{0, 1\}^n$, we find open sets V_t in Y such that

(i) $V_t \cap A \neq \emptyset$.
(ii) $\overline{V}_{t^0} \cap \overline{V}_{t^1} = \emptyset$.
(iii) $\overline{V}_{t^0} \cup \overline{V}_{t^1} \subset V_t \cap U_n$.

Set $K = \bigcap_n \bigcup_{t\in\{0,1\}^n} \overline{V}_t \subset \bigcap_n U_n = X$. Then, K is compact in X without isolated points, a contradiction. The converse implication is clear. □

This implies immediately that if X is a Čech-complete space which cannot be mapped onto $[0, 1]$ by a continuous map, then X is scattered. We recall also here the following result of Banakh, Bokalo, and Tkachuk [59]; see also [56, Theorem 3], which will be also used below.

Proposition 17.4.5 ([56, 59]) *Let X be a K-analytic space. Then, every compact subset of X is scattered if and only if there is no continuous surjection from X onto $[0, 1]$.*

We are ready to prove the following:

Theorem 17.4.6 ([332]) *Every Čech-complete (in particular, compact) Δ-space is scattered.*

Proof *Step 1*: Assume that X is compact but X is not scattered. By Theorem 17.4.1, there is a continuous mapping π from X onto the segment $[0, 1]$. By Proposition 17.4.2, there exists a closed subset F of X such that $\pi(F) = [0, 1]$ and the restriction $\pi|F : F \to [0, 1]$ is irreducible. Since X is a Δ-space, the compact space F also is a Δ-space, by Corollary 17.3.3. For simplicity, we may assume that F is X itself and $\pi : X \to [0, 1]$ is irreducible.

Let $\{X_n : n \in \mathbb{N}\}$ be a partition of $[0, 1]$ into dense sets. Put $Y_n = \bigcup_{k\geq n} X_k$ and $Z_n = \pi^{-1}(Y_n)$ for all $n \in \mathbb{N}$. Then, all sets Z_n are dense in X by Proposition 17.4.3, and the intersection $\bigcap_{n\in\mathbb{N}} Z_n$ is empty. Every compact space X is a Baire; hence, if $\{U_n : n \in \omega\}$ is any open expansion of $\{Z_n : n \in \mathbb{N}\}$, then the intersection $\bigcap_{n\in\omega} U_n$ is dense in X. In view of our Theorem 17.3.2, this conclusion contradicts the assumption that X is a Δ-space.

Step 2: Assume X is a Čech-complete space. By the first step, we deduce that every compact subset of X is scattered. Now, it is enough to apply Proposition 17.4.4. □

Proposition 17.4.7 *If X is a first-countable compact space, then X is a Δ-space if and only if X is countable.*

Proof If X is a Δ-space, then X is scattered, by Theorem 17.4.6. On the other hand, a first-countable compact space is scattered if and only if it is countable; see [552,

Theorem 8.6.10]. This proves (i) $\Rightarrow$ (ii). The converse is known since $C_p(X)$ is distinguished for countable X, and we apply Theorem 17.3.2. □

Corollary 17.4.8 ([368]) *Every countable Čech-complete space is scattered.*

Combining Theorems 17.3.2 and 17.2.8, we deduce the following:

Theorem 17.4.9 ([213]) *A Corson compact space X is a Δ-space if and only if X is a scattered Eberlein compact space.*

Applying the above Theorem 17.4.6 and Proposition 17.4.5, we note.

Corollary 17.4.10 *Let X be a K-analytic space which is mapped onto* $[0, 1]$ *by a continuous map. Then, X is not a Δ-space.*

Being motivated by the previous results, one can ask if there exist scattered compact Δ-spaces which are not Eberlein compact. In order to answer this question, we will need the following:

Theorem 17.4.11 *Let $Z = C_0 \cup C_1$ be a space such that*

(1) $C_0 \cap C_1 = \emptyset$.
(2) *C_0 is an open F_σ subset of Z.*
(3) *both C_0 and C_1 are Δ-spaces.*

Then, Z is also a Δ-space.

Proof By assumption, we know that $C_0 = \bigcup\{F_n : n \in \mathbb{N}\}$, where each F_n is closed in Z. Let $\{X_n : n \in \mathbb{N}\}$ be any countable collection of pairwise disjoint subsets of Z. We will define open sets $U_n \supseteq X_n$, $n \in \mathbb{N}$ that the collection $\{U_n : n \in \mathbb{N}\}$ is point-finite. We decompose the sets $X_n = X_n^0 \cup X_n^1$, where $X_n^0 = X_n \cap C_0$ and $X_n^1 = X_n \cap C_1$. By Theorem 17.3.2, the collection $\{X_n^0 : n \in \mathbb{N}\}$ expands to a point-finite open collection $\{U_n^0 : n \in \mathbb{N}\}$ in C_0. The set C_0 is open in Z; therefore, U_n^0 are open in Z as well.

Consider the disjoint collection $\{X_n^1 : n \in \mathbb{N}\}$ in C_1. By assumption, C_1 is a Δ-space; therefore, applying again Theorem 17.3.2, we find a point-finite expansion $\{V_n^1 : n \in \mathbb{N}\}$ in C_1 of sets which are open in C_1. Every set V_n^1 is a trace of some set W_n^1, which is open in Z, i.e. $V_n^1 = W_n^1 \cap C_1$, and every W_n^1 is open in Z. We refine the sets W_n^1 by the formula:

$$U_n^1 = W_n^1 \setminus \bigcup\{F_i : i \leq n\}.$$

Since all sets F_i are closed in Z, the sets U_n^1 are open in Z. Since all sets F_i are disjoint with C_1, the collection $\{U_n^1 : n \in \mathbb{N}\}$ remains to be an expansion of $\{X_n^1 : n \in \mathbb{N}\}$. Furthermore, the collection $\{U_n^1 : n \in \mathbb{N}\}$ is point-finite, as $\{V_n^1 : n \in \mathbb{N}\}$ is point-finite, and every point $z \in C_0$ belongs to some F_n, hence $z \notin U_m^1$ for every $m \geq n$. Now, define $U_n = U_n^0 \cup U_n^1$. The collection $\{U_n : n \in \mathbb{N}\}$ is a point-finite open expansion of $\{X_n : n \in \mathbb{N}\}$. □

Denote by $A^{(1)}$ the set of all non-isolated (in A) points of $A \subset X$. For ordinal numbers α, the α-th *derivative* of a topological space X is defined by transfinite induction as follows.

$X^{(0)} = X$; $X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}$; $X^{(\gamma)} = \bigcap_{\alpha<\gamma} X^{(\alpha)}$ for limit ordinals γ.

For a scattered space X, the smallest ordinal α such that $X^{(\alpha)} = \emptyset$ is called the *scattered height* of X and is denoted by $ht(X)$. For instance, X is discrete if and only if $ht(X) = 1$.

This yields the following:

Corollary 17.4.12 *Let Z be any separable scattered space such that its scattered height $ht(Z)$ is equal to 2. Then, Z is a Δ-space.*

Proof We present the space Z as follows: $Z = C_0 \cup C_1$, where C_0 is a countable dense in Z set consisting of isolated in Z points and C_1 consists of all accumulation points. Moreover, the space C_1 with the topology induced from Z is discrete. Hence, the assumptions included in Theorem 17.4.11 are satisfied. □

We provide a concrete example of a compact scattered Δ-space which is not Eberlein compact.

Let $\mathcal{A}$ be an almost disjoint family of subsets of the set of natural numbers $\mathbb{N}$ and let $\Psi(\mathcal{A})$ be the set $\mathbb{N} \cup \mathcal{A}$ equipped with the topology defined as follows. For each $n \in \mathbb{N}$, the singleton $\{n\}$ is open, and for each $A \in \mathcal{A}$, a base of neighbourhoods of A is the collection of all sets of the form $\{A\} \cup B$, where $B \subset A$ and $|A \setminus B| < \omega$. The space $\Psi(\mathcal{A})$ is then a first-countable separable locally compact Tychonoff space.

If $\mathcal{A}$ is a maximal almost disjoint (MAD) family, then the corresponding Isbell–Mrówka space $\Psi(\mathcal{A})$ would be in addition pseudocompact; see also [317, Chapter 8] which surveys various topological properties of such spaces.

Theorem 17.4.13 ([332]) *There exists a separable scattered compact space X with the following properties:*

(a) *The scattered height of X is equal to 3.*
(b) *X is a Δ-space.*
(c) *X is not an Eberlein compact space.*

Proof Let $\mathcal{A}$ be any uncountable almost disjoint family of subsets of $\mathbb{N}$. Let Z be the corresponding first-countable separable locally compact Isbell–Mrówka space $\Psi(\mathcal{A})$. It is easy to see that Z satisfies the assumptions of Corollary 17.4.12. Hence, Z is a Δ-space. Denote by X the one-point compactification of the separable locally compact space Z. Then, the scattered height of X is equal to 3.

On the other hand, X is a Δ-space by applying Proposition 17.3.4. Finally, X is not Eberlein compact, since every separable Eberlein compact space is metrizable, while $\Psi(\mathcal{A})$ is metrizable if and only if $\mathcal{A}$ is countable. □

In order to present an example of a compact scattered space which is not a Δ-space, we recall the classical pressing down lemma; see [389].

A subset S of $[0, \omega_1]$ is called a *stationary* subset if S has non-empty intersection with every closed and unbounded set in $[0, \omega_1]$. A mapping $\varphi : S \to [0, \omega_1]$ is called *regressive* if $\varphi(\alpha) < \alpha$ for each $\alpha \in S$.

Theorem 17.4.14 *Pressing down lemma Let $\varphi : S \to \omega_1$ be a regressive mapping, where S is a stationary subset of $[0, \omega_1]$. Then, for some $\gamma < \omega_1$, $\varphi^{-1}\{\gamma\}$ is a stationary subset of $[0, \omega_1]$.*

Every stationary set can be partitioned into countably many pairwise disjoint stationary sets [389]. Note that $[0, \omega_1]$ is a scattered locally compact and first-countable space. For simplicity, we identify $[0, \omega_1)$ with ω_1. Next statement resolves an open question which was posed in [213]. Another proof of this result can be found in [395, Lemma 6.2].

Theorem 17.4.15 ([332]) *The space $[0, \omega_1)$ is not a Δ-space. Consequently, the compact scattered space $[0, \omega_1]$ is not a Δ-space.*

Proof Assume that $\omega_1 = [0, \omega_1)$ is not a Δ-space. Evidently, L is a closed unbounded set in ω_1. Take any representation of L as the union of countably many pairwise disjoint stationary sets $\{S_n : n \in \mathbb{N}\}$. By Theorem 17.3.2, there exists a point-finite open expansion $\{U_n : n \in \mathbb{N}\}$ in ω_1.

For every $\alpha \in U_n$, there is an ordinal $\beta(\alpha) < \alpha$ such that

$$[\beta(\alpha), \alpha] \subset U_n.$$

In fact, for every $n \in \mathbb{N}$, we can define a regressive mapping $\varphi_n : S_n \to \omega_1$ by the formula: $\varphi_n(\alpha) = \beta(\alpha)$. Since S_n is a stationary set for every n, we can apply to φ_n the pressing down lemma. Hence, for each n, there are a countable ordinal γ_n and an uncountable subset $T_n \subset S_n$ with the following property: $[\gamma_n, \alpha] \subset U_n$ for every $\alpha \in T_n$. Denote

$$\gamma = \sup\{\gamma_n : n \in \mathbb{N}\} \in \omega_1.$$

Because all T_n are unbounded, for all natural n we have an ordinal $\alpha_n \in T_n$ such that $\gamma < \alpha_n$ and $[\gamma_n, \alpha_n] \subset U_n$. This implies that $\gamma \in U_n$ for every $n \in \mathbb{N}$. However, a collection $\{U_n : n \in \mathbb{N}\}$ is point-finite. This contradiction completes the proof. □

For non-scattered spaces X, our Theorem 17.4.6 implies the following corollary. For $X = \omega$, this is also [224, Corollary 2.2].

Corollary 17.4.16 *If X is a non-scattered space, the Čech–Stone compactification βX is not a Δ-space.*

Note that every scattered metrizable X is a Δ-space. Indeed, every such space X homeomorphically embeds into a scattered Eberlein compact space [63]. Apply Theorem 17.4.9 and Corollary 17.3.3. We extend this result.

A topological space X is called *σ-scattered* if X can be represented as a countable union of scattered subspaces, and X is called *strongly σ-discrete* if it is a

union of countably many of its closed discrete subspaces. Strongly σ-discreteness of X implies that X is σ-scattered but for metrizable X, by Stone [574], these two properties are equivalent.

Proposition 17.4.17 *Any σ-scattered metrizable space is a Δ-space.*

Proof We know already that every subset of X is F_σ, but then X is a Δ-space. Indeed, we show that X satisfies the condition (2) of Theorem 17.3.2. Let $\{X_n : n \in \mathbb{N}\}$ be any countable disjoint partition of X. Denote $X_n = \bigcup\{F_{n,m} : m \in \mathbb{N}\}$, where each $F_{n,m}$ is closed in X. Define open sets U_n as follows: $U_0 = X$ and

$$U_n = X \setminus \bigcup\{F_{k,m} : k < n, m < n\},\ \ n \geq 1.$$

Then, $\{U_n : n \in \mathbb{N}\}$ is a point-finite open expansion of $\{X_n : n \in \mathbb{N}\}$ in X. □

In [406, Theorem 3.1] Leiderman and Tkachuk proved the following result.

Theorem 17.4.18 ([406]) *If X is a pseudocompact Δ-space, then every countable subset of X is scattered.*

Proof Assume that some countable subset of X is not scattered. Then, there is a countable crowded set $A = \{a_n : n \in \mathbb{N}\} \subset X$ (i.e. without isolated points). Since X is a Δ-space of X, there exists a point-finite open expansion $\mathfrak{U} = \{U_n : n \in \mathbb{N}\}$ of the set A. Set $k_0 = 0$ and choose a set

$$V_0 \in \tau(a_{k_0}, X) = \{U \in \tau : a_{k_0} \in U\}$$

such that $\overline{V_0} \subset U_{k_0}$, where τ means the topology of X.

Assume that, for some $n \in \mathbb{N}$, we have open sets $V_0, \ldots, V_n$ in the space X together with $k_0, \ldots, k_n \in \mathbb{N}$ such that:

(1) $\overline{V_{i+1}} \subset V_i$ and $k_i < k_{i+1}$ whenever $0 \leq i < n$.
(2) $a_{k_i} \in V_i \subset \overline{V_i} \subset U_{k_0} \cap \ldots \cap U_{k_i}$ for every $i \leq n$.

Since $V_n \cap A$ is infinite, we can find a number $k_{n+1} \in \mathbb{N}$ such that $k_{n+1} > k_n$ and $a_{k_{n+1}} \in V_n$. There exists a set $V_{n+1} \in \tau(a_{k_{n+1}}, X)$ such that

$$\overline{V_{n+1}} \subset V_n \cap U_{k_{n+1}}.$$

The items (1) and (2) hold if replace n with $n + 1$. Hence, the inductive procedure can be continued to construct a family $\{V_n : n \in \mathbb{N}\}$ and a sequence $\{k_n : n \in \mathbb{N}\}$ such that the conditions (1) and (2) are satisfied for each $n \in \mathbb{N}$.

The property (1) with the pseudocompactness of the space X implies $F = \bigcap_n V_n \neq \emptyset$; take a point $x \in F$. By (2), we know that $x \in U_{k_n}$ for every $n \in \mathbb{N}$. Hence, the family $\mathfrak{U}$ is not point-finite. This contradiction proves that every countable subset of X is scattered. □

This implies the following:

Corollary 17.4.19 ([406]) *Any pseudocompact Δ-space X with a countable tightness is scattered.*

Proof Let X be a pseudocompact Δ-space with $t(X) \leq \aleph_0$. If $A \subset X$ and $x \in \overline{A}$, then take a countable $B \subset A$ such that $x \in \overline{B}$. The set B is scattered by Theorem 17.4.18. Hence, there is a discrete set $D \subset B$ such that $B \subset \overline{D}$. Consequently, $x \in \overline{D}$. This shows that

(3) if $A \subset X$ and $x \in \overline{A}$, then there exists a countable discrete set $D \subset A$ such that $x \in \overline{D}$.

If X is not scattered, fix a crowded subset $Y \subset X$. Then, there exists a countably infinite discrete subset $D_0 \subset Y$. Assume that $n \in \mathbb{N}$ and we have disjoint countable discrete subsets $D_0, \ldots, D_n$ in the space Y such that

(4) $D_0 \cup \ldots \cup D_i \subset \overline{D_{i+1}}$ for any $i < n$.

The set $D' = D_0 \cup \ldots \cup D_n$ is nowhere dense in Y so $Y \setminus D'$ is dense in Y. Let $\{O_x : x \in D_n\}$ be a disjoint open expansion of D_n in the space Y. By property (3), we know that there exists a countable discrete set $E_x \subset O_x \setminus D'$ such that $x \in \overline{E_x}$ for every $x \in D_n$.

The set $D_{n+1} = \bigcup\{E_x : x \in D_n\}$ is discrete, disjoint from D' and $D_n \subset \overline{D_{n+1}}$. This shows that we can construct a disjoint family $\{D_n : n \in \mathbb{N}\}$ of countably infinite discrete subsets of Y such that the condition (4) is satisfied for all $n \in \mathbb{N}$. The same condition (4) implies that $D = \bigcup\{D_n : n \in \mathbb{N}\}$ is a countable crowded subspace of X. This contradiction with Theorem 17.4.18 shows that X is scattered. □

Still, it is unknown if the assumption concerning the countable tightness can be removed.

In [332], we observed that the existence of an uncountable separable metrizable space X such that $C_p(X)$ is distinguished is independent of ZFC, and it is equivalent to the existence of a separable countably paracompact non-normal Moore space. Here, we refer readers to the article of Nyikos [473] for the interesting history of the normal Moore problem.

17.5 Some Examples of Non-Distinguished Spaces $C_p(X)$

In this section, we show some recent results due to Leiderman and Szeptycki gathered in the article [404]. This cited paper extends also several obtained results from [213] concerning certain examples of spaces $C_p(X)$ which are not distinguished.

Note first that (as we have already observed above) the one-point compactification of any Isbell–Mrówka space is a separable Δ-space. It follows that separable compact spaces $X \in \Delta$ with $|X| = \mathfrak{c}$ do exist in ZFC. Nevertheless, no Δ-set of reals can have cardinality $\mathfrak{c}$; see [509].

Below, we show the following main result of this section.

Theorem 17.5.1 *If $o(X)^{\aleph_0} \leq |X|$, then X is not a Δ-space, where $o(X)$ means the cardinality of the family of all open sets in X.*

Proof The proof uses some idea mentioned in [509]. Let $o(X)^{\aleph_0}$ by λ and set $X = \{x_\alpha : \alpha < \tau\}$. Enumerate by $\{\{U_n^\alpha : n \in \mathbb{N}\} : \alpha < \lambda\}$ all countable sequences of open subsets of X with empty intersection. Assume that X is a Δ-space. By assumption $\lambda \leq \tau$. For every $\alpha < \lambda$ choose an $n(\alpha) \in \mathbb{N}$ such that

$$x_\alpha \notin U_{n(\alpha)}^\alpha.$$

Next, define

$$A_n = \{x_\alpha : n(\alpha) \geq n\}.$$

Clearly, $\bigcap_n A_n = \emptyset$. If there existed an $\alpha < \lambda$ such that $A_n \subset U_n^\alpha$, for each $n \in \mathbb{N}$, then we would have $x_\alpha \in A_{n(\alpha)} \subset U_n^\alpha$, a contradiction. □

Theorem 17.5.1 applies to get another Proposition 17.5.2 which extends [213, Corollary 30, Corollary 37].

Proposition 17.5.2 (a) *Let X be a hereditarily separable space. If $|X| = \mathfrak{c}$, then X is not a Δ-space.*

(b) *Let X be a separable hereditarily Lindelöf space. If $|X| = \mathfrak{c}$, then X is not a Δ-space.*

Proof (a) First observe that for any X, the inequality $o(X) \leq |X|^{hd(X)}$ always holds; see [311]. Since $hd(X) = \aleph_0$, we deduce that

$$o(X)^{\aleph_0} \leq C_p(K)^{\aleph_0 \times \aleph_0} = |X|.$$

Now, Theorem 17.5.1 applies.

(b) Recall also that $w(X) \leq 2^{d(X)}$ and $o(X) \leq w(X)^{hL(X)}$ always hold; see [311]; hence

$$o(X)^{\aleph_0} \leq C_p(K)^{\aleph_0 \times \aleph_0} = |X|.$$

Again Theorem 17.5.1 applies. □

By S denote the Sorgenfrey line. By Ferrando et al. [213, Example 34], it is known that S is not a Δ-space. The proof presented in [213] is valid for any subset of S containing a segment although fails for more complicated subspaces of S. By Proposition 17.5.2(a), we deduce

Corollary 17.5.3 *Let X be any subspace of the Sorgenfrey line S with $|X| = \mathfrak{c}$. Then, X is not a Δ-space.*

A topological space X is *resolvable* (*ω-resolvable*), if it can be partitioned into 2 (countably many) dense subsets; see [116, 132, 235, 309] and references

therein. Recall that every regular countably compact space without isolated points is ω-resolvable [143, Theorem 6.9], and every countably compact Δ-space is compact [333, Theorem 4.7]. Finally, recall every compact Δ-space is scattered. Hence, if X is countably compact without isolated points, then X is not a Δ-space. In [404], Leiderman and Szeptycki, using a characterization of Δ-spaces [332, Theorem 2.1], showed the following [550, Proposition 6.6].

Proposition 17.5.4 ([550]) *If X is a Baire space and ω-resolvable, then X is not a Δ-space.*

Proof Assume that X is a Δ-space. Fix a partition of X into countably many dense sets D_n. If $U_n \supseteq D_n$, then U_n is dense open and so $\bigcap_n U_n \neq \emptyset$ and so $\{D_n : n \in \mathbb{N}\}$ has no point-finite open expansion, a contradiction. □

As the Souslin line is a Baire space and is ω-resolvable, we have.

Corollary 17.5.5 *A Souslin line is not a Δ-space.*

A Lindelöf regular space X with all open sets uncountable is resolvable, [235]. This fact has been extended (see [326]) to get the conclusion that X is ω-resolvable. Since Baire spaces without isolated points have all open subsets uncountable, we note the following:

Corollary 17.5.6 *Lindelöf Baire spaces without isolated points are not Δ-spaces.*

Note that from [116, Theorem 1.3] it follows that every Baire topological space with countable tightness but without isolated points is a ω-resolvable space. This results motivated Leiderman and Szeptycki to formulate the following [404, Problem 6.9].

Problem 17.5.7 ([404]) Does every Baire Δ-space have an isolated point?

We know already that the space $[0, \omega_1)$ is a first countable locally compact scattered space which is not a Δ-space, Theorem 17.4.15. In [404, Theorem 6.11], Leiderman and Szeptycki characterized those subspaces X of $[0, \omega_1)$ which are Δ-spaces and showed that X is a Δ-space if and only if X is a non-stationary set.

On the other hand, as proved in [486], the Axiom of Constructibility, $V = L$, implies that every Baire space without isolated points is ω-resolvable. Hence, by Proposition 17.5.4 (under $V = L$), every Δ-space which is Baire has isolated points. Also, in [116], it was shown (under MA) the following statement: *a topological space X without isolated points is ω-resolvable if it satisfies one of the following properties $\mathcal{P}$:*

(i) X contains a π-network $|\mathcal{U}| < \mathfrak{c}$ of infinite sets.
(ii) $\chi(X) < \mathfrak{c}$.
(iii) X is a Baire space and $c(X) \leq \aleph_0$.
(iv) X is a Baire space and has a network $|\mathcal{U}| < \mathfrak{c}$ such that the collection of the finite elements in it constitutes a σ-locally finite family.

Hence, under MA, every topological space X with the property $\mathcal{P}$ mentioned above is ω-resolvable if and only if X has no isolated points. This combined with Proposition 17.5.4 implies the following:

Corollary 17.5.8 *Assuming MA, if X is a Baire space without isolated points that satisfies one of the mentioned above properties $\mathcal{P}$, then X is not a Δ-space.*

Problem 17.5.9 Find possible the weakest topological property $\mathcal{P}$ under which (in ZFC) every Baire space X with property $\mathcal{P}$ is ω-resolvable.

Note another application of above Theorem 17.5.1. First recall that $nw(X)$ denotes the *network weight* of X. Note that always $nw(X) \geq \aleph_0$ holds because X is assumed to be infinite. Next, Corollary 17.5.10 extends [213, Corollary 31].

Corollary 17.5.10 *Let X be a topological space. If $2^{nw(X)} \leq |X|$, then X is not a Δ-space.*

Proof Fix a network $\mathcal{F}$ in X with $|\mathcal{F}| = nw(X)$. If $U \subset X$ is any open set, then there is a subfamily $\mathcal{F}_U \subset \mathcal{F}$ such that $U = \bigcup \mathcal{F}_U$. This means that always $o(X) \leq 2^{nw(X)}$. It follows that

$$o(X)^{\aleph_0} \leq 2^{nw(X)\times\aleph_0} = 2^{nw(X)}.$$

Hence, assuming $2^{nw(X)} \leq |X|$, we obtain that $o(X)^{\aleph_0} \leq |X|$ and then Theorem 17.5.1 applies. □

17.6 Basic Operations for Δ-Spaces

In this section, we analyse the question whether the class of Δ-spaces is invariant under the following basic topological operations like subspaces, continuous images, quotient continuous images, finite/countable unions, and finite products; see [332] for more detailed information.

First note that the property of being a Δ-space is inherited by subspaces because of Corollary 17.3.3. Clearly, every topological space is a continuous image of a discrete one. On the other hand, the following assertion has been noticed in [474].

Proposition 17.6.1 ([474]) *There exists in ZFC a MAD family $\mathcal{A}$ on $\mathbb{N}$ such that the corresponding Isbell–Mrówka space $\Psi(\mathcal{A})$ admits a continuous mapping onto the closed interval* $[0, 1]$.

For possible constructions of such MAD families $\mathcal{A}$, we refer to [72] and [587]. Thus, the class Δ is not invariant under continuous images even for first-countable separable locally compact spaces. However, a continuous mapping in Proposition 17.6.1 cannot be a quotient map. We have however the following:

Proposition 17.6.2 ([332]) *Every quotient continuous image of any Isbell–Mrówka space is a Δ-space.*

We start with the following one of main results of this section; see [333] and remarks in Introduction therein.

Theorem 17.6.3 ([333]) *Let X be a Δ-space and let $\varphi : X \to Y$ be a continuous surjection onto a topological space Y such that $\varphi(F)$ is an F_σ-set in Y for every closed set $F \subset X$. Then, Y is a Δ-space.*

Proof Assume that $\{D_n : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of Y with empty intersection. Then, $\{\varphi^{-1}(D_n) : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of X and

$$\bigcap_n \varphi^{-1}(D_n) = \varphi^{-1}(\bigcap_n D_n) = \emptyset.$$

By assumption, there is a decreasing sequence of open sets $\{U_n : n \in \mathbb{N}\}$ such that

$$\varphi^{-1}(D_n) \subset U_n$$

for each $n \in \mathbb{N}$ and

$$\bigcap_n U_n = \emptyset.$$

Now, define

$$H_n = Y \setminus \varphi(X \setminus U_n)$$

for each $n \in \mathbb{N}$. Observe that $D_n \subset H_n$ for each $n \in \mathbb{N}$. Indeed, $\varphi^{-1}(y) \subset U_n$ for every $y \in D_n$; hence, $y \notin \varphi(X \setminus U_n)$. Thus, $y \in H_n$. Note that $\varphi^{-1}(H_n) \subset U_n$. Hence, $\bigcap_{n\in\mathbb{N}} H_n = \emptyset$ because

$$\varphi^{-1}(\bigcap_{n\in\mathbb{N}} H_n) = \bigcap_{n\in\mathbb{N}} \varphi^{-1}(H_n) \subset \bigcap_{n\in\mathbb{N}} U_n = \emptyset.$$

Clearly, the sets $X \setminus U_n$ are closed; the sets $\varphi(X \setminus U_n)$ are F_σ in Y; hence, the sets H_n are G_δ in Y.

We can refine the sets H_n further and construct another decreasing sequence consisting of open sets $\{V_n : n \in \mathbb{N}\}$ such that $H_n \subset V_n$ for each $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} V_n = \emptyset$. Indeed, denote by H_n^k open subsets of Y such that $H_{n+1}^k \subset H_n^k$ and $\bigcap_{n\in\mathbb{N}} H_n^k = H_k$ for every $n, k \in \mathbb{N}$. Since the sequence $\{H_k : k \in \mathbb{N}\}$ is decreasing, by induction over upper index k without loss of generality, we may assume also that $H_n^{k+1} \subset H_n^k$ for every $n, k \in \mathbb{N}$.

We declare now that $V_n = H_n^n$ for $n \in \mathbb{N}$. It is clear that $D_n \subset V_n$ and each V_n is an open set. Note that the intersection of all sets V_n is empty. Indeed, let y be any

element of Y. There exists $k \in \mathbb{N}$ such that $y \notin H_k$; hence, there exists $n(k) \in \mathbb{N}$ such that $y \notin H^k_{n(k)}$. Fix any $m \geq \max\{k, n(k)\}$. Then,

$$H^m_m \subset H^k_m \subset H^k_{n(k)}$$

which implies that $y \notin H^m_m$. Finally, $\bigcap_{n\in\mathbb{N}} V_n = \emptyset$. This completes the proof. □

This immediately yields the following consequence.

Corollary 17.6.4 *Any continuous image of a compact Δ-space is also a Δ-space.*

Next Proposition 17.6.5 from [333] provides some corollaries below.

Proposition 17.6.5 *Assume that X is a countable union of closed subsets X_n, where each X_n is a Δ-space. Then, X also is a Δ-space. In particular, a countable union of compact Δ-spaces is also a Δ-space.*

Proof Denote by Z the free topological union of the spaces X_n for $n \in \mathbb{N}$. It is easy to see that Z is a Δ-space by applying Theorem 17.3.2. The space Z admits a natural continuous mapping φ onto X. Since $\varphi(F)$ is an F_σ-set in X for every closed set $F \subset Z$, we deduce that X is a Δ-space by applying Theorem 17.6.3. □

Corollary 17.6.6 *Let X be a σ-compact Δ-space and Y be a continuous image of X. Then Y also is a Δ-space.*

Corollary 17.6.7 *Let X be a Lindelöf Čech-complete Δ-space and let Y be a continuous image of X. Then, Y also is a Δ-space.*

Proof We know already that any Čech-complete Δ-space is scattered. Now, we use the well-known fact stating that every Lindelöf Čech-complete scattered space is σ-compact (see, e.g. [45, Theorem 4.5]). Apply Corollary 17.6.6. □

A topological space X is called *ω-bounded* if the closure of every countable subset of X is compact. Another application of Theorem 17.6.3 is the following theorem [333, Theorem 4.1].

Theorem 17.6.8 ([333]) *Every ω-bounded Δ-space is compact.*

Proof Assume that the claim fails. Then, by a result of Burke and Gruenhage [289, Lemma 1], X contains a subset Z which is a perfect preimage of the ordinal space $[0, \omega_1)$. We conclude that a Δ-space Z can be mapped by a continuous closed mapping onto $[0, \omega_1)$. By Theorem 17.6.3, this would mean that $[0, \omega_1) \in \Delta$, a contradiction since $[0, \omega_1)$ is not in Δ by Theorem 17.4.15. □

A very similar argument as presented in Theorem 17.6.8 can be applied to get the following theorem motivated by Corollary 17.4.4.

Theorem 17.6.9 ([550]) *Every compact Δ-space has countable tightness.*

Proof A compact space has countable tightness if and only if it does not contain a perfect preimage of the space $[0, \omega_1)$ (see [50] or [51]). To complete the proof, see the argument presented in the proof of Theorem 17.6.8. □

A very natural question arises whether Theorem 17.6.9 can be generalized for countably compact spaces.

A positive answer ([333, Theorem 4.3]) follows from the Proper Forcing Axiom (PFA), due to the results of Balogh [50] (see also [51]).

Theorem 17.6.10 ([333]) Assume PFA.

(1) *Every countably compact Δ-space is compact.*
(2) *Every countably compact Δ-space has countable tightness.*
(3) *Every countably compact Δ-space is sequential.*

The following result does not require extra set-theoretic assumptions. The present form of this lemma (and its proof) is also presented in [331, Lemma 3.3].

Lemma 17.6.11 *Let X be a pseudocompact non-scattered space. Then, there are a closed G_δ-subset F of X and a continuous mapping φ from F onto the unit interval* $[0, 1]$.

Proof The proof uses the argument from [405, Proposition 5.5]. If a Tychonoff space X is not zero-dimensional, then X admits a continuous mapping onto $[0, 1]$ (see, for instance, Fact 4 of T.063 [596]). Without loss of generality, we may assume that X has a base consisting of clopen sets.

Let $\mathcal{P} = \bigcup_{n\in\mathbb{N}} 2^n$ be the usual binary tree of height $\mathbb{N}$, where $2 = \{0, 1\}$. If $f \in 2^n$, then $f^\frown 0$ and $f^\frown 1$ denote two finite sequences in 2^{n+1} extending f. If $h \in 2^{\mathbb{N}}$, then $h|n \in 2^n$ denotes the first n elements of the infinite binary sequence h.

Assume that Y is a non-empty closed subset of X without isolated points. We can define by induction a family $\{U_f : f \in \mathcal{P}\}$ of non-empty clopen subsets of X satisfying the following conditions for all $f \in \mathcal{P}$:

(i) $U_f \cap Y \neq \emptyset$.
(ii) $U_{f^\frown 0}$ and $U_{f^\frown 1}$ are disjoint.
(iii) $U_{f^\frown 0} \cup U_{f^\frown 1} \subset U_f$.

Note that the set $F = \bigcap_{n\in\mathbb{N}} \bigcup_{f\in 2^n} U_f$ admits a continuous mapping onto $[0, 1]$. Indeed, identifying the Cantor set $\mathcal{C}$ with $2^{\mathbb{N}}$, one obtains a mapping $\pi : F \to \mathcal{C}$ by letting $\pi(x) = h$ for each $x \in \bigcap_{n\in\mathbb{N}} U_{h|n}$, where $h \in 2^{\mathbb{N}}$. Clearly, (ii) implies that for every $x \in F$, there exists a unique element $h \in 2^{\mathbb{N}}$ with $x \in \bigcap_{n\in\mathbb{N}} U_{h|n}$.

Since X is pseudocompact, it follows from (iii) that the set $\bigcap_{n\in\mathbb{N}} U_{h|n}$ is non-empty for each $h \in 2^{\mathbb{N}}$. Hence, $\pi(F) = \mathcal{C}$. One can verify the continuity of π which follows from (ii) and (iii). Observe that the closed interval $[0, 1]$ is a continuous image of the Cantor set $\mathcal{C}$. Hence, the closed G_δ-set $F \subset X$ admits a continuous mapping φ onto $[0, 1]$. □

Now, we prove the following:

Theorem 17.6.12 (Kąkol–Leiderman) *Every countably compact Δ-space X is scattered.*

Proof Suppose that a countably compact space X is not scattered. Recall that every countably compact space X is pseudocompact. Hence, there exists a closed subset $K \subset X$ and a continuous surjective mapping φ from K onto $[0, 1]$, by applying Lemma 17.6.11. Since very closed subset F of K is a countably compact space, its continuous image $\varphi(F)$ is a countably compact subset of $[0, 1]$; therefore, $\varphi(F)$ is compact. We derive that $\varphi|K$ is a closed continuous mapping from K onto $[0, 1]$. This contradicts Theorem 17.6.3, since $[0, 1]$ is not a Δ-space. □

This can be used to show the following:

Example 17.6.13 The space $\beta\mathbb{N} \setminus \mathbb{N}$ contains a dense countably compact not scattered subspace K which is not a Δ-space and every countable subset of K is scattered.

Indeed, by Juhász and van Mill [325, Theorem 1.1, Examples], the space $\beta\mathbb{N} \setminus \mathbb{N}$ contains a dense countably compact space K which is not scattered although all countable subsets of K are scattered. Now, it is enough to apply Theorem 17.6.12.

17.7 Δ-Spaces vs. Properties of Spaces $C_p(X)$

In this section, we study the following question:

Problem 17.7.1 Let X and Y be Tychonoff spaces and assume there exists a continuous linear surjection $T : C_p(X) \to C_p(Y)$. Which topological properties related to being a Δ-space on X are preserved onto Y?

We start with the following lemma from [333] which will be used below.

Lemma 17.7.2 *Let X and Y be two sets and let $E \subset \mathbb{R}^X$ and $F \subset \mathbb{R}^Y$ be dense vector subspaces of $\mathbb{R}^X$ and $\mathbb{R}^Y$, respectively. Assume that $T : E \longrightarrow F$ is a continuous linear surjection between lcs E and F. Then, T admits a continuous linear surjective (unique) extension $\widehat{T} : \mathbb{R}^X \longrightarrow \mathbb{R}^Y$.*

Proof Let us recall the following classical properties of the product space $\mathbb{R}^X$ which will be used below.

Property 1. Every closed vector subspace H of $\mathbb{R}^X$ is complemented in the space $\mathbb{R}^X$, and the quotient $\mathbb{R}^X/H$ is linearly homeomorphic to the product $\mathbb{R}^Z$ for some set Z; see [491, Corollary 2.6.5, Theorem 2.6.4].

Property 2. The product topology on $\mathbb{R}^X$ is minimal, i.e. $\mathbb{R}^X$ does not admit a weaker Hausdorff locally convex topology; see [491, Corollary 2.6.5(i)].

Property 3. $\mathbb{R}^Y$ fulfills the extension property, i.e. if M is a vector subspace of a lcs L, then every continuous linear mapping $T : M \longrightarrow \mathbb{R}^Y$ admits a continuous linear extension $\widehat{T} : L \longrightarrow \mathbb{R}^Y$; see [464, Theorem 10.1.2 (a)].

By Property 3, there exists a continuous linear extension:

$$\widehat{T} : \mathbb{R}^X \longrightarrow \mathbb{R}^Y$$

of T such that $F \subset \widehat{T}(\mathbb{R}^X)$. We prove that $\widehat{T}$ is a surjective mapping.

Now, denote by

$$\varphi : \mathbb{R}^X / \ker(T) \longrightarrow \mathbb{R}^Y$$

the injective mapping associated with the quotient mapping

$$Q : \mathbb{R}^X \longrightarrow \mathbb{R}^X / \ker(\widehat{T}),$$

where $\ker(\widehat{T})$ is the kernel of $\widehat{T}$ and $\varphi \circ Q = \widehat{T}$. On the other hand, by Property 1, we derive that the space $\mathbb{R}^X / \ker(T)$ is linearly homeomorphic to the product $\mathbb{R}^Z$ for some set Z. So, we may assume that φ is a continuous linear bijection from $\mathbb{R}^Z$ onto a dense subspace $\widehat{T}(\mathbb{R}^X)$ of $\mathbb{R}^Y$. This implies that on the space $\widehat{T}(\mathbb{R}^X)$ there exists a stronger locally convex topology ξ such that $(\widehat{T}(\mathbb{R}^X), \xi)$ is linearly homeomorphic with $\mathbb{R}^Z$. However, by Property 2, the space $\mathbb{R}^Z$ does not admit a weaker Hausdorff locally convex topology; hence, $\widehat{T}(\mathbb{R}^X)$ is isomorphic to the complete lcs $\mathbb{R}^Z$. Finally, $\widehat{T}(\mathbb{R}^X)$ is closed in $\mathbb{R}^Y$ and then $\widehat{T}$ is a surjection. The proof is completed. □

If $C_p(X)$ is homeomorphic to a retract of $\mathbb{R}^\kappa$ for some cardinal κ, then X must be a discrete space; we refer to [596, Problem 500]. On the other hand, there exists a continuous mapping from $\mathbb{R}^{\mathbb{N}}$ onto $C_p[0, 1]$; see also [596, Problem 486].

Using Lemma 17.7.2, we note the following consequence.

Corollary 17.7.3 *Let a dense subspace of $C_p(X)$ be a continuous linear image of $\mathbb{R}^\kappa$, for some cardinal κ. Then, X is a discrete space.*

We are at the position to prove the following main result of this section.

Theorem 17.7.4 ([333]) *If X is a Δ-space and there exists a continuous linear surjection $T : C_p(X) \to C_p(Y)$, then Y is a Δ-space.*

Proof Let $T : C_p(X) \longrightarrow C_p(Y)$ be a continuous linear surjection. Denote by

$$\widehat{T} : \mathbb{R}^X \longrightarrow \mathbb{R}^Y$$

the extension of T which exists by Lemma 17.7.2. By Theorem 17.3.2, the space $C_p(X)$ is distinguished.

Take arbitrary $f \in \mathbb{R}^Y$. There exists $g \in \mathbb{R}^X$ with $\widehat{T}(g) = f$. Then, there is a bounded set $B \subset C_p(X)$ such that $g \in \overline{B}^{\mathbb{R}^X}$. We define $A = T(B)$. Clearly, A is bounded and $f \in \overline{A}^{\mathbb{R}^Y}$ which means that $C_p(Y)$ is distinguished; equivalently, Y is a Δ-space, by Theorem 17.3.2. □

Note that Lemma 17.7.2 applies also to get the following theorem. Recall first that a topological space X is called a *Q-space* if each subset of X is F_σ, or, equivalently, each subset of X is G_δ in X.

Theorem 17.7.5 ([333]) *Let X and Y be normal spaces and assume that there exists a continuous linear surjection $T : C_p(X) \to C_p(Y)$. If X is a Q-space, then Y also is a Q-space.*

Proof Normal X is a Q-space if and only X is *strongly splittable*, i.e. for every $f \in \mathbb{R}^X$ there exists a sequence $S = \{f_n : n \in \mathbb{N}\} \subset C_p(X)$ such that $f_n \to f$ in $\mathbb{R}^X$; see, for example, by Tkachuk [597, Problems 445, 447]. Denote by

$$\widehat{T} : \mathbb{R}^X \longrightarrow \mathbb{R}^Y$$

the extension of T by using Lemma 17.7.2. Fix $f \in \mathbb{R}^Y$. Then, there exist $g \in \mathbb{R}^X$ with $\widehat{T}(g) = f$ and a sequence $B \subset C_p(X)$ converging to g in $\mathbb{R}^X$. We define $A = T(B)$. It is easy to see that $A \subset C_p(Y)$ converges to f in $\mathbb{R}^Y$. □

The following problem seems to be open. *Let X be a scattered space and assume that there exits a continuous linear surjection from $C_p(X)$ onto $C_p(Y)$. Is the space Y scattered?* We provide some partial solutions also involving the concept of Δ-spaces. A space $X = \bigcup_{n\in\mathbb{N}} X_n$ is called *σ-scattered* (*σ-discrete*) if every X_n is scattered (discrete, respectively).

Proposition 17.7.6 *Assume that X is σ-scattered (σ-discrete) and there exists a continuous linear surjection $T : C_p(X) \longrightarrow C_p(Y)$. Then, Y also is σ-scattered (σ-discrete, respectively).*

Proof The proof uses some arguments from [402, Theorem 3.4] and [362, Proposition 2.1]. For each natural $n \in \mathbb{N}$, consider the subspace $A_n(X)$ of $L_p(X)$ formed by all words of the reduced length precisely n. It is known that $A_n(X)$ is homeomorphic to a subspace of the Tychonoff product $(\mathbb{R}^*)^n \times X^n$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

The adjoint mapping T^* of the map T embeds $L_p(Y)$ into $L_p(X)$. Therefore, Y can be represented as a countable union of subspaces Y_i, $i \in \mathbb{N}$, such that each Y_i is homeomorphic to a subspace of $(\mathbb{R}^*)^n \times X^n$ for some $n = n(i)$.

Now, let p_i be the projection of each of the above pieces $Y_i \subset (\mathbb{R}^*)^n \times X^n$ to the second factor X^n. Since T is a surjection, we have that $p_i : Y_i \longrightarrow X^n$ is a finite-to-one mapping. Clearly, X^n is scattered/discrete provided X is. Since p_i is continuous, for every isolated point $z \in X^n$, its finite fibre $p_i^{-1}(z)$ consists of points isolated in Y_i and the claim follows. □

The proof of the next result uses the concept of the Δ-space.

Proposition 17.7.7 *Let X and Y be metrizable spaces and assume that there exists a continuous linear surjection from $C_p(X)$ onto $C_p(Y)$. If X is scattered, then Y also is scattered.*

Proof If X is metrizable and scattered, then X is a Δ-space by Proposition 17.4.17; consequently, Y is a Δ-space by Theorem 17.7.4. But every metrizable and scattered space is completely metrizable; see [238, Corollary 2.2].

On the other hand, one can use a general result due to Baars, de Groot, and Pelant [46] stating that if X and Y are metrizable and X is completely metrizable, then Y is completely metrizable provided there exists a continuous linear map from $C_p(X)$ onto $C_p(Y)$. This shows that Y is a Čech-complete Δ-space; hence, Y is scattered. □

Recall that if X and Y both are compact spaces and there is a continuous mapping from $C_p(X)$ onto $C_p(Y)$, then Y is Eberlein whenever X is Eberlein (see [32, Theorem IV.1.7])) and Y is Corson whenever X is Corson; see also [32, Theorem IV.3.1]).

Finally, we show the next proposition being a combination of a few known results (mostly contained in [32]) while we apply Theorem 17.7.4 in order to obtain the scatteredness for Y.

Proposition 17.7.8 ([333]) *Assume that there exists a continuous linear surjection from $C_p(X)$ onto $C_p(Y)$:*

(1) *If X is an Eberlein compact space, then Y also is Eberlein compact.*
(2) *If X is a scattered Eberlein compact space, then Y also is a scattered Eberlein compact space.*

Proof (1): Since the space $C_p(X)$ contains a dense σ-compact subspace by applying [32, Theorem IV.1.7], the space $C_p(Y)$ enjoys the same property, and consequently, $C_p(Y)$ contains a compact subset K which separates points of the space Y. On the other hand, the space $C_p(Y)$ is covered by a sequence of bounded sets. Hence, $C_p(Y)$ does not contain a copy of the space $\mathbb{R}^{\mathbb{N}}$ (since the last space is not normed). This shows that Y is a pseudocompact space. By means of evaluation mapping, we define a continuous injective mapping $\varphi : Y \longrightarrow C_p(K)$. Set $B = \varphi(Y)$. Then, B is a pseudocompact subspace of $C_p(K)$. By Arkhangel'skii [32, Theorem IV.5.5], we get that B is an Eberlein compact space. We showed that the pseudocompact space Y is mapped by a continuous injective mapping φ onto the Eberlein compact space B. However, the mapping φ must be a homeomorphism, by Arkhangel'skii [32, Theorem IV.5.11], and then, Y is Eberlein compact.

(2): We know already that every scattered Eberlein compact is a Δ-space. Hence, by above Theorem 17.7.4, the space Y is a Δ-space, so Y must be scattered by Theorem 17.4.6. □

We complete this section with the following result from [333].

Proposition 17.7.9 *Let X be a Čech-complete Lindelöf space. Then, the following assertions are equivalent:*

(1) *X is scattered.*
(2) *X is σ-scattered.*
(3) *$C_p(X)$ is a Fréchet–Urysohn space.*

Proof The implication (2) $\Rightarrow$ (1) follows from the fact that every closed subspace of X satisfies Baire category theorem. Finally, for the equivalence (1) $\Leftrightarrow$ (3), we refer to [260, Corollary 2.12]. □

Corollary 17.7.10 *Let X be a Čech-complete Lindelöf space. If X is Δ-space, then the space $C_p(X)$ is a Fréchet–Urysohn space.*

Chapter 18
Generalized Metric Spaces with $\mathfrak{G}$-Bases

Abstract This chapter deals with several concepts related with networks with applications both in Topology and Functional Analysis. Especially we will discuss this concepts for topological groups with $\mathfrak{G}$-bases. Recall that topologists are seeking several possible general sequential concepts, which together under an additional topological condition, force a topological space X to be metrizable. Following this line of research Tsaban and Zdomskyy introduced a property (the strong Pytkeev property) which implies the countable tightness, and in the frame of Fréchet–Urysohn topological groups yields the metrizability.

18.1 Selected Types of Generalized Metric Spaces

In this chapter, we recall briefly several concepts related with *networks* and provide their applications both in topology and functional analysis. Especially, we will discuss these concepts for topological groups with $\mathfrak{G}$-bases. This line of research attracted already several specialists; see, for example, [57, 65, 265, 269], and references therein.

Topological properties of lcs E endowed with the weak topology $\sigma(E, E')$ have been intensively studied from many years (see, e.g. [198, 265]).

Although $(E, \sigma(E, E'))$ is never a metrizable space for an infinite-dimensional normed E, every $\sigma(E, E')$-compact set is $\sigma(E, E')$-metrizable for a normed separable space E. Moreover, for many natural and important classes of separable metrizable lcs E, the space $(E, \sigma(E, E'))$ is a generalized metric space of some type (see, e.g. [265, 445]). Such types of topological spaces are defined by different types of networks.

In [21], Arkhangel'skii defined the concept of a *network* in a topological space, which was used in [22] for several aspects and problems in general topology, topological algebra, and functional analysis. Michael [445] introduced a large class of separable and paracompact spaces (under the name $\aleph_0$-spaces) containing all

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separable metric spaces. Apparently, these mentioned articles motivated several specialists to study various generalizations of metric spaces in the frame of certain types of networks. We refer to survey papers [289, 461, 585]. On the other hand, in 1961, Corson [147] started a systematic study of certain topological properties of the weak topology of Banach spaces. This line of research motivated also specialists to apply the concepts around generalized metric spaces to functional analysis. It turned out that for many classes of lcs E in functional analysis the weak topology of E is a generalized metric space; we refer to recent papers [58, 265, 268], and references therein. The concept of a network is one of a well-recognized good tool, coming from the pure set topology, which turned out to be of great importance to study successfully renorming theory in Banach spaces; see the survey paper [129]; especially [129, Theorem 13] for $\sigma(E, E')$-slicely networks. Note also that in [271] Garcia, Oncina, and Orihuela characterized the classes of Gul'ko and Talagrand compact spaces through a network condition and answered two questions posed by Gruenhage in [291].

Recall here several types of networks.

Definition 18.1.1 Let $\mathcal{N}$ be a family of subsets of a topological space X. Then, $\mathcal{N}$ is called

(i) A network at $x \in X$ if for each neighbourhood U_x of x there is $B \in \mathcal{N}$ with $x \in B \subset U_x$. We call $\mathcal{N}$ a network in X if $\mathcal{N}$ is a network at each $x \in X$; see [21].

(ii) A k-network in X if for compact $K \subset X$ and open U in X with $K \subset U$ we have $K \subset \bigcup \mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{N}$; see [445].

(iii) A cs-network at $x \in X$ if for each sequence $(x_n)_n \subset X$ which converges to x and each neighbourhood U_x of x there exists $B \in \mathcal{N}$ and $k \in \mathbb{N}$ such that $\{x_n : n \geq k\} \cup \{x\} \subset B \subset U_x$. Then, $\mathcal{N}$ is a cs-network if it is a cs-network at each $x \in X$; see [297].

(iv) A cs^*-network at $x \in X$ if for each sequence $(x_n)_n \subset X$ which converges to x and each neighbourhood U_x of x there is $B \in \mathcal{N}$ with $x \in B \subset U_x$ and $\{n : x_n \in B\}$ is infinite. $\mathcal{N}$ is a cs^*-network if it is a cs^*-network at each $x \in X$; see [270].

(v) A Pytkeev network (or a cp-network) at $x \in X$ if for each $A \subset X$ with $x \in \overline{A} \setminus A$ and each neighbourhood U_x of x there is $B \in \mathcal{N}$ such that $x \in B \subset U_x$ and $B \cap A$ is infinite. Then, it is a Pytkeev network if $\mathcal{N}$ is a Pytkeev network for each $x \in X$; see [54].

(vi) A cn-network at $x \in X$ if for each neighbourhood U_x of x the set $\bigcup\{B \in \mathcal{N} : x \in B \subset U_x\}$ is a neighbourhood of X. Then, $\mathcal{N}$ is a cn-network if it is a cn-network for each $x \in X$; see [261] and [262].

(vii) A ck-network at $x \in X$ if for each neighbourhood U_x of x there is a neighbourhood V_x of x such that for each compact $K \subset V_x$ there exists a finite subfamily $\mathcal{F} \subset \mathcal{N}$ with $x \in \bigcap \mathcal{F}$ and $K \subset \bigcup \mathcal{F} \subset U_x$. Then, $\mathcal{N}$ is a ck-network if it is a ck-network for each $x \in X$; see [261].

Recall also the following concepts.

Definition 18.1.2 A topological space X is said to be:

(i) A cosmic space, if X is regular and admits a countable network; see [445].
(ii) An $\aleph_0$-space, if X is regular and has a countable k-network; see [445].
(iii) A space with the strong Pytkeev property, if X has a countable Pytkeev network at any point of X; see [605].
(iv) A $\mathfrak{B}_0$-space, if X is regular and has a countable Pytkeev network; see [54].

Recall, following Pytkeev, that a topological space X is said to have the *Pytkeev property* [511] if for any set $A \subset X$ and each $x \in \overline{A} \setminus A$ there are infinite subsets $A_1, A_2, \ldots$ of A such that each neighbourhood of x contains some A_n.

Note here only that a regular space X is an $\aleph_0$-space if and only if X has a countable cs-network [297] if and only if it has a countable cs^*-network [270], although $\aleph_0$-spaces which are not Pytkeev $\aleph_0$-spaces do exist, [54]. For other relations between these concepts, we refer to articles mentioned above. Moreover, we recall also the following result due to Banakh and Leiderman [64].

Theorem 18.1.3 (Banakh) *Let X be a topological space.*

(i) If X has the strong Pytkeev property, then X has a k-network.
(ii) If $\mathcal{N}$ is a cs^*-network in X and X is a Fréchet–Urysohn space, then $\mathcal{N}$ is a Pytkeev network.
(iii) X is first countable at $x \in X$ if and only if X has a countable Pytkev network at x and X has a countable fan tightness.

Okuyama [478] and O'Meara [479] introduced the concepts of σ-spaces and $\aleph$-spaces, respectively. Both classes contain all metrizable spaces.

Definition 18.1.4 A topological space X is called a *σ-space* if X is regular and has a σ-locally finite (equivalently, σ-discrete [300]) network. X is called an *$\aleph$-space* if X is regular and has a σ-locally finite k-network.

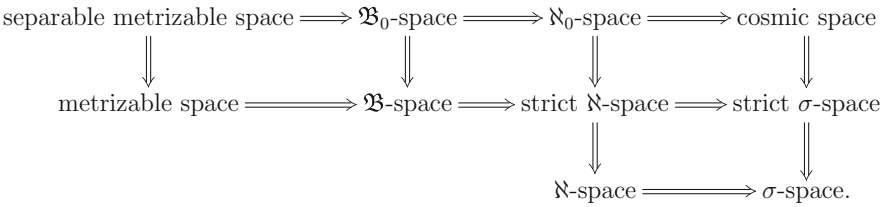
A family of subsets of a topological space X is said to be *locally finite* if each point in X has a neighbourhood that intersects only finitely many of the sets in the collection. A family of subsets of a topological space X is *σ-locally finite* if it is the union of a countable family of locally finite collections of subsets of X. This motivates the following concepts [261] and [262].

Definition 18.1.5 A topological space X is called a *$\mathfrak{B}$-space* if X has a σ-locally finite cp-network.

Each $\mathfrak{B}$-space X has the strong Pytkeev property.

Definition 18.1.6 A topological space X is called a strict σ-space if X has a σ-locally finite cn-network. X is called a strict $\aleph$-space, if X has a σ-locally finite ck-network.

The following diagram describes the relation between these classes of spaces.



None of the implications in this diagram can be reversed.

We recall also the following useful fact; see [264]; see also [479] for (ii).

Proposition 18.1.7 *Let X be a topological space.*

(i) *X is a cosmic space if and only if X is a Lindelöf σ-space.*
(ii) *X is an $\aleph_0$-space if and only if X is a Lindelöf $\aleph$-space.*

Proof Let X be a Lindelöf σ-space (respectively, X is an $\aleph$-space) with a σ-locally finite network (respectively, with a k-network) $\mathfrak{D} = \bigcup_n \mathfrak{D}_n$. We prove that every $\mathfrak{D}_n$ is countable. For every $x \in X$, find an open neighbourhood U_x of x such that U_x intersects with a finite subfamily $T(x)$ of $\mathfrak{D}_n$. Since X is a Lindelöf space, we have a countable set $\{x_k : k \in \mathbb{N}\}$ in X such that $X = \bigcup_{k\in\mathbb{N}} U_{x_k}$. Hence, any $D \in \mathfrak{D}_n$ intersects with some U_{x_k} and hence $D \in T(x_k)$. Thus, $\mathfrak{D}_n = \bigcup_{k\in\mathbb{N}} T(x_k)$ is countable.

Conversely, if X is a cosmic (respectively, X is an $\aleph_0$-space), then the space X is Lindelöf (see [445]), and it is a σ-space (respectively, an $\aleph$-space). □

Corollary 18.1.8 *Let E be a Lindelöf lcs. Then, E is a weakly $\aleph$-space if and only if E is a weakly $\aleph_0$-space.*

Arhangel'skii asked for which compact spaces K the space $C_p(K)$ is a σ-space? Assume that $C_p(K)$ is a σ-space. Is K metrizable? It is proved by Corson [445] that $C_w(K)$ is an $\aleph_0$-space if and only if K is countable, where $C_w(K)$ means the Banach space $C(K)$ endowed with the weak topology. It is natural to ask for which compact spaces K the space $C_w(K)$ is a σ-space. Clearly, if K is scattered, then $C_w(K)$ is a σ-space if and only if $C_p(K)$ is a σ-space. We show the following:

Proposition 18.1.9 *Let K be a compact space such that $C_p(K)$ is a Lindelöf space. Then, $C_p(K)$ is a σ-space if and only if K is metrizable.*

Proof If K is metrizable, then $C_p(K)$ is cosmic. Now, assume that $C_p(K)$ is a σ-space. By Proposition 18.1.7, the space $C_p(K)$ is cosmic. This implies however that K is cosmic by Michael [445, Proposition 10.5]. Consequently, K is metrizable. □

A concrete example illustrating the last proposition is every Corson compact space K since then $C_p(K)$ is a Lindelöf space; see [32, IV.2.22]. This leads to the following:

Corollary 18.1.10 *If K is a compact Corson space, then $C_p(K)$ is a σ-space if and only if K is metrizable.*

Marciszewski and Pol provided various examples of non-separable σ-spaces $C_p(K)$. Among the others, they proved that the space $C_p(K(2^{<\omega}))$ is a σ-space, where $K(2^{<\omega})$ is a AU-compactum associated with the Cantor tree; see [428, Section 2.3 and Remark 3.1, Section 2.4]. This result answers a question of Arhangelskii (see [292, p.206]), who asked if for a σ-space $C_p(X)$ the spaces X and $C_p(X)$ must be cosmic.

We recall also the following concepts; see [300] and [451]. First recall [451] that a family of a topological space is said be σ-discrete (σ-isolated) if it can be decomposed into a countable union of discrete (isolated) families.

Definition 18.1.11 A Banach space E is said:

(i) To be *descriptive*, if E has a network for its norm-topology which is σ-isolated in the weak topology
(ii) To have the *JNR-property*, if for every $\epsilon > 0$ the space E can be written as $E = \bigcup_n M_{n,\epsilon}$ that for every n and $x \in M_{n,\epsilon}$ there exists a weak neighbourhood U of x such that the norm diameter of $U \cap M_{n,\epsilon}$ is less than ϵ
(iii) To have a *Kadec norm* if the weak topology coincides with the norm topology on the unit sphere

Following Hansell [300] and Moltó at el. [450], we have the following implications for each Banach space:

$$\text{Kadec norm} \longrightarrow \text{JNR-property} \longleftrightarrow \text{descriptive}.$$

It is still unknown whether the first implication is reversible [451]. Recall that a Tychonoff space X is called *perfect*, if each closed set in X is a G_δ-set. It is well known that every σ-space is perfect; see [289]. Applying [428] we deduce that if $C(K)$ has the JNR-property and $C_p(K)$ is perfect, the space $C_p(K)$ is a σ-spaces. In [428], Marciszewski and Pol introduced a stronger concept under the name JNR_c which asserts that additionally the sets M_n are pointwise closed. We provide the following important characterization of the JNR_c-property due to Marciszewski and Pol [428, Lemma 2.3.1].

Proposition 18.1.12 (Marciszewski–Pol) *For a compact space K, the following assertions are equivalent:*

(i) The Banach space $C(K)$ has the JNR-property and $C_p(K)$ is perfect.
(ii) $C(K)$ has the JNR_c-property.
(iii) There exists a σ-discrete collection in $C_p(K)$ which is a network for $C(K)$.

Proof (i) $\Rightarrow$ (ii): Fix $\epsilon > 0$. Let τ_p be the pointwise topology of $C(K)$. Then, $C(K) = \bigcup_n M_n$, where each M_n has a cover $\mathfrak{W}_n$ by sets open in (M_n, τ_p) with diameters less than ϵ in the sup-norm. For a fixed $n \in \mathbb{N}$ and $A \subset M_n$, let

$$A^* = \overline{M_n} \setminus \overline{M_n \setminus A} \subset \overline{A},$$

where the closure is taken in $C_p(K)$. Then

$$\mathfrak{W}_n^* = \{V^* : V \in \mathfrak{W}_n\}$$

is an open cover of M_n in $\overline{M_n}$ by sets of diameter less then ϵ. Applying the fact that $C_p(K)$ is perfect, we have pointwise closed sets M_{nm} such that $\bigcup \mathfrak{W}_n^* = \bigcup_m M_{nm}$. This set M_{nm} shows the JNR_c-property with ϵ.

(ii) $\Rightarrow$ (iii): Fix a base $\mathfrak{B} = \bigcup_{m,n} \mathfrak{B}_{m,n}$ for $C(K)$ with the norm-distance between each two members of $\mathfrak{B}_{m,n}$ at least $\epsilon_{m,n} > 0$. We apply the JNR_c-property to deduce that for each pair $\{m, n\}$ we obtain C_p-closed sets M_{mnk} and covers $\mathfrak{U}_{mnk}$ of M_{mnk} by open sets in the pointwise topology with diameters less than ϵ_{mn} and such that $C(K) = \bigcup_k M_{mnk}$. Now, set

$$\mathfrak{S}_{mnk} = \{B \cap M_{mnk} : B \in \mathfrak{B}_{mn}\}.$$

This family is discrete in $C_p(K)$ and $\mathfrak{S} = \bigcup_{m,n,k} \mathfrak{S}_{mnk}$ composes a network for the Banach space $C(K)$.

(iii) $\Rightarrow$ (i): Fix a network $\mathfrak{S} = \bigcup_n \mathfrak{S}_n$ for $C(K)$, where $\mathfrak{S}_n$ is discrete in $C_p(K)$. Note that the C_p-closure does not increase the diameter. Hence, we can assume that all elements of $\mathfrak{S}$ are closed in $C_p(K)$. If U is an open set in $C_p(K)$, then the sets $F_n = \bigcup\{S \in \mathfrak{S}_n : S \subset U\}$ are open in the normed topology and closed in $C_p(K)$. We need to show that $C(K)$ has the JNR-property. Fix $\epsilon > 0$ and let M_n be the union of elements from the family of $\mathfrak{S}_n$ of diameter less then ϵ. By the construction of M_n, we derive that $C(K) = \bigcup_n M_n$ and sets M_n justify the JNR-property for ϵ. □

Observe (as it was shown in [428, Remark 2.3.2]) that if $C(K)$ has the JNR_c-property and $C_p(K)$ is Lindelöf, then K is metrizable.

Let X be a topological space and $x \in X$. Let $\mathfrak{n} \in \{cp, ck, cn\}$. The smallest size $|\mathcal{N}|$ of an $\mathfrak{n}$-network $\mathcal{N}$ at x is called the $\mathfrak{n}$-character of X at x and is denoted by $\mathfrak{n}_\chi(X, x)$. Then, $\mathfrak{n}_\chi(X) = sup\{\mathfrak{n}_\chi(X, x) : x \in X\}$ is called the $\mathfrak{n}$-character of X.

We show the following result from [261] and [262].

Theorem 18.1.13 (Gabriyelyan–Kąkol) *Let G be a topological group. Then, G is a cosmic space (an $\aleph_0$-space, $\mathfrak{B}_0$-space) if and only if G is separable and has a countable cn-character (countable ck-character, countable cp-character).*

Proof Assume that G is non-discrete. If X is cosmic (an $\aleph_0$-space, $\mathfrak{B}_0$-space), then X is hereditary separable, so we apply the fact that a regular space is cosmic (an $\aleph_0$-space, $\mathfrak{B}_0$-space) if and only if X has a countable cn-network (countable ck-network, countable cp-network). Therefore, we need to prove the sufficiency. Let $\mathfrak{D} = \{D_n\}_n$ be a countable $\mathfrak{n}$-network at the unit e of G, where $\mathfrak{n} \in \{cp, ck, cn\}$, and let $\{g_n\}_n$ be a dense subset of G. Without loss of generality, we can assume that $\mathfrak{D}$ is closed under taking finite products. We show that the countable family

$$\mathfrak{N} := \{g_n D_m : n, m \in \mathbb{N}\}$$

is an $\mathfrak{n}$-network in G.

Fix $g \in G$ and let gU be an open neighbourhood of g. Take an open symmetric neighbourhood W of e such that $W^3 \subseteq U$. In all three cases, $\mathfrak{D}$ is also a cn-network at e. Hence, the set

$$W_0 := \bigcup\{D \in \mathfrak{D} : e \in D \subseteq W\}$$

is a neighbourhood of e. As $G = \cup_n g_n W_0$, we can find $r, t \in \mathbb{N}$ such that $g = g_r \cdot h$ and $h \in D_t \subseteq W_0$.

(1) Assume that $\mathfrak{D}$ is a cn-network at e. Clearly

$$\bigcup\{g_r D_t \cdot D_m : \ D_m \in \mathfrak{D}, D_m \subseteq W\} = g_r D_t \cdot W_0 \subseteq g_r W_0^2 \subseteq g W^3 \subseteq gU,$$

and $g = g_r h \in \bigcap\{g_r D_t \cdot D_m : \ D_m \in \mathfrak{D}, D_m \subseteq W\}$. So $\mathfrak{N}$ is a cn-network at g.

(2) Assume that $\mathfrak{D}$ is a ck-network at e. Take an open neighbourhood $W_1 \subseteq W$ of e such that for every compact subset K of W_1 there exists a finite subfamily $\mathfrak{F}$ of $\mathfrak{D}$ satisfying $e \in \bigcap \mathfrak{F}$ and $K \subseteq \bigcup \ \subseteq W$. As $G = \cup_n g_n W_1$, we can take $a, b \in \mathbb{N}$ such that $g = g_a \cdot h$ and $h \in D_b \subseteq W$.
Now, for each compact subset gK of gW_1, we have

$$g_a D_b \cdot \mathfrak{F} \subseteq \mathfrak{N}, \quad g \in \bigcap g_a D_b \cdot \mathfrak{F}$$

and

$$gK = g_a h \cdot K \subseteq \bigcup g_a D_b \cdot \mathfrak{F}.$$

Thus, $\mathfrak{N}$ is a ck-network at g.

(3) Assume that $\mathfrak{D}$ is a cp-network at e. Let $A \subseteq G$ be such that $g \in \overline{A} \setminus A$. Since $e \in \overline{g^{-1}A} \setminus g^{-1}A$, there is $D_s \in \mathfrak{D}$ such that $e \in D_s \subseteq W_0$ and $D_s \cap g^{-1}A$ is infinite. Hence

$$g_r h D_s \cap A \subseteq g_r(D_t \cdot D_s) \cap A$$

is infinite. As $g \in g_r(D_t \cdot D_s) \in \mathfrak{N}$ and

$$g_r(D_t \cdot D_s) = g(h^{-1} \cdot D_t \cdot D_s) \subseteq g \cdot W_0^{-1} \cdot W_0^2 \subseteq g \cdot W^3 \subseteq g \cdot U,$$

we obtain that $\mathfrak{N}$ is a cp-network at g. □

Recall that the space $\mathfrak{D}'(\Omega)$ of distributions over an open subset $\Omega \subset \mathbb{R}^n$ is separable. We apply Theorem 18.1.13 to show that the space $D'(\Omega)$ of distributions has a countable cp-network, which essentially improves a well-known fact stating that $D'(\Omega)$ has a countable tightness.

Corollary 18.1.14 *The space of distributions $\mathfrak{D}'(\Omega)$ is a $\mathfrak{B}_0$-space; hence, it has the strong Pytkeev property.*

18.2 Topological Groups with a $\mathfrak{G}$-Base

Following previous sections, a topological group G will be said to have a $\mathfrak{G}$-base if G admits a base of neighbourhoods (of the unit) $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ such that $U_\alpha \subset U_\beta$ whenever $\beta \leq \alpha$ for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$. Most of the results included in this section were already presented in [267]. Some additional facts used below are also taken from [135] and [39].

In this section, we show that a topological group G is metrizable if and only if G is Fréchet–Urysohn and has a $\mathfrak{G}$-base. We also show that any precompact set in a topological group with a $\mathfrak{G}$-base is metrizable, and hence, G is strictly angelic. We deduce from this result that an almost metrizable group G is metrizable if and only if G has a $\mathfrak{G}$-base. We provide a few classes of topological groups with a $\mathfrak{G}$-base which are non-metrizable. Moreover, some examples of countable precompact abelian groups G with $\chi(G) = \aleph_1$ which do not admit a $\mathfrak{G}$-base will be shown.

The classical metrization theorem of Birkhoff and Kakutani asserts that a topological group G is metrizable if and only if G is *first-countable*, i.e. there exists a *decreasing* sequence $\{U_n\}_{n\in\mathbb{N}}$ which forms a base of neighbourhoods at the unit e of G.

We prove the following theorem due to Gabriyelyan, Kąkol, and Leiderman [267].

Theorem 18.2.1 (Gabriyelyan–Kąkol–Leiderman) *A topological group G is metrizable if and only if it is Fréchet–Urysohn and has a $\mathfrak{G}$-base.*

Clearly, all metrizable topological groups admit a $\mathfrak{G}$-base. Last theorem is related with the Malykhin problem (1978): *is there a separable Fréchet–Urysohn topological group that is not metrizable?* See [315] discussing this problem and paper [316] in which Hrušák and Ramos-García constructed a model of ZFC, where every separable Fréchet topological group is metrizable.

Hence, Malykhin's problem can be presented as follows: *does every countable Fréchet–Urysohn group admit a $\mathfrak{G}$-base?*

Let G be a topological group with a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and each $k \in \mathbb{N}$, set

$$D_k(\alpha) := \bigcap_{\beta\in I_k(\alpha)} U_\beta, \quad \text{where } I_k(\alpha) = \left\{\beta \in \mathbb{N}^\mathbb{N} : \beta_i = \alpha_i \text{ for } i = 1, \ldots, k\right\}.$$

Clearly, $\{D_k(\alpha)\}_{k\in\mathbb{N}}$ is an increasing sequence of subsets of G containing the unit.

In order to prove Theorem 18.2.1, we need the following:

Lemma 18.2.2 *Let $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and $\beta_k = (\beta_i^k)_{i\in\mathbb{N}} \in I_k(\alpha)$ for every $k \in \mathbb{N}$. Then, there is $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\alpha \leq \gamma$ and $\beta_k \leq \gamma$ for every $k \in \mathbb{N}$.*

Indeed, for every $i \in \mathbb{N}$, put $\gamma_i = \max\{\alpha_i, \beta_i^l : l = 1, \dots, i\} = \max\{\beta_i^l : l \in \mathbb{N}\}$. Set $\gamma := (\gamma_i)_{i\in\mathbb{N}}$.

In [135, Lemma 1.3], it was shown that every Fréchet–Urysohn topological group G enjoys the following condition (AS):

(AS) For any family $\{x_{n,k} : (n,k) \in \mathbb{N}\times\mathbb{N}\} \subset G$, with $\lim_n x_{n,k} = x \in G, k = 1, 2, \dots,$ there exists a strictly increasing sequences of natural numbers $(n_i)_{i\in\mathbb{N}}$ and $(k_i)_{i\in\mathbb{N}}$, such that $\lim_i x_{n_i,k_i} = x$.

Lemma 18.2.3 *For a topological group G with a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, the following are equivalent:*

(i) *G is metrizable.*
(ii) *G is Fréchet–Urysohn.*
(iii) *For every $\alpha \in \mathbb{N}^\mathbb{N}$, there exists $k \in \mathbb{N}$ such that $D_k(\alpha)$ is a neighbourhood of the unit e.*

Proof (i)⇒(ii) is obvious.

(ii)⇒(iii): Assume that there exists $\alpha \in \mathbb{N}^\mathbb{N}$ such that $D_k(\alpha)$ is not a neighbourhood of the unit e for every $k \in \mathbb{N}$. Hence, e belongs to the closure of the set $G \setminus D_k(\alpha)$. Since G is Fréchet–Urysohn, for every $k \in \mathbb{N}$, there is a sequence $\{x_{n,k}\}_{n\in\mathbb{N}}$ in $G \setminus D_k(\alpha)$ converging to e. By (AS), we find strictly increasing sequences of natural numbers $(n_i)_{i\in\mathbb{N}}$ and $(k_i)_{i\in\mathbb{N}}$ with $\lim_i x_{n_i,k_i} = e$.

For every $i \in \mathbb{N}$, choose $\beta_{k_i} \in I_{k_i}(\alpha)$ with $x_{n_i,k_i} \notin U_{\beta_{k_i}}$. By Lemma 18.2.2 (with $\beta_k = \alpha$ if $k \neq k_i$ for all i), take $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\beta_{k_i} \leq \gamma$ for every $i \in \mathbb{N}$. Hence, $x_{n_i,k_i} \notin U_\gamma$ for every $i \in \mathbb{N}$. Thus, $x_{n_i,k_i} \not\to e$, a contradiction. Hence, there is $k \in \mathbb{N}$ for which $D_k(\alpha)$ is a neighbourhood of e.

(iii)⇒(i) For every $\alpha \in \mathbb{N}^\mathbb{N}$, choose the minimal natural number k_α such that $D_{k_\alpha}(\alpha)$ is a neighbourhood of the unit e. By the construction of $D_k(\alpha)$, the family $\left\{\operatorname{int}\left(D_{k_\alpha}(\alpha)\right)\right\}_{\alpha\in\mathbb{N}^\mathbb{N}}$ is a countable base of neighbourhoods of e, so G is metrizable. □

We are ready to provide the proof of Theorem 18.2.1.

Proof of Theorem 18.2.1 It is enough to apply Lemma 18.2.3. □

We prove also the following theorem due to Gabriyelyan, Kąkol, and Leiderman [267].

Theorem 18.2.4 *If a topological group G has a $\mathfrak{G}$-base, then every precompact subset K in G is metrizable.*

Proof We may assume that G is complete and K is compact. Let $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be an open $\mathfrak{G}$-base in G. We may assume that all sets U_α are symmetric. We show that K is metrizable. By Proposition 6.2.9, it is enough to show that the set $W := (K \times K) \setminus \Delta$ has a compact resolution swallowing compact sets.

For each $\alpha \in \mathbb{N}^\mathbb{N}$, set

$$W_\alpha := \{(x, y) \in W : xy^{-1} \notin U_\alpha\}.$$

Then, W_α is closed in $K \times K$, and hence, it is compact for every $\alpha \in \mathbb{N}^{\mathbb{N}}$. Observe that the family $\mathcal{W} := \{W_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution in W. Indeed, if $(x, y) \in W$, then $x \neq y$. Hence, there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $xy^{-1} \notin U_\alpha$. So, $(x, y) \in W_\alpha$. Thus, $W = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} W_\alpha$ and $\mathcal{W}$ is a compact resolution.

We show that the family $\mathcal{W}$ swallows compact sets in W. Let $C \subset W$ be a compact set. Suppose that $C \nsubseteq W_\alpha$ for each $\alpha \in \mathbb{N}^{\mathbb{N}}$. Then, for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, there exists

$$(x_\alpha, y_\alpha) \in C \setminus W_\alpha.$$

So, we obtained the net $(x_\alpha, y_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ in C such that $x_\alpha y_\alpha^{-1} \in U_\alpha$ for all $\alpha \in \mathbb{N}^{\mathbb{N}}\}$. Since C is compact, this net has a cluster point $(x, y) \in C$, so $x \neq y$. Hence, there exists symmetric $U \in \mathcal{N}(X)$ such that $xy^{-1} \notin U$. Choose $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that

$$U_\alpha \cdot U_\alpha \cdot U_\alpha \subset U.$$

Then, there exists α and there exists $\beta \geqslant \alpha$ in $\mathbb{N}^{\mathbb{N}}$ such that

$$(x_\beta, y_\beta) \in (U_\alpha \times U_\alpha) \cdot (x, y).$$

Hence, $x_\beta \in U_\alpha \cdot x$ and $y_\beta \in U_\alpha \cdot y$. So, $xx_\beta^{-1}, y_\beta y^{-1} \in U_\alpha$ because U_α is symmetric. Since $x_\beta y_\beta^{-1} \in U_\beta \subset U_\alpha$, we obtain

$$xy^{-1} = (xx_\beta^{-1})(x_\beta y_\beta^{-1})(y_\beta y^{-1}) \in U_\alpha \cdot U_\alpha \cdot U_\alpha \subset U.$$

This contradicts the choice of U. We showed that $\mathcal{W}$ swallows compact sets in W. Therefore, C is metrizable by Proposition 6.2.9. □

This yields the following:

Corollary 18.2.5 *Assume that a topological group G has a $\mathfrak{G}$-base. Then, G is a k-space if and only if it is sequential.*

Proof Let G be a k-space. Each compact subset K of G is metrizable, by Theorem 18.2.4. Thus, G is sequential by Chasco et al. [135, Lemma 1.5]. If G is a sequential space, then G it is a k-space by Engelking [195, 3.3.20]. □

A topological group G is called *almost metrizable* if it contains a non-empty compact set K of countable character in G; see [39]. Every locally compact group is almost metrizable. Every almost metrizable group can be embedded as a subgroup into a Čech-complete group, [39, 4.3.16]. By applying Theorem 18.2.4, we have the following:

Theorem 18.2.6 *If G is an almost metrizable group, then G has a $\mathfrak{G}$-base if and only if G is metrizable.*

Proof Let G have a $\mathfrak{G}$-base. By Pasynkov's theorem (see [39, 4.3.20]), the group G contains a compact subgroup H such that the left quotient space G/H is metrizable.

By Theorem 18.2.4, we deduce that the subgroup H is metrizable. Thus, G is metrizable by Hewitt and Ross [310, 5.38]. □

Corollary 18.2.7 *A locally precompact group G has a $\mathfrak{G}$-base if and only if G is metrizable.*

Proof The completion of G is locally compact and has a $\mathfrak{G}$-base. Now, Theorem 18.2.6 applies. □

We provide an alternative proof for abelian groups. First, we need to recall some concepts and definitions.

For an abelian topological group G by $\widehat{G}$, we denote the group of all continuous characters on G. The group $\widehat{G}$ with the compact-open topology is denoted by $G^\wedge$. The homomorphism $\alpha_G : G \to G^{\wedge\wedge}$, $x \mapsto (\chi \mapsto (\chi, x))$, is called *the canonical homomorphism*. If α_G is a topological isomorphism, the group G is called *reflexive*. The Pontryagin–van Kampen duality theorem states that every locally compact abelian group is reflexive.

The group G is called *maximally almost periodic* (MAP) if $\widehat{G}$ separates the points of G. For a MAP abelian group G, we denote by $\sigma(G, \widehat{G})$ or τ^+ the weak topology on G, i.e. the smallest topology in G for which the elements of $\widehat{G}$ are continuous. The topology τ^+ is called *the Bohr modification* of τ. Set $G^+ := (G, \tau^+)$.

It is well known that the groups G and G^+ have the same set of continuous characters and $G^+ = G$ if and only if G is precompact; see [39].

For a subset A of an abelian topological group G, *the polar* of A is $A^\triangleright := \{\chi \in G^\wedge : \chi(A) \subseteq \mathbb{T}_+\}$, where $\mathbb{T}_+ := \{z \in \mathbb{T} : \mathrm{Re}(z) \geq 0\}$. The set A is called *locally quasi-convex* if for every $x \in G \setminus A$, there is a $\chi \in A^\triangleright$ such that $\mathrm{Re}(\chi, x) < 0$. For $B \subseteq X^\wedge$, *the inverse polar* of B is the set $B^\triangleleft := \{x \in X : x(B) \subseteq \mathbb{T}_+\}$.

Recall that A is *quasi-convex* if and only if we have $(A^\triangleright)^\triangleleft = A$; see [310, Theorem 2]. The group G is called *locally quasi-convex* if it has a basis at zero whose elements are quasi-convex subsets. Every locally compact abelian group is reflexive, and hence, it is locally quasi-convex. The family of the sets of the form

$$P(K, \varepsilon) := \{\chi \in \widehat{G} : \ |1 - (\chi, x)| < \varepsilon, \ \forall x \in K\},$$

where K is compact in G and $\varepsilon > 0$ forms a basis of open neighbourhoods at zero of the compact-open topology on $\widehat{G}$. For a subset D of G, set $(1)D := D$ and $(n+1)D := (n)D + D$ for $n \in \mathbb{N}$.

Next Lemma from [267] will be used below.

Lemma 18.2.8 *Let G be an abelian topological group.*

(1) *Assume G has a $\mathfrak{G}$-base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Set $\mathcal{W} = \{W_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, where $W_\alpha := U_\alpha^\triangleright$, the dual family of compact sets in $G^\wedge$. Then, the family $\mathcal{W}$ is a compact resolution in $G^\wedge$.*

(2) *The following are equivalent:*

   (a) *The dual compact resolution $\mathcal{W}$ swallows compact sets in $G^\wedge$.*
   (b) *Every compact subset of $G^\wedge$ is equicontinuous.*
   (c) *The canonical homomorphism α_G is continuous.*

(3) *If G has a compact resolution $\mathcal{W} = \{W_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ swallowing compact sets in G, then the dual family $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, where $U_\alpha := W_\alpha^\triangleright$, is a $\mathfrak{G}$-base in $G^\wedge$.*

Proof (1) Observe that the set W_α is compact in $G^\wedge$ by applying [42, 3.5]. Clearly, $G^\wedge = \bigcup\{W_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ and $W_\alpha \subset W_\beta$, whenever $\alpha \leq \beta$. This shows that $G^\wedge$ has a compact resolution.

(2) (a)⇒(b) Assume that $\mathcal{W}$ swallows compact sets in $G^\wedge$, and let K be a compact subset of $G^\wedge$. Take $\alpha \in \mathbb{N}^\mathbb{N}$ such that $K \subseteq U_\alpha^\triangleright$.
Let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{\varepsilon}{20}$. Choose $\beta \in \mathbb{N}^\mathbb{N}$ such that $(n)U_\beta \subseteq U_\alpha$. Then, for all $x \in U_\beta$ and each $0 \leq k \leq n$, we have $\chi(kx) \in \mathbb{T}_+$ for all $\chi \in K$. This shows $\arg(\chi(x)) \in \left[-\frac{\pi}{2n}, \frac{\pi}{2n}\right]$. Consequently

$$|1 - \chi(x)| \leq \left|1 - \exp\left\{i\frac{\pi}{2n}\right\}\right| \leq 2\pi \cdot \frac{\pi}{2n} < \varepsilon, \quad \forall x \in U_\beta.$$

We proved that K is equicontinuous.
(b)⇒(a) Assume that a compact subset K of $G^\wedge$ is equicontinuous. Then, for $\varepsilon = 0.1$, there is $U \in \mathcal{N}(X)$ such that

$$|1 - \chi(x)| < \varepsilon, \quad \forall \chi \in K,\ \forall x \in U.$$

Take $\alpha \in \mathbb{N}^\mathbb{N}$ such that $U_\alpha \subseteq U$. Then, for every $\chi \in K$, we have $\chi(U_\alpha) \subset \mathbb{T}_+$. This means that $K \subseteq U_\alpha^\triangleright = W_\alpha$. Thus, $\mathcal{W}$ swallows the compact sets in $G^\wedge$.
To see the equivalence of (b) and (c), we refer to [42, 5.10].

(3) For each $\alpha \in \mathbb{N}^\mathbb{N}$, set $U_\alpha := W_\alpha^\triangleright$. Then, U_α is open in $G^\wedge$. Also, $U_\beta \subseteq U_\alpha$ for $\alpha \leq \beta$. Since $\mathcal{W}$ swallows compact subsets of X, for each compact $K \subset G$, there is $\alpha \in \mathbb{N}^\mathbb{N}$ such that $K \subset W_\alpha$. This shows that $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-base in G. □

Now, we are ready to provide an alternative proof of Corollary 18.2.7 for abelian groups.

Theorem 18.2.9 (Gabriyelyn–Kąkol–Leiderman) *A locally precompact abelian group G has a $\mathfrak{G}$-base if and only if G is metrizable.*

Proof Assume that G has a $\mathfrak{G}$-base. We show that G is metrizable. Since the completion $\overline{G}$ has also a $\mathfrak{G}$-base, it is enough to show that $\overline{G}$ is metrizable. Recall here that $\overline{G} \cong \mathbb{R}^n \times G_0$, where $n \in \mathbb{N}$ and G_0 has an open compact subgroup H; see, for example, [310, 24.30]. This implies that the proof will be completed if we show that H is metrizable. Since G has a $\mathfrak{G}$-base, H has a $\mathfrak{G}$-base. By Lemma 18.2.8, we know that $H^\wedge$ has a compact resolution swallowing compact sets and is metrizable (as being discrete). Now, Corollary 6.2.5 implies that $H^\wedge$ is analytic; hence, $H^\wedge$ is countable. By Hewitt and Ross [310, 24.15], the space H is metrizable. The converse is clear. □

Corollary 18.2.10 *For a MAP abelian group G, the following assertions are equivalent.*

(i) G^+ *has a* $\mathfrak{G}$*-base.*
(ii) G^+ *is metrizable.*
(iii) $\widehat{G}$ *is countable.*

Proof Since G^+ is a precompact group, we apply the previous theorem to get (i)$\Leftrightarrow$(ii). The equivalence between items (ii) and (iii) one can deduce from the classic [145]. □

The next example illustrates Theorem 18.2.9.

Example 18.2.11 There exists an countable group (G, τ) of character $\chi(G) = \aleph_1$ without a $\mathfrak{G}$-base. For example, consider $G = \mathbb{Z}$. Take an independent subset E of $\mathbb{T}$ of cardinality $\aleph_1$ without torsion elements. Set $\tau := \sigma(G, \langle E\rangle)$, where $\langle E\rangle$ is a subgroup of $\mathbb{T}$ generated by E. Since $|\langle E\rangle| = \aleph_1$, we note that $\chi(G, \tau) = \aleph_1$. On the other hand, since (G, τ) is a precompact non-metrizable group, we conclude that it has no $\mathfrak{G}$-base by last Theorem 18.2.9.

18.3 When the Banach Space $\ell_1(\Gamma)$ Is a Weakly $\aleph$-Space?

The study of Banach (or Fréchet) spaces E in its weak topology $\sigma(E, E')$ is an interesting and well-motivated line of research that has been successfully developed from many years.

Recall here the following result from [264]. We proved the following:

Theorem 18.3.1 *Let $C_c(X)$ be a Fréchet lcs. The following assertions are equivalent:*

(i) $C_c(X)$ *is a weakly* $\aleph$*-space.*
(ii) $C_c(X)$ *is a weakly* $\aleph_0$*-space.*
(iii) X *is countable.*

We also showed in [264] that a Banach space E not containing a copy of ℓ_1 is a weakly $\aleph$-space if and only if it is a weakly $\aleph_0$-space if and only if the strong dual E' of E is separable. Consequently, for any $1 < p < \infty$ and an uncountable set Γ, the (reflexive) Banach space $\ell_p(\Gamma)$ is not a weakly $\aleph$-space. We showed even more: *a reflexive Fréchet lcs E is a weakly $\aleph$-space if and only if E is a weakly $\aleph_0$-space if and only if E is separable.*

These results motivate the following natural question:

Problem 18.3.2 Does there exist a non-separable Banach space E which is an $\aleph$-space in the weak topology of E?

We answer this question in the affirmative by proving the following theorem from [264, Theorem 1.2].

Theorem 18.3.3 *The Banach space $\ell_1(\Gamma)$ is an $\aleph$-space in the weak topology if and only if the cardinality of Γ does not exceed continuum.*

So, the non-separable Banach space $\ell_1(\mathbb{R})$ endowed with the weak topology is an $\aleph$-space but is not an $\aleph_0$-space. Moreover, the space $\ell_1(\mathbb{R})$ in the weak topology is not normal; see Proposition 18.3.13 below.

First, we provide some necessary condition for any lcs to be a weakly $\aleph$-space. We need the following useful observation.

Lemma 18.3.4 *Let E be a non-zero lcs. Then, $E_w := (E, \sigma(E, E'))$ has a countable pseudocharacter if and only if E_w admits a weaker separable metrizable lcs topology. In particular, $|E| = \mathfrak{c}$ provided E_w has countable pseudocharacter.*

Proof Assume that E_w has a countable pseudocharacter. Let $\bigcap_n U_n = \{0\}$, where the open sets U_n have the following standard form:

$$U_n = \{x \in E : |\chi_{i,n}(x)| < \delta_n, \text{ where } \chi_{i,n} \in E' \text{ for } 1 \le i \le k_n\}.$$

Let $\{\chi_n\}_n$ be an enumeration of the family $\{\chi_{i,n} : 1 \le i \le k_n, n \in \mathbb{N}\}$. Then, $\bigcap_n \ker(\chi_n) = \{0\}$. This implies that the following map

$$p : E_w \to \prod_n E/\ker(\chi_n) = \mathbb{R}^{\mathbb{N}}, \quad p(x) = (\chi_n(x))_{n\in\mathbb{N}},$$

is continuous and injective, and hence, $|E| = \mathfrak{c}$ as E is non-trivial. Now, the topology induced on E_w from $\mathbb{R}^{\mathbb{N}}$ is as desired. The converse assertion is trivial. □

The next fact is well known; see, for example, [264].

Lemma 18.3.5 *A lcs E admits a metrizable and separable locally convex topology τ weaker than $\sigma(E, E')$ if and only if $(E', \sigma(E', E))$ is separable.*

Lemmas 18.3.4 and 18.3.5 imply the following necessary conditions on a lcs E to be a weakly $\aleph$-space.

Proposition 18.3.6 *If E is a non-trivial lcs which is a weakly σ-space, then:*

(i) *$(E, \sigma(E, E'))$ admits a weaker separable metrizable lcs topology.*
(ii) *$\psi(E, \sigma(E, E')) = \aleph_0$ and $|E| = \mathfrak{c}$.*
(iii) *$(E', \sigma(E', E))$ is separable.*

Clearly, the "only if" part of Theorem 18.3.3 follows from Proposition 18.3.6 because the space $\ell_1(\Gamma)$ with the weak topology does not have countable pseudocharacter whenever $|\Gamma| > 2^{\aleph_0}$. We need to prove the "if" part. It is clear that if the cardinality of the set Γ_1 is less than or equal to the cardinality of the set Γ_2, then $\ell_1(\Gamma_1)$ embeds into $\ell_1(\Gamma_2)$; therefore, it is enough to consider the case when Γ has cardinality continuum.

We shall work with the space $\ell_1(2^\omega)$, where 2^ω denotes the Cantor set, treated just as an index set of cardinality continuum (recall that the space $\ell_1(S)$ does not depend on any extra structure of the set S).

We shall use some ideas from [428] (especially included in the proof of Lemma 2.3.1 in [428]) to provide the proof of Theorem 18.3.3.

For a Banach space E, we denote by $\mathbf{B}_E$ and $\mathbf{S}_E$ the closed unit ball and the unit sphere of E, respectively. Note the following:

Lemma 18.3.7 *The unit sphere $\mathbf{S}_{\ell_\infty(2^\omega)}$ is weak*-separable.*

Proof Let $\mathbb{P}$ be the family of all (necessarily finite) partitions of the Cantor set into finitely many open sets. As 2^ω is zero-dimensional, for every finite set $F \subset 2^\omega$, there is $P \in \mathbb{P}$ such that $P = \{U_x : x \in F\}$ and $x \in U_x$ for every $x \in F$. Obviously, the family $\mathbb{P}$ is countable, because the Cantor set has only countably many sets that are open and closed simultaneously. Define

$$D = \left\{ \sum_{U \in P} q_U \chi_U \in S_{\ell_\infty(2^\omega)} : P \in \mathbb{P},\ \{q_U : U \in P\} \subset \mathbb{Q} \right\},$$

where χ_A denotes the characteristic function of a set A. Obviously, D is countable. We claim that it is weak*-dense in $\mathbf{S}_{\ell_\infty(2^\omega)}$.

In fact, given $x_1, \ldots, x_k \in \ell_1(2^\omega)$ and $\varepsilon > 0$, a basic weak*-neighbourhood of $y \in S_{\ell_\infty(2^\omega)}$ is of the form:

$$V = \{v \in S_{\ell_\infty(2^\omega)} : |v(x_i) - y(x_i)| < \epsilon \text{ for } i = 1, 2, \ldots, k\}.$$

Fix $\delta > 0$ and let $F \subset 2^\omega$ be a finite set such that

$$\|x_i\| - \sum_{t \in F} |x_i(t)| = \sum_{t \notin F} |x_i(t)| < \delta \tag{18.1}$$

for every $i = 1, 2, \ldots, k$. Take a partition $P \in \mathbb{P}$ such that $U \cap F$ is either empty or a singleton, whenever $U \in \mathbb{P}$, and there is $U \in P$ such that $U \cap F = \emptyset$. For every $t \in F$ and each $U \in P$ containing $t \in U$, take $q_U \in [-1, 1] \cap \mathbb{Q}$ such that $|q_U - y(t)| < \delta$, and set $q_U = 1$ for every $U \in P$ such that $U \cap F = \emptyset$. Set $w = \sum_{U \in P} q_U \chi_U$. Then, $w \in D$. We show that $w \in V$ for δ small enough. Indeed, for every $i = 1, 2, \ldots, k$, the inequality (18.1) and the construction of w imply

$$\begin{aligned} |w(x_i) - y(x_i)| &\le \sum_{t \in F} |w(t)x_i(t) - y(t)x_i(t)| + \sum_{t \notin F} |w(t)x_i(t) - y(t)x_i(t)| \\ &< \delta \cdot \sum_{t \in F} |x_i(t)| + \sum_{t \notin F} 2|x_i(t)| < \delta(\|x_i\| + 2). \end{aligned}$$

Finally, it is clear that if δ is small enough then $w \in V$. Thus, $\mathbf{S}_{\ell_\infty(2^\omega)}$ is weak*-separable. □

Lemma 18.3.8 *Let E be a Banach space such that $\mathbf{S}_{E'}$ is weak*-separable. Then, for every $r > 0$, there exists a countable family $\mathcal{F}$ of weakly closed subsets of E*

contained in

$$E \setminus r\mathbf{B}_E = \{x \in E : \|x\| > r\}$$

and such that

$$E \setminus r\mathbf{B}_E = \bigcup_{F \in \mathcal{F}} \mathrm{int}_w(F),$$

where int_w *denotes the interior with respect to the weak topology.*

Proof Let D be a countable weak*-dense subset of $\mathbf{S}_{E'}$. Given $\varphi \in D$, $n \in \mathbb{N}$, define

$$F_{\varphi,n} = \{x \in E : \varphi(x) \geq r + 1/n\}.$$

Then, $\mathcal{F} = \{F_{\varphi,n} : \varphi \in D,\ n \in \mathbb{N}\}$ is the required family. □

We need another additional fact.

Lemma 18.3.9 *Let* $0 < \varepsilon < r$ *and let*

$$M(r, \varepsilon) = \{x \in \ell_1(2^\omega) : r - \epsilon < \|x\| \leq r\}.$$

Then, for every $x \in M(r, \epsilon)$*, there exists a weakly (in fact, pointwise) open set* $V \subset \ell_1(2^\omega)$ *such that* $x \in V$ *and* $\mathrm{diam}\,(V \cap M(r, \varepsilon)) \leq 4\varepsilon$.

Proof Given $A \subset 2^\omega$, denote by p_A the canonical projection from $\ell_1(2^\omega)$ onto $\ell_1(A)$, that is, $p_A(v) = v \restriction A$. Fix $x \in M(r, \varepsilon)$. There exists a finite set $F \subset 2^\omega$ such that $\|p_F(x)\| > r - \epsilon$. Choose an open set $U \subset \ell_1(F)$ such that $\|u\| > r - \varepsilon$ and $\|u - p_F(x)\| < \epsilon$ for every $u \in U$. Let $V = p_F^{-1}(U)$. We show that V is as required.

Obviously, $x \in V$ and V is pointwise (in particular, weakly) open. Fix $y_1, y_2 \in V \cap M(r, \epsilon)$. Let $A = 2^\omega \setminus F$.

Note that the ℓ_1-norm has the property that

$$\|v\| = \|p_F(v)\| + \|p_A(v)\|$$

for every $v \in \ell_1(2^\omega)$. In particular, $\|p_A(y_i)\| \leq \epsilon$, because $\|p_F(y_i)\| > r - \epsilon$ and $\|y_i\| \leq r$ for $i = 1, 2$. Using these facts, we get

$$\begin{aligned}\|y_1 - y_2\| &= \|p_F(y_1) - p_F(y_2)\| + \|p_A(y_1) - p_A(y_2)\| \\ &\leq \|p_F(y_1) - p_F(x)\| + \|p_F(x) - p_F(y_2)\| + \|p_A(y_1)\| + \|p_A(y_2)\| \\ &\leq 4\varepsilon.\end{aligned}$$

Hence, $\mathrm{diam}\,(V \cap M(r, \epsilon)) \leq 4\varepsilon$. □

The next fact has been used implicitly, e.g. in [428].

Lemma 18.3.10 *Let X be a metric space. Then, there exists an open base $\mathcal{B}$ in X such that $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ and each $\mathcal{B}_n$ is uniformly discrete, that is, for every n there is $\varepsilon_n > 0$ such that the distance of any two distinct members between $\mathcal{B}_n$ is $> \epsilon_n$.*

Proof A theorem of Stone says that every open cover of a metric space X admits a σ-discrete open refinement. The proof (see, e.g. [195, Proof of Theorem 4.4.1]) actually shows that every open cover of X has an open refinement of the form $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$, where each $\mathcal{U}_n$ is uniformly discrete. Now, let $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ be such that $\mathcal{W}_n$ is an open refinement of a cover by balls of radius $1/n$ and $\mathcal{W}_n$ is a countable union of uniformly discrete families. Then, $\mathcal{B}$ is easily seen to be an open base. □

Remark 18.3.11 The proof of the Theorem 18.3.3 uses the well-known fact that the space $\ell_1(\Gamma)$ has the Schur property (i.e. any convergent sequence in the weak topology is also a convergent sequence in the norm topology) for every set Γ; see [198]. This implies that any weakly compact set of $\ell_1(\Gamma)$ is also norm compact.

We are ready to complete the proof of Theorem 18.3.3.

Proof of Theorem 18.3.3 Let $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ be a base of open sets for the norm topology on $\ell_1(2^\omega)$ such that the distance between every two distinct members of $\mathcal{B}_n$ is $> 1/k_n$ for every $n \in \mathbb{N}$ (here, we have used Lemma 18.3.10).

For any $r > 0$, define

$$U_r = \ell_1(2^\omega) \setminus r\mathbf{B}_{\ell_1(2^\omega)}.$$

Let $\mathcal{F}_r$ be a countable family of weakly closed subsets of U_r such that $\bigcup_{F\in\mathcal{F}_r} \operatorname{int}_w F = U_r$ (Lemma 18.3.8). Let $\mathcal{F}_r = \{F_r^m\}_{m\in\mathbb{N}}$.

Given $n, m, i \in \mathbb{N}$, define

$$L(i,n) := M\left(\frac{i+2}{10k_n}, \frac{1}{5k_n}\right) = \left\{x \in \ell_1(2^\omega)\colon \frac{i}{10k_n} < \|x\| \le \frac{i+2}{10k_n}\right\}$$

and

$$C(n,m,i) = \{B \cap F^m_{i/10k_n} \cap L(i,n)\colon B \in \mathcal{B}_n\}.$$ □

Claim 18.3.12 For every $n, m, i \in \mathbb{N}$, the family $C(n,m,i)$ is discrete in the weak topology.

Proof Note that the union of $C(n,m,i)$ is contained in the weakly closed set

$$F^m_{i/10k_n} \cap ((i+2)/10k_n)\mathbf{B}_{\ell_1(2^\omega)};$$

therefore, it is enough to show that every point of this set has a weak neighbourhood meeting at most one set from $C(n,m,i)$.

Fix $x \in F^m_{i/10k_n} \cap ((i+2)/10k_n)\mathbf{B}_{\ell_1(2^\omega)} \subset L(i,n)$. By Lemma 18.3.9, there exists a weakly open set V such that $x \in V$ and

$$\operatorname{diam}\big(V \cap L(i,n)\big) \le 4/5k_n < 1/k_n.$$

The set V can intersect at most one $B \in \mathcal{B}_n$, as $\mathcal{B}_n$ is $1/k_n$-discrete. □

Let $O(r) = \{x \in \ell_1(2^\omega) \colon \|x\| < r\}$ and define

$$\mathcal{D}(n) = \{B \cap O(1/5k_n) \colon B \in \mathcal{B}_n\}.$$

Note that actually $\mathcal{D}(n)$ contains at most one non-empty set (because $2/5k_n < 1/k_n$); therefore, it is certainly discrete in the weak topology.

Define

$$\mathcal{A} = \bigcup\{C(n,m,i) \colon n,m,i \in \mathbb{N}\} \cup \{\mathcal{D}(n) \colon n \in \mathbb{N}\}.$$

Then, the family $\mathcal{A}$ is σ-discrete with respect to the weak topology. It remains to show that $\mathcal{A}$ is a k-network in $\ell_1(2^\omega)$ with the weak topology.

Fix a weakly compact set K contained in a weakly open set $U \subset \ell_1(2^\omega)$. It follows from Remark 18.3.11 that K is also compact in the norm topology. Choose a finite $\mathcal{E} \subset \mathcal{B}$ such that $K \subset \bigcup \mathcal{E} \subset U$. Choose n_0 such that $\mathcal{E} \subset \bigcup_{n=1}^{n_0} \mathcal{B}_n$.

For $i,n \in \mathbb{N}$, let $N(i,n) := \operatorname{int}\big(L(i,n)\big)$. Note that, for each $n \in \mathbb{N}$, the space $\ell_1(2^\omega)$ is covered by $O(1/5k_n)$ and the sets $N(i,n)$, $i \in \mathbb{N}$. Given $B \in \mathcal{B}_n$, by Lemma 18.3.8, we have that

$$B = \bigcup\Big\{B \cap \operatorname{int}_w(F^m_{i/10k_n}) \cap N(i,n) \colon m,i \in \mathbb{N}\Big\} \cup (B \cap O(1/5k_n)).$$

This shows that the family

$$\bigcup_{n\le n_0}\Big(\Big\{B \cap \operatorname{int}_w(F^m_{i/10k_n}) \cap N(i,n) \colon B \in \mathcal{B}_n \cap \mathcal{E},\ m,i \in \mathbb{N}\Big\} \cup \{B \cap O(1/5k_n) \colon B \in \mathcal{B}_n \cap \mathcal{E}\}\Big)$$

covers K and consists of norm open sets. Hence, it has a finite subfamily covering K. Hence, for any $n \le n_0$, we can find $i_0(n)$ and $m_0(n)$ such that the finite subfamily

$$\mathcal{F} := \bigcup_{n\le n_0}\Big(\Big\{B \cap F^m_{i/10k_n} \cap L(i,n) \colon B \in \mathcal{B}_n \cap \mathcal{E},\ m \le m_0(n),\ i \le i_0(n)\Big\} \cup \{B \cap O(1/5k_n) \colon B \in \mathcal{B}_n \cap \mathcal{E}\}\Big)$$

of $\mathcal{A}$ satisfies $K \subset \bigcup \mathcal{F} \subset \bigcup \mathcal{E} \subset U$. Thus, the σ-locally finite family $\mathcal{A}$ is also a k-network for $\ell_1(2^\omega)$ in the weak topology.

Finally, note the following:

Proposition 18.3.13 *The space $\ell_1(\mathbb{R})$ endowed with the weak topology is not normal.*

Proof Assume for a contradiction that $\ell_1(\mathbb{R})$ with the weak topology is a normal space. Then, the square $\ell_1^2(\mathbb{R})$ of $\ell_1(\mathbb{R})$ with the weak topology ω is also normal (note here that $\ell_1^2(\mathbb{R})$ and $\ell_1(\mathbb{R})$ endowed with the weak topologies are homeomorphic). Now, we apply a result of Corson [147, Lemma 7] to deduce that every ω-discrete set in $\ell_1(\mathbb{R})$ is countable, which clearly leads to a contradiction. This completes the proof. □

18.4 The Strong Pytkeev Property for Topological Groups

In this section, we describe the topology of a topological space X admitting a countable *cp*-network at a point. This approach provides a new metrization theorem for topological groups which are Baire spaces. In particular, this yields that a Baire topological group which is cosmic must be metrizable.

We will see that every topological group G with the strong Pytkeev property satisfies also a condition which seems to be "close" to have a $\mathfrak{G}$-base; that means G admits a base of neighbourhoods at the unit e of the form $\{U_\alpha : \alpha \in \mathbf{M}\}$, where $\mathbf{M}$ is a subset of the partially ordered set $\mathbb{N}^\mathbb{N}$ and $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$ for $\alpha, \beta \in \mathbf{M}$.

Let Ω be a set and let I be a partially ordered set with an order $\leq$. We will say that a family $\{A_i\}_{i\in I}$ of subsets of Ω is *I-decreasing* if $A_j \subseteq A_i$ for every $i \leq j$ in I.

Clearly, one of the most important examples of partially ordered sets is the product $\mathbb{N}^\mathbb{N}$ endowed with the natural partial order, i.e. $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $i \in \mathbb{N}$, where $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ and $\beta = (\beta_i)_{i\in\mathbb{N}}$.

Let $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and each $k \in \mathbb{N}$. Set

$$I_k(\alpha) := \left\{\beta \in \mathbb{N}^\mathbb{N} : \beta_i = \alpha_i \text{ for } i = 1, \ldots, k\right\}.$$

Let $\mathbf{M} \subseteq \mathbb{N}^\mathbb{N}$ and $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ be an $\mathbf{M}$-decreasing family of subsets of a set Ω. Then, we define the countable family $\mathcal{D}_\mathcal{U}$ of subsets of Ω by

$$\mathcal{D}_\mathcal{U} := \{D_k(\alpha) : \alpha \in \mathbf{M},\ k \in \omega\},\ \text{where } D_k(\alpha) = \bigcap_{\beta\in I_k(\alpha)\cap\mathbf{M}} U_\beta,$$

and say that $\mathcal{U}$ satisfies the *condition* $\mathbf{D}$ if $U_\alpha = \bigcup_k D_k(\alpha)$ for every $\alpha \in \mathbf{M}$. We refer to [57, 261, 265, 266], where a similar condition has been used.

Recall [261] that a topological space (X, τ) has a *small base* if there exists an M-decreasing base of τ for some $M \subset \mathbb{N}^\mathbb{N}$.

Below, we describe the topology of a topological space X at a point x at which it has countable *cn*-, *ck*- or *cp*-character; see [261].

Theorem 18.4.1 (Gabriyelyan–Kąkol) *Let x be a point of a topological space X. Then, we have:*

(i) *X has a countable cn-network at x if and only if X has a small base $\mathcal{U}(x) = \{U_\alpha : \alpha \in \mathbf{M}_x\}$ at x satisfying the condition* **D**. *In that case, the family $\mathcal{D}_{\mathcal{U}(x)}$ is a countable cn-network at x.*

(ii) *X has a countable ck-network at x if and only if X admits a* small base *$\mathcal{U}(x) = \{U_\alpha : \alpha \in \mathbf{M}_x\}$ at x satisfying the condition* **D** *such that the family $\mathcal{D}_{\mathcal{U}(x)}$ is a countable ck-network at x.*

(iii) *X has a countable cp-network at x if and only if X has a small base $\mathcal{U}(x) = \{U_\alpha : \alpha \in \mathbf{M}_x\}$ at x satisfying the condition* **D** *such that the family $\mathcal{D}_{\mathcal{U}(x)}$ is a countable cp-network at x.*

Proof We may assume that x is not isolated. Otherwise, if x is an isolated point, put $\mathbf{M}_x := \mathbb{N}^{\mathbb{N}}$ and $U_\alpha := \{x\}$ for each $\alpha \in \mathbf{M}_x$. Then, the family $\{U_\alpha : \alpha \in \mathbf{M}_x\}$ is as we need for all three cases of the theorem.

(i) Assume that X has a countable *cn*-network $\mathcal{D} = \{D_i\}_i$ at the point x. Recall that D_i contains x for every $i \in \mathbb{N}$.

Step 1. For every $k, i \in \mathbb{N}$, set

$$D_k^i := \bigcap_{l=1}^{k} D_{i-1+l}.$$

Hence, for each $i \in \mathbb{N}$, the sequence $\{D_k^i\}_{k\in\mathbb{N}}$ is decreasing. For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, set

$$A_\alpha := \bigcup_{i\in\mathbb{N}} D_{\alpha_i}^i = \bigcup_{i\in\mathbb{N}} \bigcap_{l=1}^{\alpha_i} D_{i-1+l}.$$

Then, $x \in A_\alpha$ and $A_\alpha \subseteq A_\beta$ for each $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ with $\beta \le \alpha$.

Step 2. Let V be a neighbourhood of x. Set $J(V) := \{j \in \mathbb{N} : D_j \subseteq V\}$. Since x is not isolated, the family $J(V)$ is infinite. We prove the following claim.

(**A**) If W is a neighbourhood of x and $J(W) := \{j \in \mathbb{N} : D_j \subseteq W\} = \{n_k\}_{k\in\mathbb{N}}$ with $n_1 < n_2 < \ldots$, then there is $\alpha = \alpha(W) \in \mathbb{N}^{\mathbb{N}}$ such that

(A_1) $\alpha_{n_k} = 1$ for every $k \in \mathbb{N}$.
(A_2) $A_\alpha = \bigcup_k D_{n_k} (\subseteq W)$ is a neighbourhood of x.

The construction of $\alpha = \alpha(W)$ is as follows: if $i = n_k$ for some $k \in \mathbb{N}$, we set $\alpha_i = 1$. So $D^i_{\alpha_i} = D_{n_k}$. Set $n_0 := 0$. If $n_{k-1} < i < n_k$ for some $k \in \mathbb{N}$, we set $\alpha_i := n_k - i + 1$. Then

$$D^i_{\alpha_i} = \bigcap_{l=1}^{\alpha_i} D_{i-1+l} \subseteq D_{i-1+\alpha_i} = D_{n_k}.$$

This shows that $A_\alpha = \bigcup_k D_{n_k}$. Since $\mathcal{D}$ is a cn-network at x, A_α is a neighbourhood of x. Thus, (A_1) and (A_2) are satisfied.

Step 3. Denote by $\mathbf{M}_x$ the set of all $\alpha \in \mathbb{N}^{\mathbb{N}}$ of the form $\alpha = \alpha(W)$ for some neighbourhood W of x. For each $\alpha \in \mathbf{M}_x$, set $U_\alpha := A_\alpha$. By $(\mathbf{A})$, the family $\{U_\alpha : \alpha \in \mathbf{M}_x\}$ is a small base at x.

Step 4. We prove that the condition $\mathbf{D}$ holds. It is clear that $\bigcup_k D_k(\alpha) \subseteq U_\alpha$. On the other hand, we have

$$\bigcup_k D_k(\alpha) = \bigcup_k \bigcap_{\beta \in I_k(\alpha) \cap \mathbf{M}_x} U_\beta = \bigcup_k \bigcap_{\beta \in I_k(\alpha) \cap \mathbf{M}_x} \left(\bigcup_i \bigcap_{l=1}^{\beta_i} D_{i-1+l} \right)$$
$$\supseteq \bigcup_k \bigcap_{\beta \in I_k(\alpha) \cap \mathbf{M}_x} \left(\bigcap_{l=1}^{\beta_k} D_{k-1+l} \right) = \bigcup_k \bigcap_{l=1}^{\alpha_k} D_{k-1+l} = U_\alpha.$$

For the converse, observe that, if X has a small base at x with the condition $\mathbf{D}$, then the countable family $\mathcal{D}_{\mathcal{U}(x)}$ is a cn-network at x.

(ii) Assume that X has a countable ck-network $\mathcal{D} = \{D_i\}_{i \in \mathbb{N}}$ at x. We may assume that $\mathcal{D}$ is closed under taking finite unions. Following the argument in (i), we can prove that X has a small base $\{U_\alpha : \alpha \in \mathbf{M}_x\}$ at x with the condition $\mathbf{D}$. We show that the countable family $\mathcal{D}_{\mathcal{U}(x)}$ is also a ck-network at x.

Let U_x be a neighbourhood of x. Set $W := \bigcup_{j \in J(U_x)} D_j$. Since $\mathcal{D}$ is a ck-network at x, there is a neighbourhood $V_x \subseteq U_x$ of x such that for each compact subset $K \subseteq V_x$ there is $j \in J(U_x)$ such that $K \subseteq D_j \subseteq W \subseteq U_x$. So $V_x \subseteq W$ and hence W is a neighbourhood of x.

Let K be a compact subset of V_x. Set $\alpha = \alpha(U_x)$. By the construction of W, there exists $i \in J(U_x)$ such that $K \subseteq D_i \subseteq W$. By the definition of $J(U_x)$, we have $i = n_k$ for some $k \in \mathbb{N}$. So, $x \in D_i = D_{n_k}$ and $\alpha_{n_k} = 1$ by (A_1). As

$$D_{n_k}(\alpha) = \bigcap_{\beta \in I_{n_k}(\alpha) \cap \mathbf{M}_x} A_\beta = \bigcap_{\beta \in I_{n_k}(\alpha) \cap \mathbf{M}_x} \left(\bigcup_{i \in \mathbb{N}} \bigcap_{l=1}^{\beta_i} D_{i-1+l} \right) \text{ (take } i = n_k)$$
$$\supseteq \bigcap_{\beta \in I_{n_k}(\alpha) \cap \mathbf{M}_x} \left(\bigcap_{l=1}^{\beta_{n_k}} D_{n_k-1+l} \right) \text{ (since } \beta_{n_k} = \alpha_{n_k} = 1) = D_{n_k},$$

we obtain that $K \subseteq D_i \subseteq D_{n_k}(\alpha) \subseteq W$. Thus, $\mathcal{D}_{\mathcal{U}(x)}$ is a countable *ck*-network at x.

The converse implication is clear.

(iii) Assume that X has a countable *cp*-network $\mathcal{D} = \{D_i\}_{i\in\mathbb{N}}$ at x. Without loss of generality, we may also assume that $\mathcal{D}$ is closed under taking finite unions. Similarly, as in (i), we prove that X has a small base $\{U_\alpha : \alpha \in \mathbf{M}_x\}$ at x satisfying the condition **D**. Observe that the countable family $\mathcal{D}_{\mathcal{U}(x)}$ is also a *cp*-network at x.

Let $A \subseteq X$ with $x \in \overline{A} \setminus A$ and let U_x be a neighbourhood of x. Set $\alpha = \alpha(U_x)$ and $W := \bigcup_{j\in J(U_x)} D_j$. Since $\mathcal{D}$ is also a *cn*-network, the set W is a neighbourhood of x. By the construction of W and the definition of *cp*-network, we note that there exists $i \in J(U_x)$ such that $x \in D_i \subseteq W$ and $D_i \cap A$ is infinite.

Applying the definition of $J(U_x)$, we deduce that $i = n_k \in J(U_x)$ for some $k \in \mathbb{N}$. Hence, $D_i = D_{n_k}$. In item (ii), we showed that $D_i \subseteq D_{n_k}(\alpha)$.

Consequently, $A \cap D_{n_k}(\alpha)$ is infinite. This shows that $\mathcal{D}_{\mathcal{U}(x)}$ is a countable *cp*-network at x.

The converse implication is obvious. □

Clearly, the validity of the condition **D** depends on the chosen family $\mathbf{M}_x$. The illustrating example might be the following one:

Example 18.4.2 Consider ℓ^1 with a small base at $\mathbf{0}$ which we define as follows: set $\mathbf{M}_0 := \mathbb{N}^{\mathbb{N}} \cap \ell^\infty$. For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbf{M}_0$, set

$$U_\alpha = \left\{ (x_i)_{i\in\mathbb{N}} \in \ell^1 : \sum_i \alpha_i |x_i| < 1 \right\}.$$

Clearly, $\{U_\alpha : \alpha \in \mathbf{M}_0\}$ is a small base in ℓ^1. Then, for each $\alpha = (\alpha_i) \in \mathbf{M}_0$ and every $k \in \mathbb{N}$, we have

$$D_k(\alpha) = \{(x_i) \in \ell^1 : \alpha_1|x_1| + \cdots + \alpha_k|x_k| < 1 \text{ and } 0 = x_{k+1} = x_{k+2} = \ldots\}.$$

Then, $\bigcup_k D_k(\alpha) \neq U_\alpha$. This shows the failure of **D**.

In order to prove the first application of the above theorem, we recall the following fact due to Banakh and Leiderman [54].

Lemma 18.4.3 *Each countable cp-network at a point $x \in X$ in a topological space X is a ck-network at x.*

We are ready to prove the following metrization theorem due to Gabriyelyan and Kąkol [261].

Theorem 18.4.4 *Let G be a Baire topological group. Then, the following are equivalent:*

(i) *G is metrizable.*
(ii) *G has the strong Pytkeev property.*
(iii) *G has countable ck-character.*
(iv) *G has countable cn-character.*
(v) *G has a $\mathfrak{G}$-base satisfying the condition* **D**.

Since a regular topological space is cosmic if and only if it has a countable *cn*-network, we note the following:

Corollary 18.4.5 *A Baire separable topological group G is metrizable if and only if G is cosmic.*

Proof of Theorem 18.4.4 The implications (i)⇒(ii) and (iii)⇒(iv) are obvious. (ii)⇒(iii) follows from Lemma 18.4.3. The implication (v)⇒(iv) follows from Theorem 18.4.1(i).

(i)⇒(v): Let $\{V_n\}_{n\in\mathbb{N}}$ be a decreasing base of neighbourhoods at the unit e of G. Then, family $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, where $U_\alpha := V_{\alpha_1}$ for $\alpha = (\alpha_i) \in \mathbb{N}^\mathbb{N}$, is a $\mathfrak{G}$-base with the condition **D**.

(iv)⇒(i): Let G have a countable *cn*-character. We need to prove that G is metrizable. We prove that G has a countable base of neighbourhoods of the unit e. By Theorem 18.4.1(i), there exists a small local base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ at e with **D**. We claim that the countable family

$$\{\overline{D_k(\alpha)} \cdot \overline{D_k(\alpha)}^{\,-1} : \alpha \in \mathbf{M}, k \in \mathbb{N}\}$$

contains a basis of neighbourhoods of e in G.

Indeed, let W be an open neighbourhood of e. Choose a symmetric open neighbourhood V of e such that $V \cdot V \subseteq \overline{V} \cdot \overline{V} \subseteq W$. Then, choose $\alpha \in \mathbf{M}$ with $U_\alpha = \bigcup_k D_k(\alpha) \subseteq V$. Since $\mathrm{Int}(U_\alpha)$ is open in G and G is Baire, there exists $k \in \mathbb{N}$ such that $\mathrm{Int}(U_\alpha) \cap \overline{D_k(\alpha)}$ has a non-empty interior in U_α, so also in G.

We proved that $\overline{D_k(\alpha)} \cdot \overline{D_k(\alpha)}^{\,-1}$ is a neighbourhood of e which is contained in W. □

Being motivated by above results, one can ask if for a topological group G with the the strong Pytkeev property and having a $\mathfrak{G}$-base $\mathcal{U}$ the base $\mathcal{U}$ satisfies the condition **D**. This question has been answered in [57, Theorem 6.4.4] in the affirmative.

For $\alpha \in \mathbb{N}^\mathbb{N}$, we denote $\alpha|n = (\alpha(0), \alpha(1), \ldots, \alpha(n-1))$. Recall also that $\uparrow \beta$ denotes the set $\{\alpha \in \mathbb{N}^\mathbb{N} : \alpha|n = \beta\}$. For a finite sequence $\beta \in \omega^{<\omega}$, we denote by $U_{\uparrow\beta}$ the intersection $\bigcap_{\alpha\in\uparrow\beta} U_\alpha$.

Theorem 18.4.6 (Banakh) *For a topological space X and a point $x \in X$, the following assertions are equivalent:*

(i) X has a $\mathfrak{G}$-base at x satisfying the condition **D**.
(ii) X has countable tightness at x and X has a $\mathfrak{G}$-base at the point x.

Proof (i) $\Rightarrow$ (ii): Assume that a $\mathfrak{G}$-base $\mathcal{U}$ at x satisfies the condition **D**. We prove that X has countable tightness at the point x. Take a subset $A \subset X$ with $x \in \overline{A}$. Consider the set $\Gamma = \{\beta \in \omega^{<\omega} : A \cap U_\beta \neq \emptyset\}$. For every $\beta \in \Gamma$, fix arbitrary $x_\beta \in A \cap U_\beta$. We show that $B = \{x_\beta : \beta \in \Gamma\} \subset A$ contains the point x in its closure.

Assume the contrary. Then, there exists a neighbourhood U_α of x such that $B \cap U_\alpha = \emptyset$. Applying the condition **D**, we derive that $U_\alpha = \bigcup_k U_{\alpha|k}$. Since $x \in \overline{A}$, the set intersects the set U_α, so hence $A \cap U_{\alpha|k} \neq \emptyset$ for some $k \in \mathbb{N}$. Then, $\alpha|k \in \Gamma$. Hence, $x_{\alpha|k} \in B \cap U_{\alpha|k} \subset B \cap U_\alpha$, a contradiction.

(ii) $\Rightarrow$ (i): Assume that X has countable tightness at $x \in X$ and X has a $\mathfrak{G}$-base $\mathcal{U}$ at x. For every $\beta \in \omega^{<\omega}$, consider $U_{\uparrow\beta} = \bigcap_{\alpha \in \uparrow\beta} U_\alpha$. Take arbitrary $\alpha \in \mathbb{N}^{\mathbb{N}}$. By countable tightness of X and [57, Theorem 6.4.1], we note that for every $\alpha \in \mathbb{N}^{\mathbb{N}}$ the union $V_\alpha = \bigcup_k U_{\alpha|k}$ is a neighbourhood of the point x. By the definition of the set V_α, we deduce that $V_\beta \subset V_\alpha$ whenever $\alpha \leq \beta$ from $\mathbb{N}^{\mathbb{N}}$.

We complete the proof by showing that the $\mathfrak{G}$-base $\{V_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ satisfies the condition **D**. Take $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$. Note that

$$V_{\alpha|k} = \bigcap_{\beta \in \uparrow(\alpha|k)} V_\beta = \bigcap_{\beta \in \uparrow(\alpha|k)} \bigcup_{n \geq k} U_{\beta|n} \supset U_{\alpha|k}.$$

Hence

$$V_\alpha = \bigcup_k U_{\alpha|k} \subset \bigcup_k V_{\alpha|k} \subset V_\alpha.$$

This shows that $V_\alpha = \bigcup_k V_{\alpha|k}$ and the $\{V_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ satisfies the condition **D**. □

Recall here that a quasibarrelled lcs with a $\mathfrak{G}$-base admits a $\mathfrak{G}$-base satisfying property **D**; see Lemma 15.1.2.

18.5 Spaces $C_c(X)$ with the Strong Pytkeev Property

In this section, we characterize those spaces $C_c(X)$ which have the strong Pytkeev property. Recall also the following theorem due to Tsaban and Zdomskyy [605].

Theorem 18.5.1 (Tsaban–Zdomskyy) *For each Polish space X, the space $C_c(X)$ has the strong Pytkeev property.*

This result will be extended by showing the following theorem from [266]. Recall here that a subset A of a topological space X is called *topologically bounded* if $f(A)$ is a bounded subset of $\mathbb{R}$ for every $f \in C(X)$. The space X is called a *μ-space* if every topologically bounded subset of X has compact closure. It is well known that every Lindelöf space is a μ-space. In particular, any Polish or hemicompact space is a μ-space.

Theorem 18.5.2 (Gabriyelyan–Kąkol–Leiderman) *Let X be a Tychonoff space with a fundamental compact resolution. Then, the following assertions are equivalent:*

(i) $C_c(X)$ *has the strong Pytkeev property.*
(ii) $C_c(X)$ *has the Pytkeev property.*
(iii) $C_c(X)$ *has countable tightness.*
(iv) $C_p(X)$ *has countable tightness.*
(v) $C_c(X)$ *is barrelled.*
(vi) X *is Lindelöf.*
(vii) X *is a μ-space.*

Proof Since X has a fundamental compact resolution, the space $C_c(X)$ has a $\mathfrak{G}$-base by Theorem 16.5.1.
(i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious.
(iii) $\Rightarrow$ (iv) follows from Lemma 16.4.1.
(iv) $\Rightarrow$ (vi) follows from [32, II.1.1].
(vi) $\Rightarrow$ (vii) is obvious.
(vii) $\Rightarrow$ (v) follows from Theorem 2.4.16.
(v) $\Rightarrow$ (i) follows from Lemma 15.1.2. □

Theorem 18.5.2 applies, for example, to all Čech-complete Lindelöf spaces X. In fact, every Čech-complete Lindelöf space X has a fundamental compact resolution. This easily follows from the well-known fact stating that X is a Čech-complete Lindelöf space if and only if it is a preimage of a Polish space under a perfect surjective map; see [289, Corollary 3.7].

Corollary 18.5.3 *Let X be a Čech-complete space. Then, $C_c(X)$ has the strong Pytkeev property if and only if X is Lindelöf.*

Corollary 18.5.4 *Let X be a locally compact space. Then, $C_c(X)$ has the strong Pytkeev property if and only if X is hemicompact.*

Remark 18.5.5 In [269], we studied various sequential properties of a Banach space E and its closed unit ball B, both endowed with the weak topology of E. We proved, among the others, that B has the Pytkeev property if and only if E in the norm topology contains no isomorphic copy of ℓ_1, while E has the Pytkeev property in the weak topology if and only if it is finite-dimensional. We extended in [269] Schlüchtermann and Wheeler's result from [547] by showing that B is a (separable) metrizable space if and only if it has countable cs^*-character and is a k-space. Hence, B is Polish if and only if it has countable cs^*-character and is Čech-complete. This supplements a result of Edgar and Wheeler from [194].

Chapter 19
The Grothendieck Property for $C(K)$-Spaces

Abstract We study Banach $C(K)$-spaces which are Grothendieck. Recall that a Banach space E is a Grothendieck space (or has the Grothendieck property) if every weak* convergent sequence in the dual space E' is weakly convergent. Two characterizations of Grothendieck spaces $C(K)$ will be provided: in terms of operators onto the Banach space c_0 and in terms of sequences of Radon measures related to the classical Josefson–Nissenzweig theorem from Banach space theory. We also present in detail a proof of Grothendieck's theorem asserting that for every extremely disconnected compact space K the space $C(K)$ is Grothendieck—for this purpose we present proofs of Rosenthal's "disjointification" lemma for sequences of measures and Dieudonné–Grothendieck's characterization of weakly compact subsets of dual spaces $C(K)'$.

19.1 Preliminaries: Spaces $\mathrm{ca}(\Sigma)$ and $C(K)'$

We first recall some standard notations and facts concerning Banach spaces of measures.

Let Ω be a set and Σ a σ-field of subsets of Ω. For every $A \in \Sigma$, we put $A^c = \Omega \setminus A$ and by $\chi_A \colon \Omega \to \{0, 1\}$, we denote *the characteristic function* of A (in Ω).

We say that a function $\mu \colon \Sigma \to \mathbb{R}$ is *a measure on* Σ if μ is finitely additive and *bounded* (or *finite*), that is, *the total variation* $\|\mu\|$ of μ, defined as

$$\|\mu\| = \sup\{|\mu(A)| + |\mu(B)| \colon \ A, B \in \Sigma, A \cap B = \emptyset\}, \tag{$\dagger$}$$

is finite (i.e. $\|\mu\| < \infty$). Note that we assume that μ is *signed*, that is, it may have negative values, but values $\pm\infty$ are not allowed. A measure μ such that $\mu(A) \geq 0$ for every $A \in \Sigma$ is called *non-negative*. A non-negative measure μ such that $\mu(\Omega) = 1$ is called *a probability measure*. The space of all σ-additive measures on Σ is denoted by $\mathrm{ca}(\Sigma)$ and endowed with the above-defined norm $\|\cdot\|$, called *the total variation norm*, which makes it a Banach space.

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For a measure $\mu \in \mathrm{ca}(\Sigma)$, we define its *variation* $|\mu|\colon \Sigma \to [0, \infty)$ as follows:

$$|\mu|(C) = \sup\{|\mu(A)| + |\mu(B)|\colon\ A, B \in \Sigma, A \cap B = \emptyset, A \cup B \subseteq C\},$$

for every $C \in \Sigma$. Of course, $\|\mu\| = |\mu|(\Omega)$. Recall that, by the standard Hahn and Jordan decomposition theorems, there are disjoint sets $P, N \in \Sigma$ with $P \cup N = \Omega$ and non-negative measures μ^-, μ^+ on Σ such that $\mu = \mu^+ - \mu^-$, $\mu^+(A) = \mu(A \cap P)$, and $\mu^-(A) = -\mu(A \cap N)$ for every $A \in \Sigma$. It follows that $|\mu| = \mu^+ + \mu^-$.

Let $\mu \in \mathrm{ca}(\Sigma)$ be a non-negative measure on a σ-field of subsets of a set Ω. By $L_1(\Omega, \Sigma, \mu)$, we mean as usual the Banach space of all μ-integrable real-valued functions on Ω, endowed with the standard $\|\cdot\|_1$-norm. The dual space $L_1(\Omega, \Sigma, \mu)'$ is isometrically isomorphic to the Banach space $L_\infty(\Omega, \Sigma, \mu)$ of μ-essentially bounded Σ-measurable real-valued functions on Ω, endowed with the standard $\|\cdot\|_\infty$-norm. We will often simply write $L_1(\mu)$ and $L_\infty(\mu)$ when the underlying measure space is clear. For $\nu \in \mathrm{ca}(\Sigma)$, recall that ν is *absolutely continuous* with respect to μ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every set $A \in \Sigma$ we have $|\nu(A)| < \varepsilon$ provided that $\mu(A) < \delta$. The classical Radon–Nikodym theorem states that in this case there exists a function $f \in L_1(\mu)$, called *a Radon–Nikodym derivative* of ν with respect to μ and often denoted as $d\nu/d\mu$, such that $\|\nu\|_{\mathrm{ca}(\Sigma)} = \|f\|_{L_1(\mu)}$ and $\nu(A) = \int_A f\,d\mu$ for every $A \in \Sigma$. The function f is defined up to a μ-measure zero set.

For a compact Hausdorff[1] space K by $Bor(K)$, we denote the σ-field of all Borel subsets of K. A function $\mu\colon Bor(K) \to \mathbb{R}$ is called *a measure on* K if $\mu \in \mathrm{ca}(Bor(K))$ and is *regular*, that is, its variation $|\mu|$ is *inner regular* (with respect to compact sets) and *outer regular* (with respect to open sets), i.e. for every Borel set $A \subseteq K$, it holds

$$|\mu|(A) = \sup\{|\mu|(F)\colon\ F \subseteq A,\ F \textit{ is a compact subset of } K\} =$$

$$= \inf\{|\mu|(U)\colon\ A \subseteq U,\ U \textit{ is an open subset of } K\}.$$

Note that every measure μ on K is *locally finite*, that is, for every $x \in K$, there is an open neighbourhood U of x such that $|\mu|(U) < \infty$ (e.g. $U = K$).[2] The space of all measures on K will be denoted by $M(K)$ and endowed with the total variation norm (see (†)), which makes $M(K)$ a Banach space.

For a compact space K by $C(K)$, we denote the Banach space of all continuous real-valued functions on K endowed with the supremum norm. Recall that by the classical Riesz–Markov–Kakutani representation theorem the dual Banach space $C(K)'$ is isometrically isomorphic to the space $M(K)$, that is, for every continuous

[1] We will omit the word "Hausdorff" in the sequel, assuming that all considered topological spaces are Hausdorff.

[2] Locally finite regular measures on topological spaces are usually called *Radon*. Since in this chapter, by definition, we deal only with Radon measures, we will also omit the word "Radon".

functional $\varphi \in C(K)'$ there is a unique measure $\mu \in M(K)$ with $\|\varphi\| = \|\mu\|$ and such that $\varphi(f) = \int_K f d\mu$ for every $f \in C(K)$. Thus, we will often identify both spaces and simply say that $C(K)'$ *is* the space of measures on K as well as for every $\mu \in C(K)'$ and $f \in C(K)$ we will write $\mu(f) = \int_K f d\mu$.

Let K be a compact space. For every $\mu \in M(K)$, there is the smallest closed subset of K, denoted by supp(μ) and called *the support* of μ, such that $|\mu|(U) = 0$ for every open set $U \subseteq K \setminus \text{supp}(\mu)$. For $A \in Bor(K)$ and $\mu \in M(K)$, we define *the restriction* $\mu \restriction A$ of μ to A by $(\mu \restriction A)(B) = \mu(A \cap B)$ for every $B \in Bor(K)$. If $x \in K$, then by δ_x, we denote the measure, called *the point measure* or *Dirac delta* at x, defined for every $A \in Bor(K)$ as $\delta_x(A) = \chi_A(x)$.

We will frequently use the following standard facts.

Lemma 19.1.1 *Let K be a totally disconnected compact space and $\mu \in C(K)'$. Set*

$$\|\mu\|' = \sup\{|\mu(A)| + |\mu(B)| : \ A, B \text{ are disjoint clopen subsets of } K\}.$$

Then, $\|\mu\| = \|\mu\|'$.

Proof Since clopen subsets of K are Borel, $\|\mu\| \geq \|\mu\|'$. Consequently, if $\|\mu\| = 0$, we are done. Assume thus that $\|\mu\| > 0$ and fix $\varepsilon > 0$. We will show that $\|\mu\|' \geq \|\mu\| - \varepsilon$, which will prove that $\|\mu\|' \geq \|\mu\|$ and hence finish the proof. Note that μ is a Borel measure on K, so by the Hahn decomposition theorem there exist disjoint Borel subsets P and N of K such that $P \cup N = K$ and for every $E \in Bor(K)$ we have $\mu(E \cap P) \geq 0$ and $\mu(E \cap N) \leq 0$. By the regularity of μ, there exist compact sets $F \subseteq P$ and $G \subseteq N$ such that

$$\mu(F) > \mu(P) - \varepsilon/4$$

and

$$|\mu|(G) > |\mu|(N) - \varepsilon/4.$$

Again, by the regularity of μ and the fact that K is compact and totally disconnected (hence zero-dimensional), there are disjoint clopen sets $U \supseteq F$ and $V \supseteq G$ such that

$$|\mu|(U) < \mu(F) + \varepsilon/4$$

and

$$|\mu|(V) < |\mu|(G) + \varepsilon/4.$$

Since also

$$|\mu|(U) = \mu(U \cap P) + |\mu|(U \cap N) \geq \mu(F) + |\mu|(U \cap N)$$

and

$$|\mu|(V) = \mu(V \cap P) + |\mu|(V \cap N) \geq \mu(V \cap P) + |\mu|(G),$$

we get that $|\mu|(U \cap N) < \varepsilon/4$ and $\mu(V \cap P) < \varepsilon/4$. We have

$$|\mu(U)| = |\mu(F) + \mu((P \setminus F) \cap U) + \mu(U \cap N)| \geq \tag{$*$}$$

$$\geq |\mu(F) + \mu((P \setminus F) \cap U)| - |\mu(U \cap N)| \geq \mu(F) - |\mu(U \cap N)| \geq$$

$$\geq \mu(F) - |\mu|(U \cap N) > \mu(F) - \varepsilon/4 > \mu(P) - \varepsilon/2.$$

By a similar computation, we get that

$$|\mu(V)| > |\mu(N)| - \varepsilon/2. \tag{$**$}$$

Since $\|\mu\| = \mu(P) - \mu(N) = \mu(P) + |\mu(N)|$, combining $(*)$ and $(**)$, we get

$$\|\mu\|' \geq |\mu(U)| + |\mu(V)| \geq \mu(P) - \varepsilon/2 + |\mu(N)| - \varepsilon/2 = \|\mu\| - \varepsilon.$$

□

Lemma 19.1.2 *Let K be a totally disconnected compact space. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures in $C(K)'$ and $\mu \in C(K)'$. Then, the following are equivalent:*

(1) $(\mu_n)_{n\in\mathbb{N}}$ is bounded and $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for every clopen set $A \subseteq K$.
(2) $(\mu_n)_{n\in\mathbb{N}}$ converges to μ in the weak topology of $C(K)'$.*

Proof Assume (1). Fix $f \in C(K)$ and let $\varepsilon > 0$. By the Stone–Weierstrass theorem, the space $\operatorname{span}\{\chi_A : A \text{ is a clopen subset of } K\}$ is a dense linear subspace of $C(K)$, so there exists a finite sequence $A_1, \ldots, A_k$ of pairwise disjoint clopen subsets of K and a finite sequence $\alpha_1, \ldots, \alpha_k$ of real numbers such that

$$\|f - \sum_{i=1}^{k} \alpha_i \chi_{A_i}\|_\infty < \varepsilon/(3M), \tag{$*$}$$

where $M = \sup_{n\in\mathbb{N}} \|\mu_n\| + 1$. Lemma 19.1.1 and (1) yield $\|\mu\| \leq M$. Also by (1), there is $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$\sum_{i=1}^{k} |\alpha_i| \cdot |\mu_n(A_i) - \mu(A_i)| < \varepsilon/3. \tag{$**$}$$

$(*)$ and $(**)$ yield for every $n \geq N$

$$|\mu_n(f)-\mu(f)| \le |\mu_n(f)-\mu_n\Big(\sum_{i=1}^{k}\alpha_i\chi_{A_i}\Big)|+$$

$$+|\mu_n\Big(\sum_{i=1}^{k}\alpha_i\chi_{A_i}\Big)-\mu\Big(\sum_{i=1}^{k}\alpha_i\chi_{A_i}\Big)|+|\mu\Big(\sum_{i=1}^{k}\alpha_i\chi_{A_i}\Big)-\mu(f)| \le$$

$$\le \|f-\sum_{i=1}^{k}\alpha_i\chi_{A_i}\|_\infty\cdot\|\mu_n\|+\sum_{i=1}^{k}|\alpha_i|\cdot|\mu_n(A_i)-\mu(A_i)|+$$

$$+\|f-\sum_{i=1}^{k}\alpha_i\chi_{A_i}\|_\infty\cdot\|\mu\|<$$

$$<(\varepsilon/(3M))\cdot M+\varepsilon/3+(\varepsilon/(3M))\cdot M=\varepsilon.$$

It follows that $\lim_{n\to\infty}\mu_n(f)=\mu(f)$. Since f was arbitrary, $(\mu_n)_{n\in\mathbb{N}}$ converges weak* to μ.

The implication (2)⇒(1) is trivial (boundedness of $(\mu_n)_{n\in\mathbb{N}}$ follows from the standard uniform boundedness principle). □

The following trivial lemma will be used frequently without any explicit reference.

Lemma 19.1.3 *Let E be a Banach space. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in E which is not weakly convergent to 0, then there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that its every subsequence $(x_{n_{k_l}})_{l\in\mathbb{N}}$ is not weakly convergent to 0.*

Proof Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E which is not weakly convergent to 0, so there are $\varphi\in E'$, $\varepsilon>0$, and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $|\varphi(x_{n_k})|>\varepsilon$ for every $k\in\mathbb{N}$. It follows that for each subsequence $(x_{n_{k_l}})_{l\in\mathbb{N}}$ we have $|\varphi(x_{n_{k_l}})|>\varepsilon$ for every $l\in\mathbb{N}$. Consequently, no subsequence $(x_{n_{k_l}})_{l\in\mathbb{N}}$ is weakly convergent to 0. □

19.2 Selected Basic Facts on Grothendieck Spaces

A Banach space E is *a Grothendieck space* (or has *the Grothendieck property*) if every weak* convergent sequence in the dual space E' is weakly convergent. Grothendieck spaces constitute an important class of Banach spaces and have been intensively studied. This class contains e.g. the following spaces:

- reflexive spaces (since the weak and weak* topologies coincide in their dual spaces),
- spaces of the form $\ell_\infty(\Gamma)$ or, more generally, spaces $C(K)$ for extremely disconnected compact spaces K (Grothendieck [286]),
- indecomposable $C(K)$-spaces (Cembranos [133]),

- injective Banach spaces (see [3, Proposition 4.3.8]),
- spaces $L_\infty(\Omega, \Sigma, \mu)$ for σ-finite measure spaces (Ω, Σ, μ) (see [150, Corollary 4.5.10]) or, more generally, von Neumann algebras (Pfitzner [493]),
- the space H^∞ of bounded analytic functions on the open unit disc (Bourgain [100]),
- the Baire class $B_\alpha(X)$ for any topological space X and ordinal number $1 \leq \alpha \leq \omega_1$ (Dashiell [151]).

We will study the case of $C(K)$-spaces for extremely disconnected compact spaces K in Sect. 19.4.

To the class of Banach spaces which do not have the Grothendieck property belong such spaces as e.g.:

- spaces of the form $\ell_1(\Gamma)$ for infinite sets Γ,
- spaces $C(K)$ for infinite compact spaces K carrying only measures of countable Maharam type[3] (Krupski and Plebanek [376]),
- spaces of the form $c_0(\Gamma)$ for infinite sets Γ or, more generally, infinite-dimensional Banach spaces E whose dual spaces E' have *the Schur property*, that is, such that weakly convergent sequences in E' are norm convergent (since, by Josefson–Nissenzweig's theorem, each infinite-dimensional Banach space admits a sequence of functionals in the dual unit sphere which is weak* convergent to 0, cf. Sect. 19.3),
- spaces isomorphic to $C(K)$-spaces for infinite compact spaces K containing a non-trivial convergent sequence (if a compact space K contains a non-trivial sequence $(x_n)_{n\in\mathbb{N}}$ convergent to some $x \in K$, then the sequence $(\mu_n)_{n\in\mathbb{N}}$ of functionals in $C(K)'$, given for each $n \in \mathbb{N}$ by the formula $\mu_n = \delta_{x_n} - \delta_x$, is weak* convergent to 0 but not weakly convergent), in particular, to $C(K)$-spaces for K metric, Rosenthal, Eberlein, Corson, etc.,
- spaces of the form $C(K \times L)$ for any two infinite compact spaces K and L (Cembranos [133]; see also Freniche [254]).

We will elaborate more on the latter class of non-examples in Chap. 20.

It is easy to see that if E is a Grothendieck Banach space, F is a Banach space, and $T\colon E \to F$ is a bounded surjective linear operator, then F is also Grothendieck. In particular, closed linear images, quotients, and complemented subspaces of Grothendieck spaces are again Grothendieck. Also, if E and F are Grothendieck spaces, then so is their direct sum $E \oplus F$. On the other hand, an infinite ℓ_∞-sum of a Grothendieck space (even of a reflexive one) need not be Grothendieck; see [512, Page 56, Beispiel 1] for a suitable example.

The following theorem due to Diestel [167, Theorem 1], which we provide without a proof, characterizes Grothendieck Banach spaces in terms of weakly compact operators.

[3] See Chap. 13 for the definition of the Maharam type of a measure.

Theorem 19.2.1 (Diestel) *For every Banach space E, the following are equivalent:*

(1) E is a Grothendieck space.
(2) For any weakly compactly generated Banach space F, every bounded linear operator $T : E \to F$ is weakly compact.
(3) For any separable Banach space F, every bounded linear operator $T : E \to F$ is weakly compact.
(4) Every bounded linear operator $T : E \to c_0$ is weakly compact.
(5) For any Banach space F, any sequence $(T_n)_{n\in\mathbb{N}}$ of weakly compact linear operators from E to F, and any bounded operator $T : E \to F$ such that $T(x)$ is the weak limit of $(T_n(x))_{n\in\mathbb{N}}$ for each $x \in E$, T is weakly compact.
(6) For any Banach space F, any sequence $(T_n)_{n\in\mathbb{N}}$ of weakly compact linear operators from E to F, and any bounded operator $T : E \to F$ such that $T(x)$ is the norm limit of $(T_n(x))_{n\in\mathbb{N}}$ for each $x \in E$, T is weakly compact.

Diestel's theorem immediately yields that the space c_0 does not have the Grothendieck property (which can be also easily seen by considering the canonical basic sequence $(e_n)_{n\in\mathbb{N}}$ of unit vectors in the dual space ℓ_1 of c_0). It follows that any Banach space admitting c_0 as a quotient cannot be Grothendieck as well.

For every Banach space E, its dual space E' is *weak* sequentially complete*, that is, every weak* Cauchy sequence in E' has a weak* limit. It is immediate that for Grothendieck spaces even more is true: the dual E' of a Grothendieck space E is *weakly sequentially complete*, that is, every weak Cauchy sequence in E' has a weak limit. It appears that the weak sequential completeness of the dual space together with the lack of a quotient isomorphic to c_0 actually characterizes Grothendieck spaces.

Lemma 19.2.2 *For every Banach space E, the following are equivalent:*

(1) E has a quotient isomorphic to c_0.
(2) E' contains a sequence which is weak convergent to 0 and equivalent to the canonical basis of ℓ_1.*

Proof Let $(e_n)_{n\in\mathbb{N}}$ be the canonical basis of ℓ_1, i.e. for every $n, k \in \mathbb{N}$, we have $e_n(k) \in \{0, 1\}$, and $e_n(k) = 1$ if and only if $n = k$.

Assume that $T : E \to c_0$ is a bounded linear surjection. Then, the adjoint operator $T^* : \ell_1 \to E'$ is an isomorphism onto its closed image $T^*(\ell_1)$ (by [641, Theorems 11-3.3 and 11-3.4.(b)]). The sequence $(x_n^*)_{n\in\mathbb{N}}$ in E', given by the formula $x_n^* = T^*(e_n)$, is weak* convergent to 0 and equivalent to $(e_n)_{n\in\mathbb{N}}$. Thus, (1) implies (2).

Let $(x_n^*)_{n\in\mathbb{N}}$ be a sequence in E' which is weak* convergent to 0 and equivalent to the canonical basis of ℓ_1. Define the operator $T : E \to c_0$ by setting $T(x) = (x_n^*(x))_{n\in\mathbb{N}}$. By the properties of $(x_n^*)_{n\in\mathbb{N}}$, the adjoint operator $T^* : \ell_1 \to E'$ is an isomorphism onto the closed image $T^*(\ell_1)$ such that $T^*(e_n) = x_n^*$ for every $n \in \mathbb{N}$. By [641, Theorem 11-3.4.(b)], it follows that T is onto, so (2) yields (1). □

Theorem 19.2.3 (Räbiger [512, Theorem 2.2]) *For every Banach space E, the following are equivalent:*

(1) E is a Grothendieck space.
(2) E has no quotients isomorphic to c_0 and the dual space E' is weakly sequentially complete.

Proof Implication (1)⇒(2) was justified before Lemma 19.2.2.

Assume (2) and suppose that E is not a Grothendieck space. Then, there exists a sequence $(x_n^*)_{n\in\mathbb{N}}$ in the dual space E' which is weak* convergent to 0, but its every subsequence $(x_{n_k}^*)_{k\in\mathbb{N}}$ is not weakly convergent to 0. As E' is weakly sequentially complete, this means that no subsequence $(x_{n_k}^*)_{k\in\mathbb{N}}$ is weakly Cauchy. Since by the uniform boundedness principle $(x_n^*)_{n\in\mathbb{N}}$ is norm bounded, we may appeal to Rosenthal's classical ℓ_1-lemma and conclude that there exists a subsequence $(x_{n_k}^*)_{k\in\mathbb{N}}$ which is equivalent to the canonical basis of ℓ_1. By Lemma 19.2.2, E admits a quotient isomorphic to c_0, which is a contradiction, so (1) holds, too. □

It turns out that in the case of $C(K)$-spaces Räbiger's characterization can be simplified. Namely, it appears that studying the Grothendieck property for $C(K)$-spaces we may only focus on the issue of the existence of quotients isomorphic to c_0, as for every compact space K the dual space $C(K)'$ is always weakly sequentially complete. The latter statement might be proved in several ways, e.g. by appealing to Pełczyński's properties (V) and (V*) (see [488]), we will however proceed in a more direct way using the following two results, essentially due to Nikodym ([467, 468]; cf. also Andô [9] and Valdivia [622]).

Theorem 19.2.4 *Let Σ be a σ-field of subsets of a given set Ω. Then, the following two theorems hold:*

(1) **The Nikodym uniform boundedness theorem**. *For every sequence $(\mu_n)_{n\in\mathbb{N}}$ of finitely additive measures on Σ, if*

$$\sup_{n\in\mathbb{N}} |\mu_n(A)| < \infty$$

for every $A \in \Sigma$, then

$$\sup_{n\in\mathbb{N}} \|\mu_n\| < \infty.$$

(2) **The Nikodym convergence theorem**. *For every sequence $(\mu_n)_{n\in\mathbb{N}}$ of σ-additive measures on Σ, if for every $A \in \Sigma$ the limit $\lim_{n\to\infty} \mu_n(A)$ exists and is equal to some real number $\mu(A)$, then the function $A \mapsto \mu(A)$ is a σ-additive measure on Σ and $\mu \in \mathrm{ca}(\Sigma)$.*

The first of Nikodym's theorems, the uniform boundedness theorem, and its proof will be detailedly discussed in Chap. 21. For the proof of the second result, we refer the reader to [170, page 90].

Proposition 19.2.5 *Let Ω be a set, Σ a σ-field of subsets of Ω, $(\mu_n)_{n\in\mathbb{N}}$ a sequence in $\mathrm{ca}(\Sigma)$, and $\mu \in \mathrm{ca}(\Sigma)$. Then, $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ if and only if for every $A \in \Sigma$ we have*

$$\lim_{n\to\infty} \mu_n(A) = \mu(A).$$

Proof The "only if" part is obvious, so let us assume that $\mu_n(A) \to \mu(A)$ for every set $A \in \Sigma$. By the Nikodym uniform boundedness principle (Theorem 19.2.4.(1)), we get that $\sup_{n\in\mathbb{N}} \|\mu_n\| < \infty$, so the series

$$\lambda = \sum_{n\in\mathbb{N}} |\mu_n|/2^n$$

is absolutely convergent in $\mathrm{ca}(\Sigma)$ and hence $\lambda \in \mathrm{ca}(\Sigma)$. By the Radon–Nikodym theorem, for each $n \in \mathbb{N}$, there is $f_n \in L_1(\lambda)$ such that

$$\mu_n(A) = \int_A f_n d\lambda$$

for every $A \in \Sigma$, and $\|f_n\|_1 = \|\mu_n\|$. It follows that $\sup_{n\in\mathbb{N}} \|f_n\|_1 < \infty$. Similarly, there is $f \in L_1(\lambda)$ such that

$$\mu(A) = \int_A f d\lambda$$

for every $A \in \Sigma$, and $\|f\|_1 = \|\mu\|$.

As $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for every $A \in \Sigma$

$$\lim_{n\to\infty} \int_A f_n d\lambda = \int_A f d\lambda$$

for every $A \in \Sigma$, and hence

$$\lim_{n\to\infty} \int_\Omega f_n \cdot g d\lambda = \int_\Omega f \cdot g d\lambda$$

for every simple function $g \in L_\infty(\lambda)$. Since simple functions are dense in $L_\infty(\lambda)$ and $\sup_{n\in\mathbb{N}} \|f_n\|_1 < \infty$, by the standard $\varepsilon/3 + \varepsilon/3 + \varepsilon/3$ argument, we get that

$$\lim_{n\to\infty} \int_\Omega f_n \cdot g d\lambda = \int_\Omega f \cdot g d\lambda$$

for every function $g \in L_\infty(\lambda)$. But this simply means that f_n converges to f weakly in $L_1(\lambda)$, which implies that μ_n converges weakly to μ in $\mathrm{ca}(\Sigma)$ (as the mapping

assigning to measures in $\mathrm{ca}(\Sigma)$ which are absolutely continuous with respect to λ their Radon–Nikodym derivatives in $L_1(\lambda)$ is an isometry). □

Theorem 19.2.6 *For every σ-field Σ on a set Ω, the space $\mathrm{ca}(\Sigma)$ is weakly sequentially complete.*

Proof Let $(\mu_n)_{n\in\mathbb{N}}$ be a weakly Cauchy sequence of measures in $\mathrm{ca}(\Sigma)$. By the completeness of $\mathbb{R}$, for each $A \in \Sigma$, there is $\mu(A) \in \mathbb{R}$ such that $\mu(A) = \lim_{n\to\infty} \mu_n(A)$. By the Nikodym convergence theorem (Theorem 19.2.4.(2)), $\mu \in \mathrm{ca}(\Sigma)$. By Proposition 19.2.5, the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to μ weakly in $\mathrm{ca}(\Sigma)$. □

We immediately obtain the following important characterization of the weak convergence in a space $C(K)'$, from which the weak sequential completeness of the latter space of measures quickly follows.

Corollary 19.2.7 *Let K be a compact space, $(\mu_n)_{n\in\mathbb{N}}$ a sequence of measures in $C(K)'$, and $\mu \in C(K)'$. Then, $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ if and only if for every Borel set $B \subseteq K$ we have*

$$\lim_{n\to\infty} \mu_n(B) = \mu(B).$$

Proof Since the weak topology on $C(K)'$ coincides with the subspace topology inherited from the weak topology of $\mathrm{ca}(Bor(K))$, $\mu_n \to \mu$ weakly in $C(K)'$ if and only if $\mu_n \to \mu$ weakly in $\mathrm{ca}(Bor(K))$. The thesis thus follows from Proposition 19.2.5. □

Theorem 19.2.8 *For every compact space K, the dual space $C(K)'$ is weakly sequentially complete.*

Proof Note that $C(K)'$ is a closed subspace of $\mathrm{ca}(Bor(K))$ and recall that a closed subspace of a weakly sequentially complete Banach space is weakly sequentially complete. □

Räbiger's characterization of Grothendieck $C(K)$-spaces now shortly reads as follows.

Corollary 19.2.9 *For every compact space K, the following are equivalent:*

(1) $C(K)$ is a Grothendieck space.
(2) $C(K)$ has no quotients isomorphic to c_0.

If a Banach space has no quotients isomorphic to c_0, then in particular it does not contain any complemented copies of c_0. As we will show in Theorem 19.2.11, in the case of $C(K)$-spaces, the converse (and even more) also holds. Let us however first recall the following well-known fact, which will appear useful in the sequel.

Theorem 19.2.10 *Let K be an infinite compact space. Then, the Banach space $C(K)$ contains an isometric copy of c_0.*

Proof Since K is infinite, there is a sequence $(U_n)_{n\in\mathbb{N}}$ of pairwise disjoint non-empty open subsets of K. By the Urysohn lemma, for each $n \in \mathbb{N}$ there is a function $f_n \in C(K, [0, 1])$ such that $\operatorname{supp}(f_n) \subseteq U_n$ and $\|f_n\|_\infty = 1$. Then, for every sequence $(\alpha_n)_{n\in\mathbb{N}}$ in the normed space c_{00}, we have

$$\|\sum_{n\in\mathbb{N}} \alpha_n f_n\|_\infty = \max_{n\in\mathbb{N}} |\alpha_n|,$$

so $(f_n)_{n\in\mathbb{N}}$ is a sequence in $C(K)$ isometrically equivalent to the canonical basis of c_0. It follows that $\overline{\operatorname{span}}\{f_n : n \in \mathbb{N}\}$ is isometrically isomorphic to c_0 (recall that c_0 is the completion of c_{00} with respect to the supremum norm). □

Theorem 19.2.11 *For every compact space K, the following are equivalent:*

(1) $C(K)$ is a Grothendieck space.
(2) $C(K)$ has no quotients isomorphic to c_0.
(3) $C(K)$ does not contain any complemented subspaces isomorphic to c_0.
(4) $C(K)$ does not contain any complemented subspaces isometrically isomorphic to c_0.

Proof Implication (1)⇒(2) follows from Corollary 19.2.9 and implications (2)⇒(3) and (3)⇒(4) are immediate. To prove (4)⇒(1), we need to show that if $C(K)$ does not have the Grothendieck property, then it contains a complemented isometric copy of c_0. Assume thus that K carries a sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ which is weak* convergent to 0, but not weakly convergent. By Dieudonné–Grothendieck's theorem (Theorem 19.4.8), we might assume that there exist a sequence $(U_n)_{n\in\mathbb{N}}$ of pairwise disjoint open sets in K and $\varepsilon > 0$ such that $|\mu_n(U_n)| > \varepsilon$ for every $n \in \mathbb{N}$. Exchanging each μ_n for $2\mu_n/\mu_n(U_n)$, we might additionally assume that $\mu_n(U_n) = 2$ for every $n \in \mathbb{N}$.

By the regularity of measures $|\mu_n|(\cdot)$, for each $n \in \mathbb{N}$, there is a compact set $K_n \subseteq U_n$ such that $|\mu_n|(U_n \setminus K_n) < 1/2$. We have

$$2 = |\mu_n(U_n)| = |\mu_n(U_n \setminus K_n) + \mu_n(K_n)| \leq$$

$$\leq |\mu_n|(U_n \setminus K_n) + |\mu_n(K_n)| \leq 1/2 + |\mu_n(K_n)|,$$

so $|\mu_n(K_n)| \geq 3/2$. By the normality of K, we can find a function $f_n \in C(K, [0, 1])$ such that $\operatorname{supp}(f_n) \subseteq U_n$ and $f_n(x) = 1$ for every $x \in K_n$ (so $\|f_n\|_\infty = 1$). Set $Y = \overline{\operatorname{span}}\{f_n : n \in \mathbb{N}\}$. It follows that Y is a closed subspace of $C(K)$ isometrically isomorphic to c_0 (cf. the proof of Theorem 19.2.10).

Define the operator $T : C(K) \to c_0$ by setting $T(f) = (\mu_n(f))_{n\in\mathbb{N}}$ for each $f \in C(K)$. The Banach–Steinhaus theorem, applied to the family $\{\mu_n : n \in \mathbb{N}\}$ of bounded functionals on $C(K)$, implies that T is bounded. For every $n \in \mathbb{N}$, we have

$$\|T(f_n)\|_\infty \geq |\mu_n(f_n)| = |\int_{U_n} f_n d\mu_n| = |\int_{K_n} f_n d\mu_n + \int_{U_n\setminus K_n} f_n d\mu_n| =$$

$$= |\mu_n(K_n) + \int_{U_n \setminus K_n} f_n d\mu_n| \geq |\mu_n(K_n)| - |\int_{U_n \setminus K_n} f_n d\mu_n| \geq$$

$$\geq |\mu_n(K_n)| - \|f_n\|_\infty \cdot |\mu_n|(U_n \setminus K_n) > 3/2 - 1/2 = 1.$$

For the operator $T \restriction Y : Y \to c_0$, by the classical result of Rosenthal [524, Theorem 3.4 and Remark 1] (recall that Y is isometrically isomorphic to c_0), there is an infinite subset $M \subseteq \mathbb{N}$ such that for the space $Z = \overline{\operatorname{span}}\{f_n : n \in M\}$ the operator $T_0 = T \restriction Z$ is an isomorphism onto its image.

Note that Z is also an isometric copy of c_0. On the other hand, $T_0(Z)$ is a linear subspace of c_0, isomorphic to c_0 itself. By the classical Sobczyk theorem (see [321, Corollary 29.22]), there is a bounded linear projection $P : c_0 \to T_0(Z)$. The operator $S = T_0^{-1} \circ P \circ T$ is a bounded linear projection from $C(K)$ onto Z, yielding a contradiction. □

The argument for implication (4)⇒(1) in the proof of Theorem 19.2.11 comes essentially from Schachermayer [542, Proposition 5.3]. The equivalence of the Grothendieck property for $C(K)$-spaces and the lack of complemented copies of c_0 was proved by Cembranos [133].

Theorem 19.2.11 yields the following characterization of Grothendieck $C(K)$-spaces in terms of spaces $C(L)$ for L containing non-trivial convergent sequences (first observed in [343]). We use the following notation: $\alpha\mathbb{N}$ denotes the one-point compactification of the discrete space $\mathbb{N}$ of natural numbers, and $X \sqcup Y$ denotes the topological disjoint union of topological spaces X and Y. For two Banach spaces E and F, $E \simeq F$ means that E and F are isomorphic.

Corollary 19.2.12 *Let $\mathcal{C}$ be the class of all compact spaces containing non-trivial convergent sequences and $\mathcal{D} \subseteq \mathcal{C}$ its any subclass such that $\alpha\mathbb{N} \in \mathcal{D}$. For every compact space K, the following are equivalent:*

(1) $C(K)$ is not a Grothendieck space.
(2) $C(K) \simeq C(K \sqcup \alpha\mathbb{N})$.
(3) $C(K) \simeq C(K \sqcup L)$ for some $L \in \mathcal{D}$.
(4) $C(K) \simeq C(L)$ for some $L \in \mathcal{C}$.

Proof Assume that $C(K)$ is not a Grothendieck space. By Theorem 19.2.11, $C(K)$ contains a closed subspace H such that $C(K) \simeq H \oplus c_0$. Since $c_0 \simeq c_0 \oplus c_0$ and $c_0 \simeq C(\alpha\mathbb{N})$, we have

$$C(K) \simeq H \oplus c_0 \simeq H \oplus c_0 \oplus c_0 \simeq$$

$$\simeq C(K) \oplus c_0 \simeq C(K) \oplus C(\alpha\mathbb{N}) \simeq C(K \sqcup \alpha\mathbb{N}),$$

so (1) implies (2). Implications (2)⇒(3)⇒(4) are obvious. If (4) holds, that is, $C(K) \simeq C(L)$ for some $L \in \mathcal{C}$, then $C(L)$ is not a Grothendieck space; hence, $C(K)$ is not Grothendieck as well, meaning that implication (4)⇒(1) also holds. □

19.3 The Grothendieck Property for $C(K)$-Spaces and Josefson–Nissenzweig's Theorem

In the previous section, we studied Grothendieck $C(K)$-spaces from the point of view of the existence of mappings onto the Banach space c_0. In this section, we will investigate them more in the terms of sequences of measures (i.e. of functionals from the dual spaces $C(K)'$), especially those related to the classical Josefson–Nissenzweig theorem.

We start with the following classical lemma and its important consequence. Recall that a norm-bounded sequence $(f_n)_{n\in\mathbb{N}}$ of functions in a given space $L_1(X, \Sigma, \mu)$ is *equi-integrable* if

$$\lim_{\mu(E)\to 0} \sup_{n\in\mathbb{N}} \|f_n \restriction E\|_{L_1(\mu)} = 0,$$

i.e. for every $\varepsilon > 0$ there is $\delta > 0$ such that for every set $E \subseteq X$ with $\mu(E) < \delta$ we have

$$\sup_{n\in\mathbb{N}} \int_E |f_n| d\mu < \varepsilon.$$

By [3, Theorem 5.2.8], a norm-bounded sequence is equi-integrable if and only if it is relatively weakly compact (in $L_1(X, \Sigma, \mu)$).

Lemma 19.3.1 (Kadec–Pełczyński–Rosenthal's Subsequential Splitting Lemma) *Let (X, Σ, μ) be a measure space, where μ is a non-negative finite σ-additive measure on a σ-field Σ of subsets of a set X. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions in $L_1(\mu)$. Then, there are a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and a sequence $(A_k)_{k\in\mathbb{N}}$ of pairwise disjoint subsets in Σ such that the sequence $(f_{n_k} \cdot \chi_{A_k^c})_{k\in\mathbb{N}}$ is equi-integrable.*

Proof See [3, Lemma 5.2.7]. □

Proposition 19.3.2 *Let $(\nu_n)_{n\in\mathbb{N}}$ be a bounded sequence of measures on a compact space K. Then, there exist a subsequence $(\nu_{n_k})_{k\in\mathbb{N}}$ and two sequences $(\lambda_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ of measures on K such that:*

- $\nu_{n_k} = \lambda_k + \theta_k$ *for every* $k \in \mathbb{N}$,
- $\operatorname{supp}(\lambda_k) \cup \operatorname{supp}(\theta_k) \subseteq \operatorname{supp}(\nu_{n_k})$ *for every* $k \in \mathbb{N}$,
- $(\theta_k)_{k\in\mathbb{N}}$ *is weakly convergent,*
- $\operatorname{supp}(\lambda_k) \cap \operatorname{supp}(\lambda_l) = \emptyset$ *for every* $k \neq l \in \mathbb{N}$,
- *there is* $r \geq 0$ *such that* $\|\lambda_k\| = r$ *for every* $k \in \mathbb{N}$.

Proof Without loss of generality, we may assume that $\nu_n \neq 0$ and $\|\nu_n\| \leq 1$ for every $n \in \mathbb{N}$. Set

$$\mu = \sum_{n\in\mathbb{N}} |\nu_n|/2^{n+1}.$$

Then, μ is a non-negative Borel measure on K such that $\|\mu\| \leq 1$. Moreover, each ν_n is absolutely continuous with respect to μ. Let f_n be a Radon–Nikodym derivative of ν_n with respect to μ such that $\operatorname{supp}(f_n) \subseteq \operatorname{supp}(\nu_n)$. It follows that $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L_1(\mu)$, so by Lemma 19.3.1 there are a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and a sequence $(A_k)_{k\in\mathbb{N}}$ of pairwise disjoint Borel subsets of K such that the sequence $(f_{n_k} \cdot \chi_{A_k^c})_{k\in\mathbb{N}}$ is equi-integrable.

By [3, Theorem 5.2.8], the set $\{f_{n_k} \cdot \chi_{A_k^c} : k \in \mathbb{N}\}$ is relatively weakly compact, so, by the Eberlein–Šmulian theorem and by passing to a subsequence if necessary, we may assume that the sequence $(f_{n_k} \cdot \chi_{A_k^c})_{k\in\mathbb{N}}$ is weakly convergent and the sequence $(\|f_{n_k} \restriction A_k\|_{L_1(\mu)})_{k\in\mathbb{N}}$ of norms converges to some $r \in [0, 1]$.

If $r = 0$, then for every $k \in \mathbb{N}$ simply set $\lambda_k = 0$ and $\theta_k = \nu_{n_k}$. Since, for every $A \in Bor(K)$

$$\theta_k(A) = \nu_{n_k}(A) = \int_A f_{n_k} \cdot \chi_{A_k} d\mu + \int_A f_{n_k} \cdot \chi_{A_k^c} d\mu,$$

$(\theta_k)_{k\in\mathbb{N}}$ is a sum of a norm-null sequence and a weakly convergent sequence; hence, it is itself weakly convergent, too. The sequences $(\lambda_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ are thus as required.

So assume that $r > 0$. Let $k \in \mathbb{N}$. Define the auxiliary measures λ'_k and θ'_k as follows (for $A \in Bor(K)$):

$$\lambda'_k(A) = \int_A f_{n_k} \cdot \chi_{A_k} d\mu \quad \text{and} \quad \theta'_k(A) = \int_A f_{n_k} \cdot \chi_{A_k^c} d\mu.$$

Obviously, $\nu_{n_k} = \lambda'_k + \theta'_k$ and $\operatorname{supp}(\lambda'_k) \cup \operatorname{supp}(\theta'_k) \subseteq \operatorname{supp}(\nu_{n_k})$. Also, $\|\lambda'_k\| = \|f_{n_k} \restriction A_k\|_{L_1(\mu)} \to r$ as $k \to \infty$. By passing to a subsequence if necessary, we may additionally assume that

$$\left| \|\lambda'_k\| - r \right| < r/2^{k+2}$$

for every $k \in \mathbb{N}$. By the regularity of each λ'_k, for every $k \in \mathbb{N}$, there is a closed set $B_k \subseteq A_k \cap \operatorname{supp}(\lambda'_k)$ such that

$$\left| |\lambda'_k|(B_k) - \|\lambda'_k\| \right| < r/2^{k+2},$$

so, by the triangle inequality

$$\left| |\lambda'_k|(B_k) - r \right| < r/2^{k+1}.$$

It follows that $\|\lambda'_k \restriction B_k\| > 0$. For every $k \in \mathbb{N}$, define the final measures λ_k and θ_k as follows:

$$\lambda_k = \frac{r}{\|\lambda'_k \restriction B_k\|}(\lambda'_k \restriction B_k)$$

and

$$\theta_k = \theta'_k - \Big(\frac{r}{\|\lambda'_k \restriction B_k\|} - 1\Big)(\lambda'_k \restriction B_k) + (\lambda'_k \restriction (A_k \setminus B_k)).$$

Of course, $\operatorname{supp}(\lambda_k) \subseteq B_k$ and $\operatorname{supp}(\lambda_k) \cup \operatorname{supp}(\theta_k) \subseteq \operatorname{supp}(\nu_{n_k})$. Simple computations also show that $\|\lambda_k\| = r$ and $\nu_{n_k} = \theta_k + \lambda_k$.

Since

$$\| - \Big(\frac{r}{\|\lambda'_k \restriction B_k\|} - 1\Big)(\lambda'_k \restriction B_k) + (\lambda'_k \restriction (A_k \setminus B_k))\| \leq$$

$$\leq \Big|\frac{r}{\|\lambda'_k \restriction B_k\|} - 1\Big| \cdot \|\lambda'_k \restriction B_k\| + \|\lambda'_k \restriction (A_k \setminus B_k)\| \longrightarrow 0$$

as $k \to \infty$, $(\theta_k)_{k\in\mathbb{N}}$ still converges weakly. Also, since the sets B_k's are pairwise disjoint and closed, $\operatorname{supp}(\lambda_k) \cap \operatorname{supp}(\lambda_l) = \emptyset$ for every $k \neq l \in \mathbb{N}$. □

Let us recall the following well-known theorem from general Banach space theory.

Theorem 19.3.3 (Josefson–Nissenzweig) *For every infinite-dimensional Banach space E, there exists a sequence $(x^*_n)_{n\in\mathbb{N}}$ in the dual space E' which is weak* convergent to 0 and such that $\|x^*_n\| = 1$ for every $n \in \mathbb{N}$.*

By the Riesz–Markov–Kakutani representation theorem, we immediately get the following translation of Josefson–Nissenzweig's theorem in the case $E = C(K)$ for some compact space K.

Corollary 19.3.4 (Josefson–Nissenzweig's Theorem for $C(K)$-Spaces) *For every infinite compact space K, there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures on K which is weak* convergent to 0 and such that $\|\mu_n\| = 1$ for every $n \in \mathbb{N}$.*

For a proof of Josefson–Nissenzweig's theorem, see [170, Chapter XII]; a short proof of the theorem in the case of $C(K)$-spaces can be found in [354].

Given a compact space K, let us say that a sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures on K is *a Josefson–Nissenzweig sequence* (*a JN-sequence* in short) if it is weak* convergent to 0 and $\|\mu_n\| = 1$ for every $n \in \mathbb{N}$. Thus, Josefson–Nissenzweig's theorem implies that every infinite compact space K admits a JN-sequence. In the sequel, we will study properties of JN-sequences on compact spaces and their relations to the Grothendieck property of $C(K)$-spaces.

For a measure μ on a compact space K, we also say that a pair (B^+, B^-) of disjoint Borel subsets of K is *a Hahn–Jordan pair for* μ if $B^+, B^- \subseteq \operatorname{supp}(\mu)$,

$\|\mu\| = |\mu|(B^+ \cup B^-)$, and for the Jordan decomposition $\mu = \mu^+ - \mu^-$ of μ into non-negative measures μ^+ and μ^-, we have $\mu^+(A) = \mu(A \cap B^+)$ and $\mu^-(A) = -\mu(A \cap B^-)$ for every Borel set $A \subseteq K$.

Theorem 19.3.5 *For every infinite compact space K, the following statements are equivalent:*

(1) $C(K)$ is a Grothendieck space.
(2) For every JN-sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures on K, it holds

$$\lim_{k\to\infty} \mu_k\Big(B_k^+ \cap \bigcup_{n\in\mathbb{N}} B_n^-\Big) = \frac{1}{2} \quad \textit{and} \quad \lim_{k\to\infty} \mu_k\Big(B_k^- \cap \bigcup_{n\in\mathbb{N}} B_n^+\Big) = -\frac{1}{2},$$

where for each $n \in \mathbb{N}$ the pair (B_n^+, B_n^-) is a Hahn–Jordan pair for μ_n.
(3) For every JN-sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures on K, it holds

$$\lim_{k\to\infty} |\mu_k|\Big(\bigcup_{n\in\mathbb{N}} B_n^+ \cap \bigcup_{n\in\mathbb{N}} B_n^-\Big) = 1,$$

where for each $n \in \mathbb{N}$ the pair (B_n^+, B_n^-) is a Hahn–Jordan pair for μ_n.
(4) For every JN-sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures on K, it holds

$$\bigcup_{n\in\mathbb{N}} B_n^+ \cap \bigcup_{n\in\mathbb{N}} B_n^- \neq \emptyset,$$

where for each $n \in \mathbb{N}$ the pair (B_n^+, B_n^-) is a Hahn–Jordan pair for μ_n.
(5) There is no JN-sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures on K such that for every $n \neq n' \in \mathbb{N}$, it holds

$$\operatorname{supp}(\mu_n) \cap \operatorname{supp}(\mu_{n'}) = \emptyset.$$

Proof Assume first that $C(K)$ is a Grothendieck space and $(\mu_n)_{n\in\mathbb{N}}$ is a JN-sequence on K. By the Grothendieck property, $(\mu_n)_{n\in\mathbb{N}}$ is weakly convergent to 0. For each $n \in \mathbb{N}$, let (B_n^+, B_n^-) be a Hahn–Jordan pair for μ_n. Considering the constant characteristic function χ_K, the definition of a JN-sequence easily yields that

$$\lim_{n\to\infty} \mu_n(B_n^+) = -\lim_{n\to\infty} \mu_n(B_n^-) = 1/2.$$

For the sake of contradiction, assume that there exists a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ and $\alpha \in [0, 1/2)$ for which we have

$$\lim_{k\to\infty} \mu_{n_k}\Big(B_{n_k}^+ \cap \bigcup_{n\in\mathbb{N}} B_n^-\Big) = \alpha.$$

Let $K \in \mathbb{N}$ be such that for every $k \geq K$ it holds

$$|\mu_{n_k}\Big(B_{n_k}^+ \cap \bigcup_{n\in\mathbb{N}} B_n^-\Big) - \alpha| < \frac{1}{8} - \frac{2\alpha}{8}$$

and

$$|\mu_{n_k}(B_{n_k}^-)| > \frac{4\alpha+2}{8}.$$

It follows that for every $k \geq K$ we have

$$\mu_{n_k}\Big(B_{n_k}^+ \cap \bigcup_{n\in\mathbb{N}} B_n^-\Big) < \frac{6\alpha+1}{8},$$

and so

$$|\mu_{n_k}\Big(\bigcup_{n\in\mathbb{N}} B_n^-\Big)| = |\mu_{n_k}(B_{n_k}^-) + \mu_{n_k}\Big(B_{n_k}^+ \cap \bigcup_{n\in\mathbb{N}} B_n^-\Big)| \geq$$

$$\geq |\mu_{n_k}(B_{n_k}^-)| - |\mu_{n_k}\Big(B_{n_k}^+ \cap \bigcup_{n\in\mathbb{N}} B_n^-\Big)| >$$

$$> \frac{4\alpha+2}{8} - \frac{6\alpha+1}{8} = \frac{1-2\alpha}{8} > 0,$$

which, since $\bigcup_{n\in\mathbb{N}} B_n^-$ is a Borel subset of K, contradicts the fact that $(\mu_n)_{n\in\mathbb{N}}$ is weakly convergent to 0 (cf. Corollary 19.2.7). It follows that

$$\lim_{k\to\infty} \mu_k\Big(B_k^+ \cap \bigcup_{n\in\mathbb{N}} B_n^-\Big) = \frac{1}{2};$$

the proof of the other equality is symmetrical. Thus, (1) implies (2).

To see that (2) implies (3), notice that for every $k \in \mathbb{N}$ we have

$$|\mu_k|\Big(\bigcup_{n\in\mathbb{N}} B_n^+ \cap \bigcup_{n\in\mathbb{N}} B_n^-\Big) = |\mu_k|\Big(B_k^+ \cap \bigcup_{n\in\mathbb{N}} B_n^-\Big) + |\mu_k|\Big(B_k^- \cap \bigcup_{n\in\mathbb{N}} B_n^+\Big) =$$

$$= \mu_k\Big(B_k^+ \cap \bigcup_{n\in\mathbb{N}} B_n^-\Big) - \mu_k\Big(B_k^- \cap \bigcup_{n\in\mathbb{N}} B_n^+\Big).$$

The implications (3)⇒(4) and (4)⇒(5) are trivial.

Assume now that (5) holds, yet $C(K)$ is not a Grothendieck space, that is, there exists a sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures on K which is weak* convergent

to 0 but its every subsequence is not weakly convergent to 0. By the Banach–Steinhaus theorem, the sequence is norm bounded, so by Proposition 19.3.2 there are a sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers and two sequences $(\lambda_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ of measures on K such that

- $\nu_{n_k} = \lambda_k + \theta_k$ for every $k \in \mathbb{N}$,
- $\operatorname{supp}(\lambda_k) \cup \operatorname{supp}(\theta_k) \subseteq \operatorname{supp}(\nu_{n_k})$ for every $k \in \mathbb{N}$,
- $(\theta_k)_{k\in\mathbb{N}}$ is weakly convergent,
- $\operatorname{supp}(\lambda_k) \cap \operatorname{supp}(\lambda_l) = \emptyset$ for every $k \neq l \in \mathbb{N}$,
- $\|\lambda_k\| = r$ for some $r \geq 0$ and every $k \in \mathbb{N}$.

If $r = 0$, then $\nu_{n_k} = \theta_k$ for every $k \in \mathbb{N}$, so $(\nu_{n_k})_{k\in\mathbb{N}}$ is weakly convergent to 0, which is impossible by our assumption. It yields that $r > 0$. Since $(\nu_{n_k})_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ are weak* convergent, $(\lambda_k)_{k\in\mathbb{N}}$ is weak* convergent as well (though not necessarily to 0). For each $k \in \mathbb{N}$, let

$$\mu_k = \frac{1}{2r}(\lambda_{2k} - \lambda_{2k+1}).$$

Of course, $\|\mu_k\| = 1$ for every $k \in \mathbb{N}$ and $\operatorname{supp}(\mu_k) \cap \operatorname{supp}(\mu_l) = \emptyset$ for every $k \neq l \in \mathbb{N}$. Also, the sequence $(\mu_k)_{k\in\mathbb{N}}$ is weak* convergent to 0, so it is a disjointly supported JN-sequence on K, which contradicts (5). Hence, (1) holds true. □

The next result, Proposition 19.3.7, asserts that in the case of totally disconnected compact spaces if there are JN-sequences with disjoint supports, then there are ones with supports constituting discrete sequences.

Lemma 19.3.6 *Let K be a totally disconnected compact space. If $(\mu_n)_{n\in\mathbb{N}}$ is a JN-sequence on K such that $\operatorname{supp}(\mu_n) \cap \operatorname{supp}(\mu_{n'}) = \emptyset$ for every $n \neq n' \in \mathbb{N}$, then there exist a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ and a sequence $(V_k)_{k\in\mathbb{N}}$ of clopen subsets of K such that $\operatorname{supp}(\mu_{n_i}) \subseteq V_i$ and $|\mu_{n_j}|(V_i) < 1/2^{i+2}$ for every $i < j \in \mathbb{N}$.*

Proof For each $l \in \mathbb{N}$, set $m_l = 2^{l+2}$.

Put $Y_0^0 = \mathbb{N}$. Since the sets $\operatorname{supp}(\mu_0), \ldots, \operatorname{supp}(\mu_{m_0})$ are closed and pairwise disjoint, there exist pairwise disjoint clopen subsets $W_0, \ldots, W_{m_0}$ of K such that $\operatorname{supp}(\mu_i) \subseteq W_i$ for every $i = 0, \ldots, m_0$. For each $i = 0, \ldots, m_0$, define the subset $Y_i^1 \subseteq Y_0^0$ as follows:

$$Y_i^1 = \{n > m_0 \colon\ |\mu_n|(W_i) < 1/m_0\}.$$

For each $n > m_0$, since $\|\mu_n\| = 1$, there is $0 \leq i \leq m_0$ such that $n \in Y_i^1$. It follows that there is $0 \leq i \leq m_0$ such that Y_i^1 is infinite. Set $n_0 = i$ and $V_0 = W_i$.

Repeat the above procedure inductively (with m_1 instead of m_0, $(\mu_n)_{n\in Y_{n_0}^1}$ instead of $(\mu_n)_{n\in Y_0^0}$, to get n_1, V_1, $Y_{n_1}^2 \subseteq Y_{n_0}^1$, and $(\mu_n)_{n\in Y_{n_1}^2}$, and so on…) to obtain desired sequences $(\mu_{n_k})_{k\in\mathbb{N}}$ and $(V_k)_{k\in\mathbb{N}}$. □

Proposition 19.3.7 *Let K be a totally disconnected compact space. If $(\mu_n)_{n\in\mathbb{N}}$ is a JN-sequence on K such that $\operatorname{supp}(\mu_n)\cap\operatorname{supp}(\mu_{n'})=\emptyset$ for every $n\neq n'\in\mathbb{N}$, then there exist a JN-sequence $(\nu_k)_{n\in\mathbb{N}}$ on K, a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$, and a sequence $(U_k)_{k\in\mathbb{N}}$ of pairwise disjoint clopen subsets of K such that $\operatorname{supp}(\nu_k)\subseteq\operatorname{supp}(\mu_{n_k})\cap U_k$ for every $k\in\mathbb{N}$.*

Proof Let $(\mu_{n_k})_{k\in\mathbb{N}}$ and a sequence $(V_k)_{k\in\mathbb{N}}$ be as in Lemma 19.3.6. For every $i\in\mathbb{N}$, set

$$W_i=\bigcup_{j=0}^{i-1}V_j,\quad U_i=V_i\setminus W_i,\quad \lambda_i=\mu_{n_i}\restriction(K\setminus W_i);$$

it follows that $\|\lambda_i\|>1/2$ and $\operatorname{supp}(\lambda_i)\subseteq\operatorname{supp}(\mu_{n_i})\cap U_i$. Also, the sets U_k's are clopen and pairwise disjoint.

We claim that the sequence $(\lambda_i)_{i\in\mathbb{N}}$ is weak* convergent to 0. To prove that, fix $f\in C(K)$ and $\varepsilon>0$. Let $\varepsilon'=\varepsilon/(\|f\|_\infty+1)$. There is $k\in\mathbb{N}$ such that $\sum_{j\geq k}1/2^{j+2}<\varepsilon'$. Set

$$g=\chi_{K\setminus W_k}\cdot f.$$

As W_k is clopen, $g\in C(K)$. Moreover, $\lambda_i(f)=\lambda_i(g)$ for every $i\geq k$. Since $\lim_{i\to\infty}\mu_{n_i}(g)=0$, there is $l\geq k$ such that for every $i\geq l$ it holds $|\mu_{n_i}(g)|<\varepsilon'$. For every $i\geq l$, we have

$$\|\mu_{n_i}\restriction(W_i\setminus W_k)\|\leq\sum_{j=k}^{i-1}|\mu_{n_i}|(V_j)<\sum_{j=k}^{i-1}1/2^{j+2}<\varepsilon',$$

so

$$|\lambda_i(g)-\mu_{n_i}(g)|=|(\mu_{n_i}\restriction W_i)(g)|=|(\mu_{n_i}\restriction(W_i\setminus W_k))(f)|\leq$$

$$\leq\|\mu_{n_i}\restriction(W_i\setminus W_k)\|\cdot\|f\|_\infty<\varepsilon'\cdot\|f\|_\infty,$$

hence

$$|\lambda_i(f)|=|\lambda_i(g)|\leq|\lambda_i(g)-\mu_{n_i}(g)|+|\mu_{n_i}(g)|<\varepsilon'\cdot(\|f\|_\infty+1)=\varepsilon,$$

which proves that $\lim_{i\to\infty}\lambda_i(f)=0$.

Finally, for every $k\in\mathbb{N}$, set $\nu_k=\lambda_k/\|\lambda_k\|$. Since $\|\lambda_k\|>1/2$ for every $k\in\mathbb{N}$, the sequence $(\nu_k)_{k\in\mathbb{N}}$ is the desired JN-sequence. □

Corollary 19.3.8 *Let K be a totally disconnected compact space. Then, the following are equivalent:*

(1) $C(K)$ *is not a Grothendieck space.*
(2) There are a JN-sequence $(\mu_n)_{n\in\mathbb{N}}$ *and a sequence* $(U_n)_{n\in\mathbb{N}}$ *of pairwise disjoint clopen subsets of* K *such that* $\operatorname{supp}(\mu_n) \subseteq U_n$ *for every* $n \in \mathbb{N}$.

Proof Implication (2)⇒(1) follows from Theorem 19.3.5 ((1)⇔(5)), whereas the proof of implication (1)⇒(2) additionally appeals to Proposition 19.3.7. □

Remark 19.3.9 The sequence $(\mu_n)_{n\in\mathbb{N}}$ from Corollary 19.3.8.(2) is easily seen to be equivalent to the canonical basis of ℓ_1, cf. Lemma 19.2.2.

The following result is a useful rephrasing of Corollary 19.3.8.

Corollary 19.3.10 *Let* K *be a totally disconnected compact space. Then, the following are equivalent:*

(1) $C(K)$ *is a Grothendieck space.*
(2) For every sequence $(\mu_n)_{n\in\mathbb{N}}$ *of measures on* K *and every sequence* $(U_n)_{n\in\mathbb{N}}$ *of pairwise disjoint clopen subsets of* K *such that* $\|\mu_n\| = |\mu_n|(U_n) = 1$ *for every* $n \in \mathbb{N}$*, there is a clopen set* $U \subseteq K$ *such that the limit* $\lim_{n\to\infty} \mu_n(U)$ *does not exist.*

Proof Assume (1). Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures on K and $(U_n)_{n\in\mathbb{N}}$ a sequence of pairwise disjoint clopen subsets of K such that $\|\mu_n\| = |\mu_n|(U_n) = 1$ for every $n \in \mathbb{N}$. For the sake of contradiction, assume that for every clopen subset U of K the limit $\lim_{n\to\infty} \mu_n(U)$ exists and is equal to some $\mu(U) \in \mathbb{R}$. For each $n \in \mathbb{N}$, set $V_n = U_{2n} \cup U_{2n+1}$ and let $\nu_n = \frac{1}{2}(\mu_{2n} - \mu_{2n+1})$. Then, $\operatorname{supp}(\nu_n) \subseteq \operatorname{supp}(\mu_{2n}) \cup \operatorname{supp}(\mu_{2n+1}) \subseteq V_n$ and $\|\nu_n\| = 1$ for every $n \in \mathbb{N}$. Moreover, $\nu_n(U) \to 0\,(= \mu(U) - \mu(U))$ for every clopen subset U of K. It follows by Lemma 19.1.2 that $(\nu_n)_{n\in\mathbb{N}}$ is a JN-sequence on K like in Corollary 19.3.8.(2), a contradiction. Thus, (2) holds.

Assume that (2) holds, yet $C(K)$ is not Grothendieck. Let $(\mu_n)_{n\in\mathbb{N}}$ be a JN-sequence from Corollary 19.3.8.(2). It follows that $\mu_n(V) \to 0$ for every clopen subset $V \subseteq K$, whereas (2) implies that there is a clopen set $U \subseteq K$ such that the limit $\lim_{n\to\infty} \mu_n(U)$ does not exist, a contradiction. Consequently, (1) holds, too. □

19.4 $C(K)$-Spaces for Extremely Disconnected K

Recall that a topological space X is *extremely disconnected* if the closure of every open subset of X is open. In this section, we prove Grothendieck's theorem stating that for every extremely disconnected compact space K its Banach space $C(K)$ is a Grothendieck space. As a corollary, we immediately get that for every set Γ the space $\ell_\infty(\Gamma)$ of bounded real-valued functions on Γ is Grothendieck.

There are several ways to prove Grothendieck's theorem. At this moment, the shortest way for us would be to assume that for a given extremely disconnected space K its space $C(K)$ is not Grothendieck and then to use Corollary 19.3.8

to get a contradiction by constructing an appropriate clopen subset of K on which the JN-sequence from the corollary would not be converging to 0, but we will intentionally follow a much longer route (though ideologically similar), the one however which will provide us with additional important results. Namely, our proof will require two powerful ingredients—Rosenthal's lemma concerning "disjointification" of sequences of measures and the characterization of relatively weakly compact subsets of dual spaces $C(K)'$ due to Dieudonné and Grothendieck.

19.4.1 Rosenthal's Lemma

Rosenthal's lemma, which we will prove now, was introduced in [524] as an important tool in the study of the structure of operators into the space ℓ_∞. There are several known proofs of it, e.g. a very short one by Kupka [392]; we present below an inductive argument, which strongly resembles the standard proof of the infinite Ramsey theorem (cf. Sobota [565]).

Lemma 19.4.1 (Rosenthal [524]) *Let $(m(k,n))_{k,n\in\mathbb{N}}$ be an infinite square matrix of non-negative real numbers for which there is $M \in \mathbb{N}$ such that*

$$\sum_{n\in\mathbb{N}} m(k,n) \leq M$$

for every $k \in \mathbb{N}$. Then, for every $\varepsilon > 0$, there is an infinite set $A \subseteq \mathbb{N}$ such that

$$\sum_{\substack{n\in A\\ n\neq k}} m(k,n) < \varepsilon$$

for every $k \in A$.

Proof The set A will be constructed by induction. Fix $\varepsilon > 0$ and for each $l \in \mathbb{N}$ set $\varepsilon_l = \varepsilon/2^{l+2}$, so $\varepsilon_0 = \varepsilon/4$. We will construct a descending sequence $(X_l)_{l\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ and a strictly increasing sequence $(a_l)_{l\in\mathbb{N}}$ of natural numbers such that for every $i \in \mathbb{N}$ the following conditions will hold:

(i) $a_i \in X_i$ and $a_i < \min X_{i+1}$,
(ii) $m(k, a_i) < \varepsilon_i$ for every $k \in X_{i+1}$,
(iii) $\sum_{n\in X_{i+1}} m(a_i, n) < \varepsilon/2$.

Put $X_0 = \mathbb{N}$. Let $K \in \mathbb{N}$ be such that $K \geq M/\varepsilon_0$. For each $0 \leq n \leq K$, set

$$Y_n = \{k \in \mathbb{N}:\ k > K,\ m(k,n) < \varepsilon_0\}.$$

It follows that

$$X_0 \setminus \{0, \ldots, K\} = \bigcup_{n=0}^{K} Y_n.$$

Indeed, if there is $k > K$ such that k does not belong to any Y_n, then it means that for every $0 \leq n \leq K$ we have $m(k, n) \geq \varepsilon_0$, so

$$M \geq \sum_{n=0}^{K} m(k, n) \geq (K + 1) \cdot \varepsilon_0 \geq (M/\varepsilon_0) \cdot \varepsilon_0 + \varepsilon_0 > M,$$

a contradiction. Consequently, there is $0 \leq a_0 \leq K$ such that Y_{a_0} is infinite. The inequality $\sum_{n\in\mathbb{N}} m(a_0, n) \leq M$ implies that there is an infinite subset $X_1 \subseteq Y_{a_0} \subseteq X_0$ such that

$$\sum_{n\in X_1} m(a_0, n) < \varepsilon/2.$$

Note that $a_0 \in X_0$ and $a_0 \leq K < \min Y_{a_0} \leq \min X_1$, so condition (i) holds. Condition (ii) is satisfied by the choice of a_0, the definition of Y_{a_0}, and the fact that $X_1 \subseteq Y_{a_0}$. Condition (iii) holds by the choice of X_1. The zeroth step of the induction is thus finished.

Assume now that for some $l \geq 1$ we have conducted l steps of the induction, that is, we have constructed sequences $X_0 \supseteq X_1 \supseteq \ldots \supseteq X_l$ of infinite subsets of $\mathbb{N}$ and $a_0 < a_1 < \ldots < a_{l-1}$ of natural numbers such that conditions (i)–(iii) hold for every $0 \leq i \leq l - 1$. We will construct an infinite set $X_{l+1} \subseteq X_l$ and find a natural number $a_l > a_{l-1}$ satisfying conditions (i)–(iii) also for $i = l$. We proceed basically in the same way as in the zeroth step. Enumerate

$$X_l = \{k_0 < k_1 < k_2 < \ldots\}.$$

Let $K' \in \mathbb{N}$ be such that $K' \geq M/\varepsilon_l$. For each $0 \leq n \leq K'$, set

$$Y'_n = \{k_j : \ j \in \mathbb{N}, \ j > K', \ m(k_j, k_n) < \varepsilon_l\}.$$

It follows, similarly as in the zeroth step, that

$$X_l \setminus \{k_0, \ldots, k_{K'}\} = \bigcup_{n=0}^{K'} Y'_n.$$

Consequently, there is $0 \leq n_0 \leq K'$ for which the set Y'_{n_0} is infinite. Let $a_l = k_{n_0}$. The inequality $\sum_{n\in\mathbb{N}} m(a_l, n) \leq M$ implies that there is an infinite subset $X_{l+1} \subseteq Y'_{n_0} \subseteq X_l$ such that

$$\sum_{n\in X_{l+1}} m(a_l, n) < \varepsilon/2.$$

As previously, note that $a_l \in X_l$ and $a_l \leq k_{K'} < \min Y'_{n_0} \leq \min X_{l+1}$, so condition (i) holds. Condition (ii) is satisfied by the choice of n_0, the definition of Y'_{n_0}, and the fact that $X_{l+1} \subseteq Y'_{n_0}$. Condition (iii) holds by the choice of X_{l+1}. The l-th step of the induction is finished.

We finish the induction with a descending sequence $(X_l)_{l\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ and a strictly increasing sequence $(a_l)_{l\in\mathbb{N}}$ of natural numbers satisfying conditions (i)–(iii). Set $A = \{a_k\colon\ k \in \mathbb{N}\}$. Note that for every pair $k > n$ we have $a_k \in X_k \subseteq X_{n+1}$, so by condition (ii) we have $m(a_k, a_n) < \varepsilon_n$. On the other hand, for every pair $k < n$, we have $a_n \in X_n \subseteq X_{k+1}$, so for every $k \in \mathbb{N}$, by condition (iii), we get that $\sum_{n>k} m(a_k, a_n) < \varepsilon/2$. Summing up, for every $k \in \mathbb{N}$, we have

$$\sum_{\substack{n\in\mathbb{N}\\ n\neq k}} m(a_k, a_n) = \sum_{n<k} m(a_k, a_n) + \sum_{n>k} m(a_k, a_n) <$$

$$< \sum_{n<k} \varepsilon_n + \varepsilon/2 = \sum_{n<k} \varepsilon/2^{n+2} + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

so the proof of the lemma is finished. □

By applying Rosenthal's lemma to the setting $m(n, k) = |\mu_k|(A_n)$ (for every $k, n \in \mathbb{N}$), we get the following measure-theoretic corollary.

Corollary 19.4.2 (Rosenthal's Lemma for Measures) *Let K be a compact space and $(A_n)_{n\in\mathbb{N}}$ a sequence of pairwise disjoint Borel subsets of K. Let $(\mu_k)_{k\in\mathbb{N}}$ be a bounded sequence of measures on K. Then, for every $\varepsilon > 0$, there is an infinite set $A \subseteq \mathbb{N}$ such that*

$$\sum_{\substack{n\in A\\ n\neq k}} |\mu_k|(A_n) < \varepsilon$$

for every $k \in A$.

Recall that two infinite subsets A and B of $\mathbb{N}$ are *almost disjoint* if their intersection $A \cap B$ is finite. A family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ is *almost disjoint* if every two distinct $A, B \in \mathcal{F}$ are almost disjoint. It is a standard fact from set theory that there exist almost disjoint families of subsets of $\mathbb{N}$ having cardinality equal to continuum $\mathfrak{c}$ (i.e. the cardinality of the real line $\mathbb{R}$). The symbol ω_1 below denotes the first uncountable ordinal number.

Lemma 19.4.3 *Let K be a totally disconnected compact space. Let $(V_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint clopen subsets of K. Assume that $\{A_\xi\colon\ \xi < \omega_1\}$ is an almost disjoint family of infinite subsets of $\mathbb{N}$ such that for every $\xi < \omega_1$ the closure*

$\overline{\bigcup_{n\in A_\xi} V_n}$ *is open in* K. *Then, for every sequence* $(\mu_n)_{n\in\mathbb{N}}$ *of measures on* K, *there exists* $\eta < \omega_1$ *such that the equality*

$$|\mu_k|\Big(\overline{\bigcup_{n\in A_\xi} V_n}\Big) = |\mu_k|\Big(\bigcup_{n\in A_\xi} V_n\Big)$$

holds for every $\eta \le \xi < \omega_1$ *and* $k \in \mathbb{N}$.

Proof For every $\xi < \omega_1$, let C_ξ be a closed subset of K given by the formula:

$$C_\xi = \overline{\bigcup_{n\in A_\xi} V_n} \setminus \bigcup_{n\in A_\xi} V_n,$$

i.e. C_ξ is the boundary of $\bigcup_{n\in A_\xi} V_n$ in K. It is easy to see that $C_\xi \cap C_\zeta = \emptyset$ whenever $\xi \neq \zeta < \omega_1$. Hence, for every $k \in \mathbb{N}$, by the finiteness of the measure $|\mu_k|$, there exists an ordinal number $\eta_k < \omega_1$ such that $|\mu_k|(C_\xi) = 0$ for every $\xi \ge \eta_k$. Then, for every $\xi \ge \eta_k$, we have

$$|\mu_k|\Big(\overline{\bigcup_{n\in A_\xi} V_n}\Big) = |\mu_k|\Big(C_\xi \cup \bigcup_{n\in A_\xi} V_n\Big) =$$

$$|\mu_k|(C_\xi) + |\mu_k|\Big(\bigcup_{n\in A_\xi} V_n\Big) = |\mu_k|\Big(\bigcup_{n\in A_\xi} V_n\Big).$$

Set $\eta = \sup_{k\in\mathbb{N}} \eta_k$. The conclusion of the lemma holds by the above calculations. □

Lemma 19.4.3 can be used to obtain the following "boundary" version of Rosenthal's lemma for measures on extremely disconnected compact spaces.

Corollary 19.4.4 *Let* K *be an extremely disconnected compact space and* $(V_n)_{n\in\mathbb{N}}$ *a sequence of pairwise disjoint clopen subsets of* K. *Let* $(\mu_k)_{k\in\mathbb{N}}$ *be a bounded sequence of measures on* K. *Then, for every* $\varepsilon > 0$, *there is an infinite set* $B \subseteq \mathbb{N}$ *such that*

$$|\mu_k|\Big(\overline{\bigcup_{\substack{n\in B\\ n\neq k}} V_n}\Big) < \varepsilon$$

for every $k \in B$.

Proof By Rosenthal's lemma for measures (Corollary 19.4.2), there is an infinite subset A of $\mathbb{N}$ such that

$$\sum_{\substack{n\in A\\ n\neq k}} |\mu_k|(V_n) < \varepsilon \qquad (*)$$

for every $k \in A$. Let $\{A_\xi : \xi < \omega_1\}$ be a family of pairwise disjoint infinite subsets of A. Then, since K is extremely disconnected, by Lemma 19.4.3, there is $\eta < \omega_1$ such that the equality

$$|\mu_k|\Big(\overline{\bigcup_{n\in A_\eta} V_n}\Big) = |\mu_k|\Big(\bigcup_{n\in A_\eta} V_n\Big) \tag{$**$}$$

holds for every $k \in A$. Set $B = A_\eta$. We have $B \subseteq A$. Let $k \in B$. The σ-additivity of $|\mu_k|$ implies that

$$|\mu_k|\Big(\bigcup_{\substack{n\in B\\ n\neq k}} V_n\Big) = \sum_{\substack{n\in B\\ n\neq k}} |\mu_k|(V_n),$$

so, by $(**)$ and $(*)$, we have

$$|\mu_k|\Big(\overline{\bigcup_{\substack{n\in B\\ n\neq k}} V_n}\Big) = |\mu_k|\Big(\bigcup_{\substack{n\in B\\ n\neq k}} V_n\Big) = \sum_{\substack{n\in B\\ n\neq k}} |\mu_k|(V_n) < \varepsilon.$$

□

19.4.2 *Dieudonné–Grothendieck's Characterization of Relatively Weakly Compact Subsets of Measures*

We now present a proof of a characterization of relatively weakly compact subsets of dual spaces $C(K)'$ (or, equivalently, of spaces $M(K)$) due to Dieudonné and Grothendieck. To achieve this goal, we first need to introduce several auxiliary results and notions.

Recall that, for a σ-field Σ of subsets of a given set Ω, a subset $\mathcal{M}$ of $\mathrm{ca}(\Sigma)$ is *uniformly countably additive* if for each decreasing sequence $(A_n)_{n\in\mathbb{N}}$ in Σ with $\bigcap_{n=0}^{\infty} A_n = \emptyset$ and every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $|\mu(A_n)| < \varepsilon$ for every $n \geq N$ and $\mu \in \mathcal{M}$. Equivalently, $\mathcal{M}$ is uniformly countably additive if for every sequence $(A_n)_{n\in\mathbb{N}}$ in Σ decreasing to a set $A \in \Sigma$ and every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $|\mu(A) - \mu(A_n)| < \varepsilon$ for every $n \geq N$ and $\mu \in \mathcal{M}$.

Lemma 19.4.5 *Let Σ be a σ-field of subsets of a given set Ω. Let $\mathcal{A}$ be a Boolean subalgebra of Σ generating Σ. Suppose that $(\mu_n)_{n\in\mathbb{N}}$ is a uniformly countably additive sequence of measures on Σ such that the limit $\lim_{n\to\infty} \mu_n(A)$ exists for every $A \in \mathcal{A}$. Then, the limit $\lim_{n\to\infty} \mu_n(A)$ exists for every $A \in \Sigma$.*

Proof Set

$$\mathcal{B} = \{A \in \Sigma : \lim_{n\to\infty} \mu_n(A) \text{ exists}\}.$$

Then, $\mathcal{A} \subseteq \mathcal{B} \subseteq \Sigma$ and hence $\mathcal{B}$ also generates Σ. We will show that $\mathcal{B}$ is a monotone class, that is, it is closed under countable decreasing intersections and under countable increasing unions. The conclusion will then follow by the standard monotone class theorem implying that $\mathcal{B} = \Sigma$.

Since $\Omega \in \mathcal{A} \subseteq \mathcal{B}$, we will only show the argument for decreasing sequences. Let $(A_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{B}$ decreasing to some $A \in \Sigma$. We claim that $A \in \mathcal{B}$. Fix $\varepsilon > 0$. Since $(\mu_n)_{n\in\mathbb{N}}$ is uniformly countably additive, there is $K \in \mathbb{N}$ such that for every $k \geq K$ and $n \in \mathbb{N}$ we have

$$|\mu_n(A) - \mu_n(A_k)| < \varepsilon/3. \tag{$*$}$$

In particular, $(*)$ holds for $k = K$. But $\lim_{n\to\infty} \mu_n(A_K)$ exists (since $A_K \in \mathcal{B}$), so there is $N \in \mathbb{N}$ such that for every $p, q \geq N$ we have

$$|\mu_p(A_K) - \mu_q(A_K)| < \varepsilon/3. \tag{$**$}$$

For every $p, q \geq N$, by $(*)$ (applied twice) and $(**)$, we get

$$|\mu_p(A) - \mu_q(A)| \leq$$

$$\leq |\mu_p(A) - \mu_p(A_K)| + |\mu_p(A_K) - \mu_q(A_K)| + |\mu_q(A_K) - \mu_q(A)| <$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

so $\lim_{n\to\infty} \mu_n(A)$ exists, that is, $A \in \mathcal{B}$. □

Lemma 19.4.6 *Let Ω be a set, Σ a σ-field of subsets of Ω, and $\mathcal{M}$ a bounded subset of* $\mathrm{ca}(\Sigma)$. *If $\mathcal{M}$ is uniformly countably additive, then $\mathcal{M}$ is relatively weakly compact.*

Proof Assume that $\mathcal{M}$ is uniformly countably additive. By the Eberlein–Šmulian theorem, it is enough to show that every sequence in $\mathcal{M}$ contains a weakly convergent subsequence. So, fix $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ and set

$$\lambda = \sum_{n\in\mathbb{N}} |\mu_n|/2^{n+1}.$$

Obviously, λ is a non-negative element of $\mathrm{ca}(\Sigma)$ with $\|\lambda\| \leq \sup_{n\in\mathbb{N}} \|\mu_n\| < \infty$ and each μ_n is absolutely continuous with respect to λ.

For each $n \in \mathbb{N}$, let $f_n \in L_1(\lambda)$ be the Radon–Nikodym derivative of μ_n with respect to λ. Each f_n is the pointwise limit of a sequence of simple functions from Ω into $\mathbb{R}$, so there is a countable family $\Gamma_n \subseteq \Sigma$ such that f_n is measurable with respect to the σ-field Σ_n generated by Γ_n. Set $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ and let $\mathcal{A}$ be the Boolean subalgebra of Σ generated by Γ. Note that $|\Gamma| = |\mathcal{A}| = \aleph_0$ and $\sup_{n\in\mathbb{N}} |\mu_n(A)| < \infty$ for every $A \in \mathcal{A}$ (recall that $\mathcal{M}$ is bounded), so, by a diagonal argument, we may find a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \mu_{n_k}(A)$ exists for every $A \in \mathcal{A}$.

Since $(\mu_{n_k})_{k\in\mathbb{N}}$ is uniformly countably additive, by Lemma 19.4.5, $\lim_{k\to\infty}\mu_{n_k}(A)$ exists for every A in the σ-field Σ' generated by $\mathcal{A}$. Proposition 19.2.5 implies that the sequence $(f_{n_k})_{k\in\mathbb{N}}$ is weakly convergent in the closed subspace $L_1(\Omega, \Sigma', \lambda \restriction \Sigma')$ of $L_1(\lambda)$, and hence, it is weakly convergent in $L_1(\lambda)$. It follows that $(\mu_{n_k})_{k\in\mathbb{N}}$ is weakly convergent in $\mathrm{ca}(\Sigma)$. □

(The converse to Lemma 19.4.6 also holds (see, e.g. [170, Theorem 13, page 92]) for the proof using the celebrated Vitali–Hahn–Saks theorem.)

Lemma 19.4.7 *Let K be a compact space and $\mathcal{M} \subseteq C(K)'$. Assume that for every sequence $(O_n)_{n\in\mathbb{N}}$ of pairwise disjoint open subsets of K, the following equality holds*

$$\lim_{n\to\infty}\sup_{\mu\in\mathcal{M}}|\mu(O_n)| = 0. \tag{$*$}$$

Then:

(1) For every $\varepsilon > 0$ and a closed subset $L \subseteq K$, there is an open set $U \subseteq K$ such that $L \subseteq U$ and $|\mu|(U \setminus L) \leq \varepsilon$ for every $\mu \in \mathcal{M}$.
(2) For every $\varepsilon > 0$ and an open subset $U \subseteq K$, there is a closed set $L \subseteq K$ such that $L \subseteq U$ and $|\mu|(U \setminus L) \leq \varepsilon$ for every $\mu \in \mathcal{M}$.

Proof To see (1), assume that it does not hold and proceed with a simple inductive argument using the regularity of elements of $\mathcal{M}$, eventually to get a contradiction with $(*)$.

To get (2), fix $\varepsilon > 0$ and an open set $U \subseteq K$. The set $U^c = K \setminus U$ is closed, so by (1) there is an open set $V \subseteq K$ such that $U^c \subseteq V$ and $|\mu|(V \setminus U^c) \leq \varepsilon$ for every $\mu \in \mathcal{M}$. Let $L = V^c = K \setminus V$. Then, L is closed, $L \subseteq U$, and

$$|\mu|(U \setminus L) = |\mu|(U) - |\mu|(V^c) = (|\mu|(K) - |\mu|(U^c)) - (|\mu|(K) - |\mu|(V)) =$$

$$= |\mu|(V) - |\mu|(U^c) = |\mu|(V \setminus U^c) \leq \varepsilon$$

for every $\mu \in \mathcal{M}$. □

We are ready to prove the main theorem of this section.

Theorem 19.4.8 (Dieudonné [171], Grothendieck [286]) *Let K be a compact space and $\mathcal{M}$ be a bounded subset of $C(K)'$. Then, $\mathcal{M}$ is relatively weakly compact if and only if for every sequence $(O_n)_{n\in\mathbb{N}}$ of pairwise disjoint open subsets of K we have*

$$\lim_{n\to\infty}\sup_{\mu\in\mathcal{M}}|\mu(O_n)| = 0. \tag{DG}$$

Proof Assume that $\mathcal{M}$ is relatively weakly compact. For the sake of contradiction, suppose that there exist a sequence $(O_n)_{n\in\mathbb{N}}$ of pairwise disjoint open subsets of K, a sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures in $\mathcal{M}$, and $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ it holds

$$|\mu_n(O_n)| \geq \varepsilon.$$

$\mathcal{M}$ is bounded, so by Rosenthal's lemma (Lemma 19.4.1) we can find a sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$|\mu_{n_k}|\Big(\bigcup_{\substack{l\in\mathbb{N}\\ l\neq k}} O_{n_l}\Big) < \varepsilon/3$$

for every $k \in \mathbb{N}$. Since $\mathcal{M}$ is relatively weakly compact, by the Eberlein–Šmulian theorem, $\mathcal{M}$ is relatively weakly sequentially compact, so there is a further subsequence $(n_{k_l})_{l\in\mathbb{N}}$ such that $(\mu_{n_{k_l}})_{l\in\mathbb{N}}$ converges weakly to some $\mu \in \overline{\mathcal{M}}$ (the closure taken in the weak topology of $C(K)'$). Since $|\mu|$ is σ-additive and finite, there is $l_0 \in \mathbb{N}$ such that

$$|\mu|\Big(\bigcup_{l\geq l_0} O_{n_{k_l}}\Big) < \varepsilon/3.$$

Set $O = \bigcup_{l\geq l_0} O_{n_{k_l}}$; then, O is an open set, so it is Borel. For every $m \geq l_0$, we have

$$\begin{aligned}
&|\mu(O) - \mu_{n_{k_m}}(O)| = \\
&= |\mu(O) - \mu_{n_{k_m}}(O_{n_{k_m}}) - \mu_{n_{k_m}}\Big(\bigcup_{\substack{l\geq l_0\\ l\neq m}} O_{n_{k_l}}\Big)| \geq \\
&\geq |\mu_{n_{k_m}}(O_{n_{k_m}})| - |\mu(O)| - |\mu_{n_{k_m}}\Big(\bigcup_{\substack{l\geq l_0\\ l\neq m}} O_{n_{k_l}}\Big)| \geq \\
&\geq |\mu_{n_{k_m}}(O_{n_{k_m}})| - |\mu|(O) - |\mu_{n_{k_m}}|\Big(\bigcup_{\substack{l\geq l_0\\ l\neq m}} O_{n_{k_l}}\Big) > \\
&> \varepsilon - \varepsilon/3 - \varepsilon/3 = \varepsilon/3,
\end{aligned}$$

which shows that $(\mu_{n_{k_l}}(O))_{l\in\mathbb{N}}$ does not converge to $\mu(O)$—a contradiction with Corollary 19.2.7, as $(\mu_{n_{k_l}})_{l\in\mathbb{N}}$ converges weakly to μ. Thus, equality (DG) holds.

Assume now that (DG) holds, yet $\mathcal{M}$ is not relatively weakly compact. By the Eberlein–Šmulian theorem, there is a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ such that none of its subsequences is weakly convergent. Set

$$\lambda = \sum_{n\in\mathbb{N}} |\mu_n|/2^{n+1},$$

so λ is a non-negative element of $C(K)'$ (recall that $\mathcal{M}$ is bounded), with respect to which every μ_n is absolutely continuous.

By Lemma 19.4.6, the set $\{\mu_n : n \in \mathbb{N}\}$ is not uniformly countably additive. It follows that there is no non-negative measure $\varrho \in C(K)'$ such that for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $B \in Bor(K)$ if $\varrho(B) < \delta$, then $|\mu_n(B)| < \varepsilon$ for every $n \in \mathbb{N}$ (otherwise, by the σ-additivity of ϱ, $\{\mu_n : n \in \mathbb{N}\}$ would be trivially uniformly countably additive). In particular, there are a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$, a sequence $(B_k)_{n\in\mathbb{N}}$ of Borel subsets of K, and $\varepsilon > 0$ such that

$$\lambda(B_k) \leq 1/2^{k+2} \quad \text{and} \quad |\mu_{n_k}|(B_k) \geq \varepsilon$$

for every $k \in \mathbb{N}$. By the regularity of λ, we may enlarge each B_k to an open set U_k such that

$$\lambda(U_k) \leq 1/2^{k+1} \quad \text{and} \quad |\mu_{n_k}|(U_k) \geq \varepsilon. \tag{19.1}$$

For each $k \in \mathbb{N}$, set

$$V_k = \bigcup_{m=k}^{\infty} U_m.$$

The sequence $(V_k)_{k\in\mathbb{N}}$ is decreasing and so, by (19.1), we have

$$\lim_{k\to\infty} \lambda(V_k) = 0. \tag{19.2}$$

Also, for each $k \in \mathbb{N}$, we have

$$|\mu_{n_k}|(V_k) \geq \varepsilon. \tag{19.3}$$

Since we assume that (DG) holds, by Lemma 19.4.7.(2), for each $k \in \mathbb{N}$, there is a compact set $K_k \subseteq V_k$ such that

$$|\mu_{n_l}|(V_k \setminus K_k) \leq \varepsilon/2^{k+2} \tag{19.4}$$

for every $l \in \mathbb{N}$. For every $k \in \mathbb{N}$, set $F_k = K_0 \cap \ldots \cap K_k$, so $F_k \subseteq K_k \subseteq V_k$. Since

$$V_k \setminus F_k \subseteq (V_0 \setminus K_0) \cup \ldots \cup (V_k \setminus K_k),$$

by (19.3) and (19.4), we have

$$|\mu_{n_k}|(F_k) = |\mu_{n_k}|(V_k) - |\mu_{n_k}|(V_k \setminus F_k) \geq$$

$$\geq |\mu_{n_k}|(V_k) - \sum_{m=0}^{k} |\mu_{n_k}|(V_m \setminus K_m) \geq$$

$$\geq \varepsilon - \sum_{m=0}^{k} \varepsilon/2^{k+2} > \varepsilon/2. \tag{19.5}$$

The sequence $(F_k)_{k\in\mathbb{N}}$ is a decreasing sequence of compact sets, so $F = \bigcap_{k\in\mathbb{N}} F_k$ is a compact set such that $\lambda(F) = 0$ (by (19.2)), and hence, $|\mu_{n_k}|(F) = 0$ for every $k \in \mathbb{N}$ (by the definition of λ). By Lemma 19.4.7.(1), there is an open set W containing F such that

$$|\mu_{n_k}|(W) = |\mu_{n_k}|(W \setminus F) \leq \varepsilon/4 \tag{19.6}$$

for every $k \in \mathbb{N}$. Since W is open and each F_k is compact, there is $m \in \mathbb{N}$ such that $F_m \subseteq W$, hence, by (19.5) and (19.6), we have

$$\varepsilon/2 \leq |\mu_{n_m}|(F_m) \leq |\mu_{n_m}|(W) \leq \varepsilon/4,$$

a contradiction. It follows that $\mathcal{M}$ is relatively weakly compact. □

The following rephrasing of Dieudonné–Grothendieck's theorem serves as a starting point for proofs of numerous theorems concerning the Grothendieck property for $C(K)$-spaces.

Corollary 19.4.9 *Let K be a compact space. Assume that $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $C(K)'$ which is weak* convergent to $\mu \in C(K)'$ but not weakly convergent. Then, there are a strictly increasing sequence $(n_p)_{p\in\mathbb{N}}$ of natural numbers, a sequence $(U_p)_{p\in\mathbb{N}}$ of pairwise disjoint open subsets of K, and $\varepsilon > 0$ such that*

$$|\mu_{n_p}(U_p)| \geq \varepsilon$$

for every $p \in \mathbb{N}$.

Proof We first show that $(\mu_n)_{n\in\mathbb{N}}$ is not relatively weakly compact, so, for the sake of contradiction, assume it is. By the Eberlein–Šmulian theorem, each subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ contains yet another subsequence $(\mu_{n_{k_l}})_{l\in\mathbb{N}}$ which is weakly convergent to some measure $\nu \in C(K)'$. Since the weak* topology is weaker than the weak topology, it follows that actually $\nu = \mu$. Consequently, each subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ contains a subsequence $(\mu_{n_{k_l}})_{l\in\mathbb{N}}$ which is weakly convergent to μ. This however implies that $(\mu_n)_{n\in\mathbb{N}}$ itself is weakly convergent to μ, a contradiction.

By Dieudonné–Grothendieck's theorem, there are a sequence $(O_k)_{n\in\mathbb{N}}$ of pairwise disjoint open subsets of K and $\varepsilon > 0$ such that

$$\limsup_{k\to\infty} \left(\sup_{n\in\mathbb{N}} |\mu_n(O_k)| \right) > \varepsilon.$$

Hence, there are sequences $(k_l)_{l\in\mathbb{N}}$ and $(k'_l)_{l\in\mathbb{N}}$ of natural numbers such that $k_l < k_{l+1}$ and

$$|\mu_{k'_l}(O_{k_l})| > \varepsilon$$

for every $l \in \mathbb{N}$. Since $(O_{k_l})_{l\in\mathbb{N}}$ consists of pairwise disjoint sets and each $\mu_{k'_l}$ is bounded, there is a strictly increasing sequence $(l_p)_{p\in\mathbb{N}}$ of natural numbers such that for every $p \in \mathbb{N}$ we have $k'_{l_p} < k'_{l_{p+1}}$. For each $p \in \mathbb{N}$, set $n_p = k'_{l_p}$ and $U_p = O_{k_{l_p}}$. □

19.4.3 Proof of Grothendieck's Theorem

We are in the position to prove Grothendieck's theorem.

Theorem 19.4.10 (Grothendieck [286, Théorème 9]) *Let K be an extremely disconnected compact space. Then, the Banach space $C(K)$ is a Grothendieck space.*

Proof Assume that $C(K)$ is not a Grothendieck space, so there is a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $C(K)'$ which is weak* convergent but not weakly convergent. By subtracting the weak* limit, we may actually assume that $(\mu_n)_{n\in\mathbb{N}}$ converges weak* to 0. Corollary 19.4.9 implies that there are a strictly increasing sequence $(n_p)_{p\in\mathbb{N}}$ of natural numbers, a sequence $(U_p)_{p\in\mathbb{N}}$ of pairwise disjoint open subsets of K, and $\varepsilon > 0$ such that

$$|\mu_{n_p}(U_p)| \geq 3\varepsilon$$

for every $p \in \mathbb{N}$. Since K is totally disconnected, it has a base consisting of clopen subsets, so, by the regularity of each μ_{n_p}, we can find for each $p \in \mathbb{N}$ a clopen set $V_p \subseteq U_p$ such that

$$|\mu_{n_p}(V_p)| \geq \varepsilon. \tag{1}$$

By Rosenthal's lemma (Corollary 19.4.4), we can find a strictly increasing sequence $(p_k)_{k\in\mathbb{N}}$ such that

$$|\mu_{n_{p_k}}|\Big(\overline{\bigcup_{\substack{l\in\mathbb{N}\\ l\neq k}} V_{p_l}}\Big) < \varepsilon/2 \tag{2}$$

for every $k \in \mathbb{N}$. Set

$$V = \overline{\bigcup_{k\in\mathbb{N}} V_{p_k}},$$

and note that for every $k \in \mathbb{N}$ it holds

$$V \setminus V_{p_k} = \overline{\bigcup_{\substack{l\in\mathbb{N}\\ l\neq k}} V_{p_l}}. \tag{3}$$

Since K is extremely disconnected, V is clopen in K. By (3) and then (1) and (2), for every $k \in \mathbb{N}$, we have

$$|\mu_{n_{p_k}}(V)| =$$

$$= |\mu_{n_{p_k}}(V_{p_k}) + \mu_{n_{p_k}}(V \setminus V_{p_k})| = |\mu_{n_{p_k}}(V_{p_k}) + \mu_{n_{p_k}}\Big(\overline{\bigcup_{\substack{l\in\mathbb{N}\\ l\neq k}} V_{p_l}}\Big)| \geq$$

$$\geq |\mu_{n_{p_k}}(V_{p_k})| - |\mu_{n_{p_k}}\Big(\overline{\bigcup_{\substack{l\in\mathbb{N}\\ l\neq k}} V_{p_l}}\Big)| \geq |\mu_{n_{p_k}}(V_{p_k})| - |\mu_{n_{p_k}}|\Big(\overline{\bigcup_{\substack{l\in\mathbb{N}\\ l\neq k}} V_{p_l}}\Big) >$$

$$> \varepsilon - \varepsilon/2 = \varepsilon/2,$$

so $\limsup_{n\to\infty} |\mu_n(V)| \geq \varepsilon/2 > 0$, which contradicts the fact that $(\mu_n)_{n\in\mathbb{N}}$ is weak* convergent to 0. This contradiction implies that $C(K)$ is a Grothendieck space. □

Recall that for any set Γ, endowed with the discrete topology, its Čech–Stone compactification $\beta\Gamma$ is an extremely disconnected compact space for which the Banach space $C(\beta\Gamma)$ is isometrically isomorphic to $\ell_\infty(\Gamma)$. Thus, Theorem 19.4.10 immediately yields the following important corollary.

Corollary 19.4.11 *For any set Γ, the Banach space $\ell_\infty(\Gamma)$ is Grothendieck.*

19.5 Grothendieck $C(K)$-Spaces of Small Density

In the previous sections, we provided characterizations of Grothendieck $C(K)$-spaces in terms of existence of complemented copies of the Banach space c_0 (Sect. 19.2) or properties of Josefson–Nissenzweig sequences of Radon measures on K (Sect. 19.3). Also, a general characterization of Grothendieck Banach spaces in terms of weakly compact operators into separable Banach spaces was given (Theorem 19.2.1). Apparently, all these characterizations appeal to objects (functions, measures, operators) which are *external* with respect to underlying

compact spaces K. Unfortunately, as of yet no characterization of Grothendieck $C(K)$-spaces solely in terms of the topology of underlying compact spaces K have been found.

Section 19.4.3 contains a proof of Grothendieck's theorem asserting that for an extremely disconnected compact space K the Banach space $C(K)$ is Grothendieck (Theorem 19.4.10). An analysis of the proof of Grothendieck's theorem suggests that we do not require full extremal disconnectedness of K, but only that the closures of countable unions of (pairwise disjoint) clopen subsets of K are open. Recall that compact spaces in which open $\mathbb{F}_\sigma$-sets have open closures are called *basically disconnected* (cf., e.g. [628]). Thus, the statement of Grothendieck's theorem can be strengthened in the following way.

Theorem 19.5.1 (Grothendieck, Basically Disconnected Version) *Let K be a basically disconnected compact space. Then, the Banach space $C(K)$ is a Grothendieck space.*

It is easy to see that a totally disconnected compact space K is extremely disconnected if and only if the Boolean algebra $Clopen(K)$, consisting of all clopen subsets of K and equipped with the standard set-theoretic operations, is complete. Recall that a Boolean algebra $\mathcal{A}$ is *complete* if every antichain[4] $(a_i)_{i\in I}$ in $\mathcal{A}$ has supremum $\bigvee_{i\in I} a_i$ in $\mathcal{A}$. Also, $\mathcal{A}$ is *σ-complete* if every countable antichain in $\mathcal{A}$ has supremum in $\mathcal{A}$. Consequently, a totally disconnected compact space K is basically disconnected if and only if $Clopen(K)$ is σ-complete.

To simplify the language in what follows, let us say that a Boolean algebra $\mathcal{A}$ has *the Grothendieck property* if the Banach space $C(St(\mathcal{A}))$ is Grothendieck, where $St(\mathcal{A})$ denotes the Stone space of $\mathcal{A}$, that is, the totally disconnected compact Hausdorff space consisting of all ultrafilters on $\mathcal{A}$. Recall that $\mathcal{A}$ is isomorphic to the Boolean algebra $Clopen(St(\mathcal{A}))$ and that for every totally disconnected compact space K the Stone space $St(Clopen(K))$ is homeomorphic to K (for more basic information on Stone spaces, see e.g. [244]).

It is natural and totally reasonable to ask how the σ-completeness of the Boolean algebra $Clopen(K)$ in Grothendieck's Theorem 19.4.10 can be weakened and whether any such weakening can actually characterize all those totally disconnected compact spaces K whose Banach spaces $C(K)$ are Grothendieck. This path of research was followed by many mathematicians who proposed several weakenings of σ-completeness for Boolean algebras implying the Grothendieck property. Those new properties may be divided into two main groups, *completeness properties* and *separation properties* (also known as *interpolation properties*). To the first group belong those properties which give suprema of some subantichains of given antichains (much like in the case of the σ-completeness) and to the second one—those which give upper bounds (so not necessarily suprema) of subantichains of given antichains and additionally require that these provided upper bounds are

[4] A collection $(a_i)_{i\in I}$ in a Boolean algebra $\mathcal{A}$ is *an antichain* if $a_i \wedge a_j = 0$ for every $i \neq j \in I$.

disjoint from the rest of the antichains. Usually, every completeness property has a corresponding weaker interpolation property.

The main completeness properties are *property (E)* (Schachermayer [542]), *the Up-Down Semi-Completeness* (in short *UDSC*; Dashiell [151]), *the Subsequential Completeness Property* (*SCP*; Haydon [303]), and *the Weak Subsequential Completeness Property* (*WSCP*; Aizpuru [2]). To the group of separation/interpolation properties belong primarily *the Interpolation Property* (or *property (I)*; Seever [551]), *property (f)* (Moltó [449]), *the Subsequential Interpolation Property* (*SIP*; Freniche [253]), *the Weak Subsequential Interpolation Property* (*WSIP*; Freniche [255]), and *the Subsequential Separation Property* (*SSP*; Haydon [304]).

As said, if a Boolean algebra $\mathcal{A}$ has any of the above-mentioned properties, then it has the Grothendieck property. To give the reader at least a flavour of those all properties, let us provide the definitions of the SCP and SSP, both due to Haydon [303, 304]:

- A Boolean algebra $\mathcal{A}$ has *the Subsequential Completeness property* (*the SCP*) if for every infinite countable antichain $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ there is an infinite set $M \subseteq \mathbb{N}$ such that the supremum $\bigvee_{n\in M} A_n$ exists in $\mathcal{A}$.
- A Boolean algebra $\mathcal{A}$ has *the Subsequential Separation Property* (*the SSP*) if for every infinite countable antichain $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ there are an infinite set $M \subseteq \mathbb{N}$ and an element $A \in \mathcal{A}$ such that $A_{2n} \leq A$ and $A_{2n+1} \wedge A = 0_{\mathcal{A}}$ for every $n \in M$.

It is immediate that the σ-completeness implies the SCP which itself further implies the SSP. For examples that the converse implications do not hold, see [303] and [253].

It appears (at least consistently) that none of the above-named properties is necessary for a Boolean algebra $\mathcal{A}$ to have the Grothendieck property. To explain this, we need yet another separation property for Boolean algebras, introduced by Koszmider and Shelah [373]: a Boolean algebra $\mathcal{A}$ has *the Weak Subsequential Separation Property* (*the WSSP*) if and only if for every infinite countable antichain $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ there is an element $A \in \mathcal{A}$ such that both of the sets $\{n \in \mathbb{N}\colon\ A_n \leq A\}$ and $\{n \in \mathbb{N}\colon\ A_n \wedge A = 0_{\mathcal{A}}\}$ are infinite. One can easily see that this property is weaker than any other property mentioned above (in particular, than the SCP and SSP). But, by contrast, it does not imply the Grothendieck property: there exists a Boolean algebra $\mathcal{A}$ which has the WSSP but the space $C(St(\mathcal{A}))$ is not Grothendieck (see [373, Proposition 2.5]). It appears however that Boolean algebras with the WSSP have another surprising property: they contain large independent families.

Recall that a subset $\mathcal{I}$ of a Boolean algebra $\mathcal{A}$ is *independent* if for every disjoint finite non-empty sets $F, G \subseteq \mathcal{I}$ we have

$$\Big(\bigwedge_{A\in F} A\Big) \wedge \Big(\bigwedge_{A\in G} A^c\Big) \neq 0_{\mathcal{A}}.$$

Koszmider and Shelah proved the following theorem ([373, Theorem 1.4]).

Theorem 19.5.2 (Koszmider–Shelah [373, Theorem 1.4]) *Every infinite Boolean algebra with the WSSP contains an independent family of size continuum* $\mathfrak{c}$ *(the cardinality of* $\mathbb{R}$*).*

Theorem 19.5.2 implies that every infinite Boolean algebra having any of the properties named in the previous paragraphs (like the SCP or SSP) contains an independent family of size $\mathfrak{c}$ and thus has itself size at least $\mathfrak{c}$. On the hand, assuming the Continuum Hypothesis CH, Talagrand [580] obtained the following striking result.

Theorem 19.5.3 (Talagrand [580]) *Assuming CH, there is a totally disconnected compact space* K *such that* $C(K)$ *is Grothendieck and* $Clopen(K)$ *does not contain uncountable independent families.*

In other words, Talagrand's result provides an example of a Boolean algebra with the Grothendieck property but without uncountable independent families. Theorems 19.5.2 and 19.5.3 imply hence together that under CH none of the above-named properties can actually characterize Boolean algebras with the Grothendieck property.

Talagrand's CH example [580] has necessarily size $\mathfrak{c}$, as no countable infinite Boolean algebra may have the Grothendieck property (since then its Stone space would be metrizable and thus contain a non-trivial convergent sequence). In the light of Theorem 19.5.2, one can thus ask whether every infinite Boolean algebra with the Grothendieck property has size at least $\mathfrak{c}$. As said, this happens under CH, but one can show that it is also a consequence of Martin's axiom (see [567]). However, in general, this is not true. Brech [102] and Sobota and Zdomskyy [568] (see also [570]) proved that consistently the inequality $\omega_1 < \mathfrak{c}$ holds and there exist infinite Boolean algebras with the Grothendieck property and of cardinality ω_1. (Since the detailed statement of the latter results requires familiarity with the set-theoretic method of forcing, we will omit it.)

Recall that for every compact space K, we have $dens(C(K)) = w(K)$, that is, the density of $C(K)$ is equal to the weight of K. If K is totally disconnected, then $w(K) = |Clopen(K)|$, so in particular $dens(C(K)) = |Clopen(K)|$. It follows that if K is a totally disconnected compact space such that the Boolean algebra $Clopen(K)$ has the WSSP, then $dens(C(K)) \geq \mathfrak{c}$. Similarly, by the aforementioned results of Brech [102], Sobota [567], and Sobota and Zdomskyy [568], we get the following theorem.

Theorem 19.5.4 (Brech–Sobota–Zdomskyy) *(1) Under the assumption of the Continuum Hypothesis or Martin's axiom, for every compact space* K*, if the Banach space* $C(K)$ *is Grothendieck, then* $dens(C(K)) \geq \mathfrak{c}$.

(2) Consistently, there exists an infinite totally disconnected compact space K *such that the Banach space* $C(K)$ *is Grothendieck and* $dens(C(K)) = \omega_1 < \mathfrak{c}$.

Chapter 20
The ℓ_1-Grothendieck Property for $C(K)$-Spaces

Abstract This chapter studies a weak variant of the Grothendieck property for Banach spaces $C(K)$, called the ℓ_1-Grothendieck property. We characterize the ℓ_1-Grothendieck property in terms of operators onto the space c_0 (this time endowed with the pointwise topology) and in terms of sequences of (finitely supported) Radon measures related to Josefson–Nissenzweig's theorem. We present a construction of a separable compact space K such that $C(K)$ has the ℓ_1-Grothendieck property but it does not have the Grothendieck property. Generalizing the classical result of Cembranos and Freniche, stating that the Banach space $C(K \times L)$ does not have the Grothendieck property for any infinite compact spaces K and L, we show that spaces of the form $C(K \times L)$ have never even the ℓ_1-Grothendieck property—the proof is constructive and based on tools from probability theory. We also study the ℓ_1-Grothendieck property for spaces $C(K)$ where K is the limit of an inverse system based on simple extensions of totally disconnected compact spaces.

20.1 The ℓ_1-Grothendieck Property and Josefson–Nissenzweig's Theorem

Recall that a Banach space E is *Grothendieck* (or has *the Grothendieck property*) if every weak* convergent sequence in the dual space E' is also weakly convergent. At the end of Chap. 19, in Sect. 19.5, we mentioned and briefly discussed a long-standing open problem of finding an internal (topological) characterization of those compact spaces K for which their Banach spaces $C(K)$ are Grothendieck. As this problem seems particularly difficult and probably sensitive to the assumed system of axioms of set theory, it is reasonable to consider simpler variants of the Grothendieck property, which potentially might be also fruitful for the full property itself. In this chapter, we will study two such variants.

Let K be a compact (Hausdorff) space and $\mathcal{M} \subseteq C(K)'$. The space $C(K)$ is *an $\mathcal{M}$-Grothendieck space* (or has *the $\mathcal{M}$-Grothendieck property*) if every weak* convergent sequence $(\mu_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{M}$ is weakly convergent. For brevity, we also say in this case that K has *the $\mathcal{M}$-Grothendieck property*. Of course,

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if $\mathcal{M} = C(K)'$, then the $\mathcal{M}$-Grothendieck property is simply *the* Grothendieck property (as defined above).

Recall that by the Riesz–Markov–Kakutani representation theorem, we may identify the dual space $C(K)'$ with the space of (Radon) measures $M(K)$ on K (see Chap. 19), so we may choose $\mathcal{M}$ to be some special class of measures on K. By $\Delta(K)$, we denote the (non-closed) linear span of the subset $\{\delta_x : x \in K\}$ of $C(K)'$, that is, the (non-closed) linear subspace of $C(K)'$ consisting of all measures on K which are finite combinations of Dirac one-point measures. Similarly, by $\ell_1(K)$, we denote the (closed) linear subspace of $C(K)'$ consisting of all measures which are countable combinations of Dirac one-point measures, that is, $\mu \in \ell_1(K)$ if and only if μ can be represented as the sum $\mu = \sum_{x\in X} \alpha_x \delta_x$ for some countable subset $X \subseteq K$ and coefficients $\{\alpha_x \in \mathbb{R} : x \in X\}$. Note that the standard ℓ_1-norm on $\ell_1(K)$ coincides with the variation norm inherited from $C(K)'$ and that $\ell_1(K)$ is a complemented subspace of $C(K)'$ (which follows by the existence of a unique decomposition of each measure $\mu \in C(K)'$ into the non-atomic and purely atomic parts; see [398, Theorem 2.8.9]). Of course, $\Delta(K) \subseteq \ell_1(K)$. If $C(K)$ (so K) has the $\Delta(K)$-Grothendieck property (resp. $\ell_1(K)$-Grothendieck property), then we simply say that $C(K)$ (and K) has *the* Δ*-Grothendieck property* (resp. *the* ℓ_1*-Grothendieck property*). In Theorem 20.1.1, we prove that the Δ-Grothendieck property and the ℓ_1-Grothendieck property are actually equivalent.

The class of all ℓ_1-Grothendieck compact spaces contains of course all compact spaces with the Grothendieck property, but not only—in Sect. 20.3 we will describe a general scheme of constructing compact spaces which have the ℓ_1-Grothendieck property but not the Grothendieck property. In particular, one can construct such a space which is separable.

The class of compact spaces which do not have the ℓ_1-Grothendieck property contains e.g. the following spaces:

- spaces containing non-trivial convergent sequences (see Sect. 19.2 for a proof; cf. also Marciszewski and Sobota [429] for a more general result),
- products $K \times L$ for infinite compact space K and L (Kąkol, Sobota, Zdomskyy [353], see Sect. 20.4; cf. also [341]),
- limits of inverse systems based on simple extensions (Kąkol, Sobota, Zdomskyy [354], see Sect. 20.5),
- the Stone space $St(\mathcal{J})$ of the Boolean algebra $\mathcal{J}$ of Jordan measurable subsets of [0, 1] (Schachermayer [542]; cf. Chap. 21).

In Sect. 19.3, we studied relations between Grothendieck $C(K)$-spaces and the classical Josefson–Nissenzweig theorem (which, recall, asserts that every infinite-dimensional Banach space admits a weak* null sequence of norm-1 functionals). It appears that both the Δ-Grothendieck property and the ℓ_1-Grothendieck property can be also characterized in terms of JN-sequences. In order to do this properly, we need to introduce the following notions.

Let X be a Tychonoff space. By $Bor(X)$, we denote the σ-field of all Borel subsets of X. A function $\mu : Bor(X) \to \mathbb{R}$ is called *a (Borel) measure on* X if $\mu \in \mathrm{ca}(Bor(X))$, is *finite*, that is, has finite *total variation*, and is *regular*, that is,

its variation $|\mu|$ is *inner regular* (with respect to compact sets) and *outer regular* (with respect to open sets) (where the respective definitions are identical to the ones provided in Sect. 19.1 for Radon measures on compact spaces); μ may get negative values, that is, it may be *signed*. For a measure μ on X and a function $f \in C(X)$, we will simply write $\mu(f) = \int_X f d\mu$ (assuming the integral exists). A measure μ on X is *finitely supported* (resp. *countably supported*) if μ can be written as a finite (resp. countable) combination of one-point measures, that is, there is a finite (resp. countable) subset F of X such that $\mu = \sum_{x\in F} \alpha_x \delta_x$ for some set $\{\alpha_x \in \mathbb{R}\setminus\{0\} : x \in F\}$ of coefficients. Note that in this case, for *the total variation norm* $\|\mu\|$ of μ, we have $\|\mu\| = \sum_{x\in F} |\alpha_x|$, as well as $\mu(f) = \sum_{x\in F} \alpha_x f(x)$ for every $f \in C(X)$. If F is finite, then it is *the support* of μ and we write $\mathrm{supp}(\mu) = F$. Notice also that the space of all finitely supported measures on X is isomorphic to the dual space $C_p(X)'$ of $C_p(X)$. Furthermore, we say that a sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures on X is *finitely supported* (resp. *countably supported*) if for each $n \in \mathbb{N}$ the measure μ_n is finitely supported (resp. countably supported).

A countably supported sequence $(\mu_n)_{n\in\mathbb{N}}$ of Borel measures on a Tychonoff space X is *a countably supported Josefson–Nissenzweig sequence*, or *a csJN-sequence* in short, if $\|\mu_n\| = 1$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu_n(f) = 0$ for every $f \in C(X)$. Additionally, a csJN-sequence $(\mu_n)_{n\in\mathbb{N}}$ is *a finitely supported Josefson–Nissenzweig sequence*, or *an fsJN-sequence* in short, if it is finitely supported. Note that in the case of X being compact, each csJN-sequence on X is also a JN-sequence on X (as defined in Sect. 19.3).

A Tychonoff space X has *the finitely supported Josefson–Nissenzweig property*, or *the fsJNP* in short, if X admits an fsJN-sequence. Similarly, X has the *the countably supported Josefson–Nissenzweig property*, or *the csJNP* in short, if there is a csJN-sequence on X.[1]

The finitely supported Josefson–Nissenzweig property was first studied by Banakh, Kąkol, and Śliwa [62], who investigated its connections with the existence of complemented copies of the space c_0, endowed with the pointwise topology, in spaces $C_p(X)$ (see Sect. 20.2) as well as proved that $\beta\mathbb{N}$ does not have the fsJNP. Let us note that the latter result simply follows from the fact that $\beta\mathbb{N}$ is extremely disconnected, so it has the Grothendieck property (by Theorem 19.4.10), and, from the following theorem, proved in [354], which establishes the main connection between the ℓ_1-Grothendieck property of compact spaces and the finitely supported Josefson–Nissenzweig property.

Theorem 20.1.1 (Kąkol–Sobota–Zdomskyy [354, Theorem 4.1]) *Let K be an infinite compact space. Then, the following are equivalent:*

(1) K has the ℓ_1-Grothendieck property.
(2) K has the Δ-Grothendieck property.

[1] Let us note that in papers [341, 429], and [430] fsJN-sequences were simply called *JN-sequences*. Also, in [429], the fsJNP is called simply *the JNP*. In this chapter, we follow the nomenclature of [354].

(3) K does not have the fsJNP.
(4) K does not have the csJNP.

Proof Let us first show equivalence (1)⇔(2). As $\Delta(K) \subseteq \ell_1(K)$, the ℓ_1-Grothendieck property immediately implies the Δ-property. Assume now that K has the Δ-Grothendieck property and let $(\mu_n)_{n\in\mathbb{N}} \subseteq \ell_1(K)$ be weak* null. For each $n \in \mathbb{N}$, find a finite set $F_n \subseteq K$ such that $\|\mu_n \restriction (K \setminus F_n)\| < 1/n$. It follows that $(\mu_n \restriction F_n)_{n\in\mathbb{N}}$ is weak* null, too. Indeed, for every $f \in C(K)$ and $n \in \mathbb{N}$, we have

$$|(\mu_n \restriction F_n)(f)| \leq |\mu_n(f)| + |(\mu_n \restriction (K \setminus F_n))(f)| < |\mu_n(f)| + \|f\|_\infty \cdot 1/n,$$

so $\lim_{n\to\infty}(\mu_n \restriction F_n)(f) = 0$.

Similarly, for every $x^{**} \in C(K)^{**}$, we have

$$|x^{**}(\mu_n)| \leq |x^{**}(\mu_n \restriction F_n)| + |x^{**}(\mu_n \restriction (K \setminus F_n))| \leq$$

$$\leq |x^{**}(\mu_n \restriction F_n)| + \|x^{**}\| \cdot 1/n,$$

so $\lim_{n\to\infty} |x^{**}(\mu_n)| = 0$, since by the Δ-Grothendieck property the sequence $(\mu_n \restriction F_n)_{n\in\mathbb{N}}$ is weakly null. This proves that K has the ℓ_1-Grothendieck property.

Next, we show equivalence (3)⇔(4). If K has the fsJNP, then K has trivially also the csJNP, since $\Delta(K) \subseteq \ell_1(K)$. Let us thus assume that K has the csJNP and let $(\mu_n)_{n\in\mathbb{N}}$ be a csJN-sequence. For each $n \in \mathbb{N}$, let F_n be a finite subset of $\operatorname{supp}(\mu_n)$ such that $\|\mu_n \restriction (K \setminus F_n)\| < 1/n$, so $\|\mu_n \restriction F_n\| > 1 - 1/n$. For every $n \in \mathbb{N}$, define the measure ν_n on K as follows:

$$\nu_n = (\mu_n \restriction F_n) \Big/ \|\mu_n \restriction F_n\|;$$

then, $\nu_n \in \Delta(K)$ and $\|\nu_n\| = 1$. For every $f \in C(K)$, we have

$$|\nu_n(f)| = |(\mu_n \restriction F_n)(f)| \Big/ \|\mu_n \restriction F_n\| \leq$$

$$\leq \Big(|\mu_n(f)| + |(\mu_n \restriction (K \setminus F_n))(f)|\Big) \Big/ \|\mu_n \restriction F_n\| <$$

$$\Big(|\mu_n(f)| + \|f\|_\infty/n\Big) \Big/ (1 - 1/n),$$

so $\lim_{n\to\infty} \nu_n(f) = 0$, since $\lim_{n\to\infty} \mu_n(f) = 0$, which implies that $(\nu_n)_{n\in\mathbb{N}}$ is weak* null. It follows that $(\nu_n)_{n\in\mathbb{N}}$ is an fsJN-sequence on K and hence K has the fsJNP.

Finally, we show equivalence (1)⇔(4). Assume that K has the ℓ_1-Grothendieck property. Suppose additionally that K has the csJNP, so there is a csJN-sequence $(\mu_n)_{n\in\mathbb{N}}$ on K. Put $L = \bigcup_{n\in\mathbb{N}} \operatorname{supp}(\mu_n)$. By the ℓ_1-Grothendieck property, $(\mu_n)_{n\in\mathbb{N}}$ is weakly null. The Hahn–Banach theorem implies then that $(\mu_n)_{n\in\mathbb{N}}$ is also weakly

null as a sequence of elements of the space $\ell_1(L)$. Since the space $\ell_1(L)$ has the Schur property, it follows that

$$\lim_{n\to\infty} \|\mu_n\|_{C(K)'} = \lim_{n\to\infty} \|\mu_n\|_{\ell_1(L)} = 0,$$

which contradicts the fact that $\|\mu_n\|_{C(K)'} = 1$ for every $n \in \mathbb{N}$. Thus, K cannot have the csJNP.

Assume now that K does not have the csJNP. Suppose additionally that K does not have the ℓ_1-Grothendieck property, so there exists a weak* null sequence $(\mu_n)_{n\in\mathbb{N}} \subseteq \ell_1(K)$ which is not weakly null. Since the weak topology is weaker than the norm topology, it follows that there exist a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ and $\varepsilon > 0$ such that for every $k \in \mathbb{N}$ we have $\|\mu_{n_k}\| > \varepsilon$. For every $k \in \mathbb{N}$, define the measure $\nu_k = \mu_{n_k}/\|\mu_{n_k}\|$ and note that $\nu_k \in \ell_1(K)$ and $\|\nu_k\| = 1$. Also, for every $f \in C(K)$, we have

$$|\nu_k(f)| = |\mu_{n_k}(f)|/\|\mu_{n_k}\| < |\mu_n(f)|/\varepsilon,$$

so $\lim_{k\to\infty} \nu_k(f) = 0$. It follows that $(\nu_k)_{k\in\mathbb{N}}$ is a csJN-sequence on K, which is a contradiction. □

Corollary 20.1.2 *If a compact space K has the Grothendieck property, then it does not have the fsJNP.*

Using the same methods as in the proof of implication (1)⇒(4), one can obtain the following fact (which also implies Corollary 20.1.2), concerning general JN-sequences on compact spaces with the Grothendieck property (see Sect. 19.3).

Proposition 20.1.3 *Let K be a compact space with the Grothendieck property. Assume that $(\mu_n)_{n\in\mathbb{N}}$ is a JN-sequence on K. For each $n \in \mathbb{N}$, write $\mu_n = \nu_n + \theta_n$, where $\nu_n \in \ell_1(K)$ and θ_n is non-atomic. Then, $\|\nu_n\| \to 0$, or equivalently $\|\theta_n\| \to 1$, as $n \to \infty$.*

We finish this introductory section mentioning the following two important and useful theorems, which essentially assert that—studying the finitely supported Josefson–Nissenzweig property of Tychonoff spaces—one can confine the attention to quite simple fsJN-sequences.

Theorem 20.1.4 (Marciszewski–Sobota–Zdomskyy [430]) *Let X be a Tychonoff space with the fsJNP. Then, there exists an fsJN-sequence $(\mu_n)_{n\in\mathbb{N}}$ on X such that:*

- $\operatorname{supp}(\mu_n) \cap \operatorname{supp}(\mu_m) = \emptyset$ *for every* $n \neq m \in \mathbb{N}$.
- $\bigcup_{n\in\mathbb{N}} \operatorname{supp}(\mu_n)$ *is a discrete subset of* X.

Theorem 20.1.5 (Marciszewski–Sobota–Zdomskyy [430]) *If a Tychonoff space X has the fsJNP, then either there is an fsJN-sequence $(\mu_n)_{n\in\mathbb{N}}$ on X such that $|\operatorname{supp}(\mu_n)| = 2$ for every $n \in \mathbb{N}$, or for every fsJN-sequence $(\mu_n)_{n\in\mathbb{N}}$ on X, we have $\lim_{n\to\infty} |\operatorname{supp}(\mu_n)| = \infty$.*

Note that the existence of an fsJN-sequence $(\mu_n)_{n\in\mathbb{N}}$ on a given Tychonoff space X such that $|\operatorname{supp}(\mu_n)| = 2$ for every $n \in \mathbb{N}$ does not imply that X contains a non-trivial convergent sequence—for a suitable example, see [430, Section 6.1].

20.2 The Finitely Supported Josefson–Nissenzweig Property and Complemented Copies of $(c_0)_p$ in $C_p(X)$-Spaces

The underlying set of the classical Banach space $c_0 = \{x \in \mathbb{R}^{\mathbb{N}}\colon \lim_{n\to\infty} x(n) = 0\}$ is naturally a subset of the space $\mathbb{R}^{\mathbb{N}}$; therefore, we can endow it with the product topology inherited from $\mathbb{R}^{\mathbb{N}}$. Let us denote the resulting space by $(c_0)_p$. Thus, $(c_0)_p$ is just the Banach space c_0 but endowed with the pointwise topology (which is, of course, weaker than the norm or weak topology).

In this section, we will prove the theorem of Banakh, Kąkol, and Śliwa ([62]) asserting that a Tychonoff space X has the finitely supported Josefson–Nissenzweig property if and only if the space $C_p(X)$ contains a complemented copy of the space $(c_0)_p$ (Theorem 20.2.4). Consequently, by Theorem 20.1.1, we will get that for a compact space K the space $C(K)$ has the ℓ_1-Grothendieck property if and only if $C_p(K)$ does not contain any complemented copies of $(c_0)_p$ (Corollary 20.2.5; cf. Theorem 19.2.11).

We first prove that every space $C_p(X)$ contains a copy of $(c_0)_p$ (not necessarily closed), as observed in [344]. Recall that in Chap. 19 we presented a standard argument that every Banach space $C(K)$ contains an isometrically isomorphic copy of the Banach space c_0 (Theorem 19.2.10).

Theorem 20.2.1 (Kąkol–Moltó–Śliwa [344, Theorem 3.1]) *Let X be an infinite Tychonoff space. Then, $C_p(X)$ contains a (not necessarily closed) copy of the space $(c_0)_p$. If X contains an infinite compact subset, then $C_p(X)$ contains a closed copy of the space $(c_0)_p$.*

Proof Fix an infinite subset W of X and let $(U_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint open subsets of X such that for every $n \in \mathbb{N}$ there exists $x_n \in U_n \cap W$. For every $n \in \mathbb{N}$, let $f_n\colon X \to [0, 1]$ be a continuous function such that $f_n(x_n) = 1$ and $f_n(x) = 0$ for every $x \in X \setminus U_n$.

Define the linear operator $T\colon (c_0)_p \to C_p(X)$ for every $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in c_0$ as follows:

$$T(\alpha) = \sum_{n\in\mathbb{N}} \alpha_n f_n.$$

We need to show that T is well defined, that is, that for every $\alpha \in c_0$ the function $T(\alpha)$ is continuous. So, fix $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ and let $f = T(\alpha)$. Let $\varepsilon > 0$. Then the set $M = \{n \in \mathbb{N}\colon |\alpha_n| \geq \varepsilon\}$ is finite. For $x \notin \bigcup_{n\in\mathbb{N}} U_n$, we have $f(x) = 0$. Consequently, the set

$$Y = \{x \in X\colon\ |f(x)| \geq \varepsilon\} = \bigcup_{n\in\mathbb{N}} \{x \in U_n\colon\ |f(x)| \geq \varepsilon\} =$$

$$= \bigcup_{n\in\mathbb{N}} \{x \in U_n\colon\ |\alpha_n| f_n(x) \geq \varepsilon\} = \bigcup_{n\in M} \{x \in U_n\colon\ |\alpha_n| f_n(x) \geq \varepsilon\} =$$

$$= \bigcup_{n\in M} \{x \in U_n\colon\ f_n(x) \geq \varepsilon/|\alpha_n|\}$$

is closed in X, and hence, $f^{-1}[(-\varepsilon, \varepsilon)] = X \setminus Y$ is open.

Let now $t \in \mathbb{R} \setminus \{0\}$ and $\varepsilon \in (0, |t|)$. Then, $0 \notin [t - \varepsilon, t + \varepsilon]$, so the set

$$f^{-1}[(t - \varepsilon, t + \varepsilon)] = \bigcup_{n\in\mathbb{N}} \{x \in U_n\colon\ t - \varepsilon < f(x) < t + \varepsilon\} =$$

$$= \bigcup_{n\in\mathbb{N}} \{x \in U_n\colon\ t - \varepsilon < \alpha_n f_n(x) < t + \varepsilon\} =$$

$$= \bigcup_{n\in\mathbb{N}} (\alpha_n f_n)^{-1}[(t - \varepsilon, t + \varepsilon)]$$

is open. It follows that f is continuous and that T is well-defined.

It is easy to see that T is a linear bijection onto its image $F = T(c_0)$. We show that T itself is continuous. Let $k \in \mathbb{N}$, $y_1, \ldots, y_k \in X$, $\varepsilon > 0$, and put

$$U = \{f \in F\colon\ |f(y_i)| < \varepsilon \text{ for } 1 \leq i \leq k\},$$

so U is a basic open neighbourhood of 0 in F. Let $m \in \mathbb{N}$ be such that $1/m < \varepsilon$ and

$$\{y_1, \ldots, y_k\} \subseteq U_0 \cup \ldots \cup U_m \cup (X \setminus \bigcup_{n\in\mathbb{N}} U_n).$$

Clearly, the sets

$$V_l = \{(\alpha_n)_{n\in\mathbb{N}} \in c_0\colon\ \max_{0\leq n\leq l} |\alpha_n| < 1/l\}, \quad l \in \mathbb{N},\ l > 0,$$

form a base of neighbourhoods of 0 in $(c_0)_p$. We have

$$T(V_m) = \Big\{ \sum_{n\in\mathbb{N}} \alpha_n f_n\colon\ (\alpha_n)_{n\in\mathbb{N}} \in c_0,\ \max_{0\leq n\leq m} |\alpha_n| < 1/m \Big\} \subseteq U.$$

Indeed, let $f = \sum_{n\in\mathbb{N}} \alpha_n f_n \in T(V_m)$ and $1 \leq i \leq k$. If $y_i \in U_j$ for some $0 \leq j \leq m$, then

$$|f(y_i)| = |\alpha_j| f_j(y_i) \leq |\alpha_j| < 1/m < \varepsilon.$$

If on the other hand $y_i \in X \setminus \bigcup_{n\in\mathbb{N}} U_n$, then $|f(y_i)| = 0 < \varepsilon$. Consequently, $f \in U$, and so $T(V_m) \subseteq U$. It follows that T is continuous.

We now show that T is open, so that $T^{-1}\colon F \to (c_0)_p$ is continuous. Let $l \in \mathbb{N}$, $l > 0$, and $\varepsilon \in (0, 1/l)$. Put

$$U' = \{f \in F\colon\ |f(x_i)| < \varepsilon \text{ for } 0 \leq i \leq l\},$$

so U' is a basic open neighbourhood of 0 in F. Let $f \in U'$, so $f = \sum_{n\in\mathbb{N}} \alpha_n f_n$ for some $(\alpha_n)_{n\in\mathbb{N}} \in c_0$ such that $|f(x_i)| = |\alpha_i| < \varepsilon < 1/l$ for every $0 \leq i \leq l$. We immediately get that $f \in T(V_l)$. Consequently, $U' \subseteq T(V_l)$. It follows that T is open, and hence, T^{-1} is also continuous.

Since both T and T^{-1} are continuous, T is an isomorphism onto its image, and hence, $C_p(X)$ contains an isomorphic (possibly non-closed) copy of $(c_0)_p$.

We now assume that W is compact and show that the above-constructed isomorphic copy F of $(c_0)_p$ in $C_p(X)$ is closed. Let $(f_\xi)_{\xi\in I}$ be a net in F converging to some $f \in C_p(X)$. For every $\xi \in I$, there is $(\alpha_n^\xi)_{n\in\mathbb{N}} \in c_0$ such that $f_\xi = \sum_{n\in\mathbb{N}} \alpha_n^\xi f_n$. For every $n \in \mathbb{N}$, let $\beta_n = f(x_n)$.

Let $n \in \mathbb{N}$. Then

$$\alpha_n^\xi = \alpha_n^\xi f_n(x_n) = f_\xi(x_n) \xrightarrow[\xi\in I]{} f(x_n) = \beta_n,$$

so

$$f_\xi(x) = \alpha_n^\xi f_n(x) \xrightarrow[\xi\in I]{} \beta_n f_n(x)$$

for every $x \in U_n$. Hence, $f = \beta_n f_n$ on U_n.

Clearly, $f(x) = 0$ for every $x \in X \setminus \bigcup_{n\in\mathbb{N}} U_n$. It follows that for every $x \in X$, we have

$$f(x) = \sum_{n\in\mathbb{N}} \beta_n f_n(x),$$

so to finish the proof, it suffices to show that $(\beta_n)_{n\in\mathbb{N}} \in c_0$.

Let $\varepsilon > 0$ and put $P = \{n \in \mathbb{N}\colon |\beta_n| \geq \varepsilon\}$. We will show that P is finite. The function f is continuous, so the set $D = \{x \in X\colon |f(x)| \geq \varepsilon\}$ is closed and hence the intersection $W \cap D$ is compact. We have

$$D = \bigcup_{n\in\mathbb{N}} \{x \in U_n\colon\ |f(x)| \geq \varepsilon\} = \bigcup_{n\in\mathbb{N}} \{x \in U_n\colon\ |\beta_n| f_n(x) \geq \varepsilon\} =$$

$$= \bigcup_{n\in P} \{x \in U_n\colon\ |\beta_n| f_n(x) \geq \varepsilon\} \subseteq \bigcup_{n\in P} U_n.$$

Consequently, $W \cap D \subseteq \bigcup_{n\in P} U_n$; hence, $\{W \cap D \cap U_n : n \in P\}$ is an open cover of the compact set $W \cap D$. Since for each $n \in P$ we have $W \cap D \cap U_n \neq \emptyset$ (as $x_n \in D$) and the sets U_k's are pairwise disjoint, it follows by compactness of $W \cap D$ that P is finite. Hence, $(\beta_n)_{n\in\mathbb{N}} \in c_0$, and so $f \in F$. Consequently, F is a closed subspace of $C_p(X)$. □

We will now focus on the issue when a given space $C_p(X)$ contains a *complemented* copy of the space $(c_0)_p$, that is, there exist two closed linear subspaces G and H of $C_p(X)$ such that G is isomorphic to $(c_0)_p$, $C_p(X) = G+H$, $G\cap H = \{0\}$, and both of the canonical projections from $C_p(X)$ onto G and H are continuous.

Recall that a subset A of a topological space X is *(topologically) bounded*[2] if for every $f \in C(X)$ the image $f(A)$ is a bounded subset of $\mathbb{R}$. Thus, every subset of a (pseudo)compact space is bounded.

In the following lemma, for each $n \in \mathbb{N}$, e_n^* denotes the canonical functional on c_0 such that $e_n^*(x) = x(n)$ for every $x \in c_0$.

Lemma 20.2.2 *Let X be a Tychonoff space and $T : C_p(X) \to (c_0)_p$ a continuous linear surjection. For every $n \in \mathbb{N}$, define the functional $\mu_n \in C_p(X)'$ by $\mu_n(f) = e_n^*(T(f))$, $f \in C(X)$. Then, the union $\bigcup_{n\in\mathbb{N}} \operatorname{supp}(\mu_n)$ is bounded in X.*

Proof For every $n \in \mathbb{N}$, let $S_n = \operatorname{supp}(\mu_n)$ and put $S = \bigcup_{n\in\mathbb{N}} S_n$. For the sake of contradiction, assume that there exists $f \in C(X)$ such that $f(S)$ is an unbounded subset of $\mathbb{R}$. Without loss of generality, we may assume that $f(x) \geq 0$ for every $x \in X$.

First, as each set S_n is finite, we can inductively find a sequence $(S_{n_k})_{k\in\mathbb{N}}$ such that

$$\max f(S_{n_k}) > 3 + \max f(S_{n_i})$$

for every $k, i \in \mathbb{N}$ such that $k > i$. For every $k \in \mathbb{N}$, choose $x_k \in S_{n_k}$ such that $f(x_k) = \max f(S_{n_k})$. Since X is Tychonoff and f is continuous, for every $k \in \mathbb{N}$, we can find an open neighbourhood U_k of x_k such that

$$U_k \subseteq \overline{U_k} \subseteq \{x \in X : \ |f(x) - f(x_k)| < 1\}$$

and $U_k \cap S_{n_k} = \{x_k\}$. Observe that for any $k > i$ we have

$$f(x_k) - f(x_i) = f(x_k) - \max f(S_{n_i}) > 3,$$

which, by the triangle inequality, yields that $\overline{U_k} \cap (S_{n_i} \cup \overline{U_i}) = \emptyset$. We also have $\overline{\bigcup_{k\in\mathbb{N}} U_k} = \bigcup_{k\in\mathbb{N}} \overline{U_k}$. Indeed, let $x \in \overline{\bigcup_{k\in\mathbb{N}} U_k} \setminus \bigcup_{k\in\mathbb{N}} U_k$, then there is an open neighbourhood U of x such that for every $y \in U$ we have $|f(x) - f(y)| < 1/3$. It follows that there is a unique $k \in \mathbb{N}$ such that $U \cap U_k \neq \emptyset$. Consequently, $x \in \overline{U_k}$.

[2] Some authors use the name *functionally bounded*.

Now, having in mind that $U_k \cap S_{n_k} = \{x_k\}$, we can inductively construct a sequence $(f_k)_{k\in\mathbb{N}} \subseteq C(X)$ such that $\operatorname{supp}(f_k) \subseteq U_k$ and

$$\mu_{n_k}(f_k) > 1 + \sum_{i=0}^{k-1} |\mu_{n_k}(f_i)| \qquad (*)$$

for every $k \in \mathbb{N}$.

Finally, since $\overline{\bigcup_{k\in\mathbb{N}} U_k} = \bigcup_{k\in\mathbb{N}} \overline{U_k}$ and the sequence $(U_k)_{k\in\mathbb{N}}$ is pairwise disjoint, the function $g = \sum_{k\in\mathbb{N}} f_k$ is well-defined and continuous, that is, $g \in C(X)$. For every $k < i$, we have $U_i \cap S_{n_k} = \emptyset$, so, by $(*)$

$$|\mu_{n_k}(g)| = |\sum_{i=0}^{k} \mu_{n_k}(f_i)| \geq |\mu_{n_k}(f_k)| - \sum_{i=0}^{k-1} |\mu_{n_k}(f_i)| > 1,$$

which contradicts the fact that $\lim_{n\to\infty} \mu_n(g) = 0$. □

Proposition 20.2.3 *Assume that a Tychonoff space X has the fsJNP. Then, $C_p(X)$ contains a complemented subspace isomorphic to $(c_0)_p$.*

Proof Let $(\mu_n)_{n\in\mathbb{N}}$ be an fsJN-sequence on X. By Theorem 20.1.4, we may assume that $\operatorname{supp}(\mu_n) \cap \operatorname{supp}(\mu_{n'}) = \emptyset$ for every $n \neq n' \in \mathbb{N}$ and that $\bigcup_{n\in\mathbb{N}} \operatorname{supp}(\mu_n)$ is a discrete subset of X. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint open subsets of X such that $\operatorname{supp}(\mu_n) \subseteq U_n$ for every $n \in \mathbb{N}$.

Let an operator $S\colon C_p(X) \to (c_0)_p$ be defined for every $f \in C_p(X)$ by the formula:

$$S(f) = (\mu_n(f))_{n\in\mathbb{N}}.$$

It is immediate that S is well-defined, linear, and continuous.

For every $n \in \mathbb{N}$, let $f_n\colon X \to [-1, 1]$ be a continuous function such that $\operatorname{supp}(f_n) \subseteq U_n$ and $f_n(x) = \operatorname{sgn}(\mu_n(\{x\}))$ for every $x \in \operatorname{supp}(\mu_n)$. Then, for every $n \neq k \in \mathbb{N}$, we have $\mu_n(f_n) = \|\mu_n\| = 1$ and $\mu_n(f_k) = 0$.

Let $T\colon (c_0)_p \to C_p(X)$ be an operator defined for every $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in c_0$ by the formula:

$$T(\alpha) = \sum_{n\in\mathbb{N}} \alpha_n f_n.$$

As in the proof of Theorem 20.2.1, T is well-defined, linear, and continuous. Moreover, it is an isomorphism onto the image $T(c_0)$.

For every $\alpha \in c_0$, we have $(S \circ T)(\alpha) = \alpha$, so S is a surjection. The operator $P\colon C_p(X) \to C_p(X)$, defined as $P = T \circ S$, is a continuous linear projection and its image $P(C_p(X)) = T(c_0)$ is a complemented subspace of $C_p(X)$. Since $(c_0)_p$ and $T(c_0)$ are isomorphic, $(c_0)_p$ is isomorphic to a complemented subspace of $C_p(X)$. □

We are in the position to prove the main result of [62].

Theorem 20.2.4 (Banakh–Kąkol–Śliwa [62]) *For every Tychonoff space X, the following are equivalent:*

(1) X has the fsJNP.
(2) $C_p(X)$ contains a complemented copy of $(c_0)_p$.
(3) $C_p(X)$ has a quotient isomorphic to $(c_0)_p$.
(4) $C_p(X)$ admits a continuous linear surjection onto $(c_0)_p$.

Proof Implication (1)⇒(2) follows from Proposition 20.2.3 and implications (2)⇒(3) and (3)⇒(4) are obvious, so it suffices to show that (4) implies (1).

Let $T\colon C_p(X) \to (c_0)_p$ be a continuous linear surjection. For every $n \in \mathbb{N}$ and $f \in C(X)$, set

$$\mu_n(f) = e_n^*(T(f)),$$

where $e_n^* \in (c_0)_p'$ is the standard n-th coordinate functional (see the paragraph before Lemma 20.2.2). Note that $\mu_n \in C_p(X)'$ for each $n \in \mathbb{N}$. Obviously, for every $f \in C(X)$, we have $\lim_{n\to\infty} \mu_n(f) = 0$. By Lemma 20.2.2, the union $S = \bigcup_{n\in\mathbb{N}} \operatorname{supp}(\mu_n)$ is a bounded subset of X, so the following value

$$\|f\|_S = \sup\{|f(x)|\colon\ x \in S\}$$

is finite for every $f \in C(X)$.

Put

$$B = \{(x_n)_{n\in\mathbb{N}} \in c_0\colon |x_n| \le \|\mu_n\|,\ \forall n \in \mathbb{N}\}.$$

We claim that B is an absorbing subset of the space $(c_0)_p$. Let $x = (x_n)_{n\in\mathbb{N}} \in c_0$. Since T is a surjection, there is $f \in C(X)$ such that $T(f) = x$. For every $n \in \mathbb{N}$, we have

$$|x_n| = |e_n^*(T(f))| = |\mu_n(f)| \le \|\mu_n\| \cdot \|f\|_S,$$

so $x \in \|f\|_S \cdot B$, showing that B is absorbing. Clearly, B is also convex, balanced, and closed in $(c_0)_p$, so it is a barrel in the Banach space $(c_0, \|\cdot\|_\infty)$. Consequently, B is a neighbourhood of 0 in $(c_0, \|\cdot\|_\infty)$ and thus $\inf_{n\in\mathbb{N}} \|\mu_n\| > 0$. For every $n \in \mathbb{N}$, define the functional in $C_p(X)'$ as follows:

$$\nu_n = \mu_n \big/ \|\mu_n\|,$$

and note that the sequence $(\nu_n)_{n\in\mathbb{N}}$ is an fsJN-sequence on X, so X has the fsJNP. □

Theorems 20.1.1 and 20.2.4 yield together the following important corollary.

Corollary 20.2.5 *For a compact space K, the following are equivalent:*

(1) $C(K)$ has the ℓ_1-Grothendieck property.
(2) K does not have the fsJNP.
(3) $C_p(K)$ does not contain any complemented copies of $(c_0)_p$.

Since $(c_0)_p \simeq (c_0)_p \oplus (c_0)_p$ and $(c_0)_p \simeq C_p(\alpha\mathbb{N})$, by a similar argument as in Corollary 19.2.12, we obtain the following characterization of ℓ_1-Grothendieck $C(K)$-spaces in terms of spaces $C_p(L)$ for L containing non-trivial convergent sequences (first observed in [343]).

Corollary 20.2.6 *Let $\mathcal{C}$ be the class of all compact spaces containing non-trivial convergent sequences and $\mathcal{D} \subseteq \mathcal{C}$ its any subclass such that $\alpha\mathbb{N} \in \mathcal{D}$. For every compact space K, the following are equivalent:*

(1) $C(K)$ is not an ℓ_1-Grothendieck space.
(2) $C_p(K) \simeq C_p(K \sqcup \alpha\mathbb{N})$.
(3) $C_p(K) \simeq C_p(K \sqcup L)$ for some $L \in \mathcal{D}$.
(4) $C_p(K) \simeq C_p(L)$ for some $L \in \mathcal{C}$.

20.3 The Grothendieck Property vs. the ℓ_1-Grothendieck Property

In his unpublished note [500], Plebanek constructed in ZFC a compact space K such that its every separable closed subspace L has the Grothendieck property, but K itself does not have the property. It follows that K is not separable, but it has the ℓ_1-Grothendieck property.

Following the ideas of [500], Kąkol, Sobota, and Zdomskyy [354] constructed a separable compact space without the Grothendieck property, but with the ℓ_1-Grothendieck property. In this section, we present this construction (Theorem 20.3.3).

Lemma 20.3.1 *Let K be a totally disconnected compact space, μ a probability measure on K, and $(A_n)_{n\in\mathbb{N}}$ a sequence of mutually disjoint clopen subsets of K such that $\mu(A_n) > 0$ for every $n \in \mathbb{N}$. Define the set F as follows: $x \in F$ if and only if for every clopen neighbourhood U of x the following inequality is satisfied:*

$$\limsup_{n\to\infty} \frac{\mu(A_n \cap U)}{\mu(A_n)} > 0.$$

Then, F is closed and non-empty, and the quotient space K/F does not have the Grothendieck property.

Proof We first show that $F \neq \emptyset$. Assume for the sake of contradiction that for every $x \in K$ there exists its clopen neighbourhood U_x such that $\lim_n \mu(A_n \cap U_x)/\mu(A_n) = 0$. By compactness of K, there exists a finite cover $U_{x_1}, \ldots, U_{x_k}$ of K. We then have

$$1 = \lim_{n\to\infty} \frac{\mu(A_n \cap K)}{\mu(A_n)} \leq \sum_{i=1}^{k} \lim_{n\to\infty} \frac{\mu(A_n \cap U_{x_i})}{\mu(A_n)} = 0,$$

a contradiction.

Let us now prove that K/F does not have the Grothendieck property. Let $\varphi\colon K \to K/F$ be the quotient map. Denote $p = \varphi(F)$. For every $n \in \mathbb{N}$, define a measure μ_n on K/F as follows:

$$\mu_n(A) = \frac{\mu(A_n \cap \varphi^{-1}(A))}{\mu(A_n)},$$

where A is a clopen subset of K/F. Then, μ_n converges weak* to δ_p on K/F. Indeed, if A is a clopen in K/F not containing p, then $\varphi^{-1}(A) \cap F = \emptyset$ and hence, by compactness of $\varphi^{-1}(A)$, we have $\limsup_n \mu(A_n \cap \varphi^{-1}(A))/\mu(A_n) = 0$, and so $\lim_n \mu_n(A) = 0$. On the other hand, if $p \in A$, then

$$\lim_{n\to\infty} \mu_n(A) = \lim_{n\to\infty} \Big(\mu_n(K/F) - \mu_n(A^c)\Big) = 1 - \lim_{n\to\infty} \mu_n(A^c) = 1 - 0 = 1.$$

Had K/F the Grothendieck property, μ_n would converge weakly to δ_p and hence $\mu_n(\{p\})$ would converge to 1, which is not the case, since $A_n \cap F = \emptyset$ for every $n \in \mathbb{N}$ as elements of $(A_n)_{n\in\mathbb{N}}$ are mutually disjoint. □

Recall that a compact space in which every open $\mathbb{F}_\sigma$-set has open closure is called *basically disconnected* (cf. Sect. 19.5).

Lemma 20.3.2 *Let K be a basically disconnected compact space, μ a probability measure on K, and $(A_n)_{n\in\mathbb{N}}$ a sequence of mutually disjoint clopen subsets of K such that $\mu(A_n) > 0$ for every $n \in \mathbb{N}$. Let F be defined as in Lemma 20.3.1. Let $\mathcal{Z}$ denote the family of all clopen subsets C of K such that $\lim_{n\to\infty} \mu(A_n \cap C)\big/\mu(A_n) = 0$, i.e.*

$$\mathcal{Z} = \{C \subseteq K\colon\ C \text{ is clopen and } C \cap F = \emptyset\}.$$

Then, $\mathcal{Z}$ has the following pseudo-union-like property: for every sequence $(C_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{Z}$, there exists $C \in \mathcal{Z}$ such that

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}\ (C_n \setminus C \subseteq \bigcup_{j\leq m} A_j).$$

Proof The proof is similar to the standard one showing that the asymptotic density zero ideal on $\mathbb{N}$ is a P-ideal. Namely, inductively find a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of indices such that

$$\frac{\mu\Big(A_n \cap \bigcup_{i=0}^{k} C_i\Big)}{\mu(A_n)} < \frac{1}{k+1}$$

for every $n > n_k$ and $k \in \mathbb{N}$. Put

$$C = \overline{\bigcup_{k\in\mathbb{N}} \Big(C_k \setminus \bigcup_{j=0}^{n_k} A_j\Big)}.$$

Since K is basically disconnected, C is a clopen set. It follows easily that for every $k \in \mathbb{N}$ we have

$$C_k \setminus C \subseteq \bigcup_{j=0}^{n_k} A_j.$$

We shall now show that $C \in \mathcal{Z}$. Fix $n \in \mathbb{N}$, $n > n_0$, and let $k \in \mathbb{N}$ be such that $n_k < n \leq n_{k+1}$. We have

$$\frac{\mu(A_n \cap C)}{\mu(A_n)} \leq \frac{\mu\Big(A_n \cap \bigcup_{i=0}^{k} C_i\Big)}{\mu(A_n)} + \frac{\mu(A_n \cap D)}{\mu(A_n)} < \frac{1}{k+1} + \frac{\mu(A_n \cap D)}{\mu(A_n)},$$

where

$$D = C \setminus \Big(\bigcup_{i\in\mathbb{N}} (C_i \setminus \bigcup_{j=0}^{n_i} A_j)\Big),$$

so it is just the set added to $\bigcup_{i\in\mathbb{N}}(C_i \setminus \bigcup_{j=0}^{n_i} A_j)$ after taking the closure. Since $k \to \infty$ if $n \to \infty$, it is enough to show that $A_n \cap D = \emptyset$, whence $\mu(A_n \cap D)/\mu(A_n) = 0$. Assume to the contrary that there exists $x \in A_n \cap D$. Since $x \in D$, every neighbourhood of x intersects $C_k \setminus \bigcup_{j=0}^{n_k} A_j$ for infinitely many k. In particular, A_n must intersect $C_k \setminus \bigcup_{j=0}^{n_k} A_j$ for some k with $n_k > n$, which is impossible. □

We are ready to present the main theorem of this section.

Theorem 20.3.3 (Kąkol–Sobota–Zdomskyy [354, Theorem 5.3]) *For every infinite basically disconnected compact space K, there exists a compact space L and a continuous surjection $\varphi\colon K \to L$ such that L does not have the Grothendieck property, but it has the ℓ_1-Grothendieck property.*

Proof Let μ, $(A_n)_{n\in\mathbb{N}}$, F, and $\mathcal{Z}$ be as in Lemmas 20.3.1 and 20.3.2. Note that since K is not scattered we may assume that μ vanishes on points (see [552, Theorem 19.7.6]). Put $L = K/F$ and let φ be the quotient map. It follows from Lemma 20.3.1 that L does not have Grothendieck property. For the sake of

contradiction, assume that L does not have the ℓ_1-Grothendieck property either, so there is a disjointly supported JN-sequence $(\mu_n)_{n\in\mathbb{N}}$ on L (see Theorem 20.1.4). We may assume that $\varphi(F) \cap \operatorname{supp}(\mu_n) = \emptyset$ for every $n \in \mathbb{N}$, and hence, for every $n \in \mathbb{N}$, we can find $C_n \in \mathcal{Z}$ such that $\operatorname{supp}(\mu_n) \subseteq \varphi(C_n)$. Let $C \in \mathcal{Z}$ be like in Lemma 20.3.2 for the sequence $(C_n)_{n\in\mathbb{N}}$, i.e. $C_n \setminus C \subset \bigcup_{j=0}^{m_n} A_j$ for some increasing number sequence $(m_n)_{n\in\mathbb{N}}$. By Proposition 20.1.3, the Grothendieck property of K (and hence of $\varphi(C)$) yields

$$\lim_{n\to\infty} \|\mu_n \restriction \varphi(C)\| = 0,$$

which together with $\|\mu_n \restriction \varphi(C_n)\| = 1$ and $C_n \setminus C \subset \bigcup_{j=0}^{m_n} A_j$ gives

$$\lim_{n\to\infty} \|\mu_n \restriction \bigcup_{j=0}^{m_n} \varphi(A_j)\| = 1.$$

On the other hand, since for every finite $Q \subseteq \mathbb{N}$ we have

$$\lim_{n\to\infty} \|\mu_n \restriction \bigcup_{j\in Q} \varphi(A_j)\| = 0,$$

it follows that there exists a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ and a sequence $(Q_k)_{k\in\mathbb{N}}$ of pairwise disjoint subsets of $\mathbb{N}$ such that for every $k \in \mathbb{N}$ we have

$$\|\mu_{n_k} \restriction \bigcup_{j\in Q_k} \varphi(A_j)\| > 1/2.$$

Let B_k be a clopen subset of $\bigcup_{j\in Q_k} A_j$ containing $\operatorname{supp} \mu_{n_k} \cap \bigcup_{j\in Q_k} A_j$ and such that $\frac{\mu(B_k\cap A_j)}{\mu(A_j)} < 1/2^k$ for all $j \in Q_k$ (this is the only place where we use that μ vanishes on points). Set $D = \overline{\bigcup_{k\in\mathbb{N}} B_k}$ and note that D is a clopen subset of K such that $D \cap A_j = B_k \cap A_j$ for all $k \in \mathbb{N}$ and $j \in Q_k$, and $D \cap A_j = \emptyset$ for $j \in \mathbb{N} \setminus \bigcup_{k\in\mathbb{N}} Q_k$. Thus

$$\lim_{j\to\infty} \frac{\mu(A_j \cap D)}{\mu(A_j)} = 0,$$

which means $D \in \mathcal{Z}$, i.e. $D \cap F = \emptyset$. It follows from the above that

$$\|\mu_{n_k} \restriction \varphi(D)\| = \|\mu_{n_k} \restriction \bigcup_{j\in Q_k} \varphi(A_j)\| > 1/2$$

for every $k \in \mathbb{N}$, which is a contradiction, since K (and hence $\varphi(D)$) has the Grothendieck property. □

Considering $K = \beta\mathbb{N}$ and recalling that continuous surjections do not increase densities, we obtain the following corollary.

Corollary 20.3.4 *There exists a separable compact space L such that it does not have the Grothendieck property but it has the ℓ_1-Grothendieck property.*

Corollary 20.3.4 and Theorems 19.2.11 and 20.2.4 imply the next result.

Corollary 20.3.5 *There exists a separable compact space L such that $C(L)$ contains a complemented copy of c_0, but $C_p(L)$ does not contain any complemented copies of $(c_0)_p$.*

Corollaries 20.3.5, 19.2.12, and 20.2.6 together have the following consequence.

Corollary 20.3.6 *There exists a separable compact space L such that $C(L) \simeq C(L \sqcup \alpha\mathbb{N})$ and $C_p(L) \not\simeq C_p(L \sqcup \alpha\mathbb{N})$.*

Note that Corollary 20.3.6, with an aid of Corollary 20.2.6, implies that isomorphisms of Banach spaces $C(K)$ do not preserve the ℓ_1-Grothendieck property.

Let us say that a Boolean algebra $\mathcal{A}$ has *the ℓ_1-Grothendieck property* if its Stone space $St(\mathcal{A})$ has the ℓ_1-Grothendieck property (cf. Sect. 19.5). Recall that every (σ-)complete Boolean algebra has the Grothendieck property (by Theorem 19.4.10). By Theorem 20.3.3 and the Stone duality, we obtain the following corollary saying that every (σ-)complete Boolean algebra may be slimmed down in such a way that it loses the Grothendieck property but still preserves the ℓ_1-Grothendieck property.

Corollary 20.3.7 *For every σ-complete Boolean algebra $\mathcal{A}$, there exists a subalgebra $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B}$ does not have the Grothendieck property, but it has the ℓ_1-Grothendieck property.*

20.4 Spaces $C(K \times L)$ and Lack of the ℓ_1-Grothendieck Property

Cembranos [133] and Freniche [254] proved that, for every two infinite compact spaces K and L, the Banach space $C(K \times L)$ always contains a complemented copy of the space c_0, so, in particular, the product $K \times L$ never has the Grothendieck property. In this section, we present the stronger result of Kąkol, Sobota, and Zdomskyy [353] stating that for every infinite compact spaces K and L their product $K \times L$ always has the finitely supported Josefson–Nissenzweig property, or, equivalently, it never has the ℓ_1-Grothendieck property (Theorem 20.4.4). The proof is totally constructive, that is, the appropriate fsJN-sequence $(\mu_n)_{n\in\mathbb{N}}$ is defined directly by a simple formula and relies on standard probabilistic tools (like the weak law of large numbers).

Before we present the proof, we need to introduce auxiliary notation. As usual, we assume that $0 \in \mathbb{N}$ and by $\mathbb{N}_+$ we denote the set of positive integers, i.e. $\mathbb{N}_+ =$

$\mathbb{N} \setminus \{0\} = \{1, 2, 3, \ldots\}$. Also, we identify every $n \in \mathbb{N}$ with the set $\{0, \ldots, n-1\}$. For a set X by $\wp(X)$, we denote its power set.

For every $n \in \mathbb{N}_+$, put $\Omega_n = \{-1, 1\}^n$ and $\Sigma_n = n \times \{n\}$ (so $|\Omega_n| = 2^n$ and $|\Sigma_n| = n$). To simplify the notation, we will usually write $i \in \Sigma_n$ instead of $(i, n) \in \Sigma_n$—this should cause no confusion. Put also $\Omega = \bigcup_{n\in\mathbb{N}_+} \Omega_n$ and $\Sigma = \bigcup_{n\in\mathbb{N}_+} \Sigma_n$ and endow these two sets with the discrete topologies. This way, we can think of the product space $\Omega \times \Sigma$ as a countable union of pairwise disjoint discrete rectangles $\Omega_k \times \Sigma_m$ of size $m2^k$—the rectangles $\Omega_n \times \Sigma_n$ will bear a special meaning, namely, they will be the supports of measures from the sequence $(\mu_n)_{n\in\mathbb{N}_+}$ on the space $\beta\Omega \times \beta\Sigma$ (i.e. on the product of the Čech–Stone compactifications of Ω and Σ) defined as follows ($n \in \mathbb{N}_+$):

$$\mu_n = \sum_{\substack{s\in\Omega_n \\ i\in\Sigma_n}} \frac{s(i)}{n2^n}\delta_{(s,i)}.$$

Then, $\operatorname{supp}(\mu_n) = \Omega_n \times \Sigma_n$, so $|\operatorname{supp}(\mu_n)| = n2^n$, $\|\mu_n\| = 1$, and

$$\pi_i(\operatorname{supp}(\mu_n)) \cap \pi_i(\operatorname{supp}(\mu_{n'})) = \emptyset$$

for every $n \neq n'$ and $i \in \{0, 1\}$ (where π_i denotes the projection on the i-th coordinate). Note that for each $n \in \mathbb{N}_+$ and any two sets $A \in \wp(\Omega)$ and $B \in \wp(\Sigma)$ we have

$$|\mu_n([A] \times [B])| \leq \frac{|A \cap \Omega_n|}{2^n} \cdot \frac{|B \cap \Sigma_n|}{n}, \qquad (\dagger)$$

where $[A]$ and $[B]$ always denote the clopen subsets of $\beta\Omega$ and $\beta\Sigma$ corresponding in the sense of the Stone duality to A and B, respectively—since $\beta\Omega$ and $\beta\Sigma$ are extremely disconnected, we have $[A] = \overline{A}^{\beta\Omega}$ and $[B] = \overline{B}^{\beta\Sigma}$.

In the next proposition, we will prove that the sequence $(\mu_n)_{n\in\mathbb{N}}$, as defined above, is weak* convergent to 0 on the space $\beta\Omega \times \beta\Sigma$, that is, it is an fsJN-sequence. However, before we do that, we need to provide a bit of explanation of probability tools we use in the proof. For every $n \in \mathbb{N}_+$ and $i \in n$, define the function $X_i \colon \Omega_n \to \{0, 1\}$ as follows: $X_i(r) = 1$ if and only if $r(i) = 1$, where $r \in \Omega_n$. Put $S_n = \sum_{i=0}^{n-1} X_i$, so $S_n \colon \Omega_n \to n + 1$ is the function computing the number of 1s in the argument sequence $r \in \Omega_n$. For a finite set $A \subseteq \mathbb{N}$, let P_A denote the standard product probability on $\{-1, 1\}^A$ (assigning $1/2^{|A|}$ to each elementary event, i.e. $P_A(\{r\}) = 1/2^{|A|}$ for each $r \in \{-1, 1\}^{|A|}$). Recall that for every $k \leq n$ it holds

$$P_n(S_n = k) = P_n(\{r \in \Omega_n \colon\ S_n(r) = k\}) = \binom{n}{k} 1/2^n.$$

We will need the following fact, being a variant of the weak law of large numbers, which estimates the probability that $S_n(r)$ has value "far" (with respect to ε) from $n/2$, i.e. that "r contains *significantly* more (with respect to ε) 1s than -1s or *vice versa*".

Lemma 20.4.1 *If $n \in \mathbb{N}_+$ and $\varepsilon \in (0, 1/12]$ are such numbers that $n \geq 48/\varepsilon$, then*

$$P_n(|S_n - n/2| \geq \varepsilon n/2) \leq \frac{\sqrt{2}}{\varepsilon\sqrt{n}}.$$

Proof See Bollobás [91, Theorem 1.7.(i)]. □

We are ready to prove the aforementioned auxiliary proposition.

Proposition 20.4.2 *The sequence $(\mu_n)_{n\in\mathbb{N}_+}$ defined above is convergent to 0 with respect to the weak* topology of the dual space $C(\beta\Omega \times \beta\Sigma)'$. In particular, it is an fsJN-sequence on $\beta\Omega \times \beta\Sigma$.*

Proof Since $\beta\Omega \times \beta\Sigma$ is a totally disconnected compact space, to prove that $(\mu_n)_{n\in\mathbb{N}_+}$ is weak* convergent to 0, by Lemma 19.1.2, it is enough to show that it converges to 0 on every clopen subset of the form $[A] \times [B]$, where $A \in \wp(\Omega)$ and $B \in \wp(\Sigma)$. So, let us fix two such sets A and B.

Fix $\varepsilon \in (0, 1/12]$ and put

$$I_0 = \{n \in \mathbb{N}_+ \colon\ |B \cap \Sigma_n| < 2/\varepsilon^4\}$$

and

$$I_1 = \mathbb{N}_+ \setminus I_0 = \{n \in \mathbb{N}_+ \colon |B \cap \Sigma_n| \geq 2/\varepsilon^4\}.$$

For each $i \in \{0, 1\}$, we will find $N_i \in \mathbb{N}_+$ such that for every $n \geq N_i$, $n \in I_i$, it holds

$$|\mu_n([A] \times [B])| = |\mu_n([A \cap \Omega_n] \times [B \cap \Sigma_n])| < 2\varepsilon.$$

We first look for N_0. By (†) for every $n \in I_0$, we have

$$|\mu_n([A] \times [B])| \leq \frac{|A \cap \Omega_n|}{2^n} \cdot \frac{|B \cap \Sigma_n|}{n} \leq \frac{|B \cap \Sigma_n|}{n} < \frac{2}{n\varepsilon^4},$$

so if I_0 is infinite, then there exists $N_0 \in \mathbb{N}_+$ such that for every $n \geq N_0$, $n \in I_0$, we have

$$|\mu_n([A] \times [B])| < 2\varepsilon.$$

If, on the other hand, I_0 is finite, then simply set $N_0 = 1 + \max I_0$.

Let us now look for N_1. We assume that I_1 is non-empty—otherwise, simply set $N_1 = 0$. For every $n \in I_1$, define the set $\Delta_{n,\varepsilon}$ as follows:

$$\Delta_{n,\varepsilon} = \left\{ s \in \Omega_n : \ \left| |\{i \in B \cap \Sigma_n : \ s(i) = 1\}| - \frac{|B \cap \Sigma_n|}{2} \right| \geq \varepsilon \frac{|B \cap \Sigma_n|}{2} \right\},$$

so $\Delta_{n,\varepsilon}$ denotes the event that $s \in \Omega_n$ is "far" (with respect to ε) from having the same numbers of 1s and -1s when restricted to the set B. If we put similarly

$$\Gamma_{n,\varepsilon} =$$

$$= \left\{ s \in \{-1, 1\}^{B \cap \Sigma_n} : \ \left| |\{i \in B \cap \Sigma_n : \ s(i) = 1\}| - \frac{|B \cap \Sigma_n|}{2} \right| \geq \varepsilon \frac{|B \cap \Sigma_n|}{2} \right\},$$

then, we trivially have

$$\Delta_{n,\varepsilon} = \Gamma_{n,\varepsilon} \times \{-1, 1\}^{\Sigma_n \setminus B}. \tag{$\times$}$$

Using this, for every $n \in I_1$, we will estimate the following values (see (1) and (3)):

$$|\mu_n([A \cap \Delta_{n,\varepsilon}] \times [B \cap \Sigma_n])| = \Big| \sum_{\substack{s \in A \cap \Delta_{n,\varepsilon} \\ i \in B \cap \Sigma_n}} \frac{s(i)}{n 2^n} \Big| \tag{$*$}$$

and

$$|\mu_n([A \cap (\Omega_n \setminus \Delta_{n,\varepsilon})] \times [B \cap \Sigma_n])| = \Big| \sum_{\substack{s \in A \cap (\Omega_n \setminus \Delta_{n,\varepsilon}) \\ i \in B \cap \Sigma_n}} \frac{s(i)}{n 2^n} \Big|. \tag{$**$}$$

Note that

$$|\mu_n([A] \times [B])| \leq$$

$$\leq |\mu_n([A \cap \Delta_{n,\varepsilon}] \times [B \cap \Sigma_n])| + |\mu_n([A \cap (\Omega_n \setminus \Delta_{n,\varepsilon})] \times [B \cap \Sigma_n])|,$$

so obtaining "good" estimations of $(*)$ and $(**)$ will finish the proof.

Fix $n \in I_1$ and let us start with the estimation of $(*)$. Note that $|B \cap \Sigma_n| \geq 48/\varepsilon$, so recall $(\times)$ and apply Lemma 20.4.1 with the set $B \cap \Sigma_n$ instead of the set $n = \{0, \ldots, n-1\}$ to get that

$$\begin{aligned} P_n(\Delta_{n,\varepsilon}) &= P_{B \cap \Sigma_n}(\Gamma_{n,\varepsilon}) \cdot P_{\Sigma_n \setminus B}\Big(\{-1, 1\}^{\Sigma_n \setminus B}\Big) = \\ &= P_{B \cap \Sigma_n}(\Gamma_{n,\varepsilon}) \leq \frac{\sqrt{2}}{\varepsilon \sqrt{|B \cap \Sigma_n|}}. \end{aligned} \tag{0}$$

It holds that

$$\Big|\sum_{\substack{s\in A\cap\Delta_{n,\varepsilon}\\ i\in B\cap\Sigma_n}}\frac{s(i)}{n2^n}\Big| \le \sum_{\substack{s\in A\cap\Delta_{n,\varepsilon}\\ i\in B\cap\Sigma_n}}\frac{1}{n2^n} = \frac{|A\cap\Delta_{n,\varepsilon}|\cdot|B\cap\Sigma_n|}{n2^n} \le$$

$$\le \frac{|\Delta_{n,\varepsilon}|\cdot n}{n2^n} = P_n(\Delta_{n,\varepsilon}),$$

so by (0), we get the following estimation of $(*)$:

$$\Big|\sum_{\substack{s\in A\cap\Delta_{n,\varepsilon}\\ i\in B\cap\Sigma_n}}\frac{s(i)}{n2^n}\Big| \le \frac{\sqrt{2}}{\varepsilon\sqrt{|B\cap\Sigma_n|}}. \tag{1}$$

We now estimate $(**)$. For every $s\in\Omega_n\setminus\Delta_{n,\varepsilon}$, we have

$$|\sum_{i\in B\cap\Sigma_n} s(i)| =$$

$$= \Big||\{i\in B\cap\Sigma_n\colon\ s(i)=1\}| - |\{i\in B\cap\Sigma_n\colon\ s(i)=-1\}|\Big| \le$$

$$\le \Big||\{i\in B\cap\Sigma_n\colon\ s(i)=1\}| - \frac{|B\cap\Sigma_n|}{2}\Big| +$$

$$+\Big||\{i\in B\cap\Sigma_n\colon\ s(i)=-1\}| - \frac{|B\cap\Sigma_n|}{2}\Big| <$$

$$< 2\cdot\varepsilon\frac{|B\cap\Sigma_n|}{2} = \varepsilon|B\cap\Sigma_n|,$$

so

$$|\sum_{i\in B\cap\Sigma_n} s(i)| < \varepsilon|B\cap\Sigma_n|. \tag{2}$$

Next, it holds

$$\Big|\sum_{\substack{s\in A\cap(\Omega_n\setminus\Delta_{n,\varepsilon})\\ i\in B\cap\Sigma_n}}\frac{s(i)}{n2^n}\Big| \le \frac{1}{n2^n}\Big|\sum_{\substack{s\in A\cap(\Omega_n\setminus\Delta_{n,\varepsilon})\\ i\in B\cap\Sigma_n}} s(i)\Big| =$$

$$= \frac{1}{n2^n}\Big|\sum_{s\in A\cap(\Omega_n\setminus\Delta_{n,\varepsilon})}\ \sum_{i\in B\cap\Sigma_n} s(i)\Big| \le \frac{1}{n2^n}\sum_{s\in A\cap(\Omega_n\setminus\Delta_{n,\varepsilon})}\ |\sum_{i\in B\cap\Sigma_n} s(i)| \le$$

$$\leq \frac{1}{n} \max\Big\{ \Big| \sum_{i \in B \cap \Sigma_n} s(i) \Big| : \ s \in \Omega_n \setminus \Delta_{n,\varepsilon} \Big\},$$

so by (2)

$$\Big| \sum_{\substack{s \in A \cap (\Omega_n \setminus \Delta_{n,\varepsilon}) \\ i \in B \cap \Sigma_n}} \frac{s(i)}{n 2^n} \Big| < \frac{\varepsilon |B \cap \Sigma_n|}{n} \leq \frac{\varepsilon n}{n} = \varepsilon. \tag{3}$$

Using (1), (3), and the fact that $|B \cap \Sigma_n| \geq 2/\varepsilon^4$, we conclude that for every $n \in I_1$ we have

$$|\mu_n([A] \times [B])| = \Big| \sum_{\substack{s \in A \cap \Omega_n \\ i \in B \cap \Sigma_n}} \frac{s(i)}{n 2^n} \Big| \leq$$

$$\leq \Big| \sum_{\substack{s \in A \cap \Delta_{n,\varepsilon} \\ i \in B \cap \Sigma_n}} \frac{s(i)}{n 2^n} \Big| + \Big| \sum_{\substack{s \in A \cap (\Omega_n \setminus \Delta_{n,\varepsilon}) \\ i \in B \cap \Sigma_n}} \frac{s(i)}{n 2^n} \Big| <$$

$$< \frac{\sqrt{2}}{\varepsilon \sqrt{|B \cap \Sigma_n|}} + \varepsilon \leq \varepsilon + \varepsilon = 2\varepsilon.$$

It follows that if for N_1 we take any number from I_1, e.g. we set $N_1 = \min I_1$, then for every $n \in I_1$, $n \geq N_1$, we have

$$|\mu_n([A] \times [B])| < 2\varepsilon.$$

We finish the proof by denoting $N = \max(N_0, N_1)$ and seeing that for every $n \geq N$ we obviously have the same inequality, i.e.:

$$|\mu_n([A] \times [B])| < 2\varepsilon.$$

Since $\varepsilon \in (0, 1/12]$ is arbitrary, it holds that $\lim_{n \to \infty} \mu_n([A] \times [B]) = 0$ and hence $(\mu_n)_{n \in \mathbb{N}_+}$ is weak* convergent to 0. □

Remark 20.4.3 Let us note that from the above proof we can deduce for every $n \in \mathbb{N}_+$ the following estimations of the value $|\mu_n([A] \times [B])|$, depending only on the size of the intersection $B \cap \Sigma_n$:

$$|\mu_n([A] \times [B])| \leq \begin{cases} \frac{|B \cap \Sigma_n|}{n} & , \text{ if } |B \cap \Sigma_n| < \frac{2}{\varepsilon^4}, \\ \frac{\sqrt{2}}{\varepsilon \sqrt{|B \cap \Sigma_n|}} + \frac{\varepsilon |B \cap \Sigma_n|}{n} & , \text{ if } |B \cap \Sigma_n| \geq \frac{2}{\varepsilon^4}. \end{cases}$$

We are in the position to prove the main result of this section.

Theorem 20.4.4 (Kąkol–Sobota–Zdomskyy [353]) *For every infinite compact spaces K and L, their product $K \times L$ has the fsJNP.*

Proof First, notice that the space $\mathbb{N}$ of all non-negative integers, endowed with the discrete topology, is homeomorphic to both Ω and Σ, so the Čech–Stone compactifications $\beta\mathbb{N}$, $\beta\Omega$, and $\beta\Sigma$ are mutually homeomorphic. Consequently, by Proposition 20.4.2, $\beta\mathbb{N} \times \beta\mathbb{N}$ admits an fsJN-sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures with pairwise disjoint supports, contained completely in $\mathbb{N}\times\mathbb{N}$ (as measures μ_n's defined above on $\beta\Omega \times \beta\Sigma$ have pairwise disjoint supports contained in $\Omega \times \Sigma$).

Let D and E be discrete countable subsets of K and L, respectively. Let $\varphi\colon \mathbb{N} \to D$ and $\psi\colon \mathbb{N} \to E$ be bijections. By the Stone extension property of $\beta\mathbb{N}$, there are continuous maps $\Phi\colon \beta\mathbb{N} \to K$ and $\Psi\colon \beta\mathbb{N} \to L$ such that $\Phi \restriction \mathbb{N} = \varphi$ and $\Psi \restriction \mathbb{N} = \psi$. For each $n \in \mathbb{N}$, define a measure ϱ_n on $K \times L$ as follows:

$$\varrho_n = \sum_{(x,y)\in \mathrm{supp}(\nu_n)} \nu_n(\{(x, y)\}) \cdot \delta_{(\varphi(x),\psi(y))},$$

it follows that $\|\varrho_n\| = 1$ and $\mathrm{supp}(\varrho_n)$ is finite. Since $(\nu_n)_{n\in\mathbb{N}}$ weak* converges to 0, for every $f \in C(K \times L)$, we have

$$\lim_{n\to\infty} \int_{K\times L} f\, d\varrho_n = \lim_{n\to\infty} \int_{\beta\mathbb{N}\times\beta\mathbb{N}} f(\Phi, \Psi)\, d\nu_n = 0,$$

where $f(\Phi, \Psi)(x, y) = f(\Phi(x), \Psi(y)) \in C(\beta\mathbb{N} \times \beta\mathbb{N})$, so $(\varrho_n)_{n\in\mathbb{N}}$ is also weak* convergent to 0. Consequently, $(\varrho_n)_{n\in\mathbb{N}}$ is an fsJN-sequence on $K \times L$ and so $K \times L$ has the fsJNP. □

By Theorems 20.4.4 and 19.2.11 and Corollary 20.2.5, we immediately get the following result (whose $C(K \times L)$-part was first obtained by Cembranos [133] and Freniche [254]).

Corollary 20.4.5 *For every infinite compact spaces K and L, the space $C_p(K \times L)$ contains a complemented copy of $(c_0)_p$, and the Banach space $C(K \times L)$ contains a complemented copy of c_0.*

Corollaries 19.2.12 and 20.2.6 imply the next consequence.

Corollary 20.4.6 *For every infinite compact spaces K and L, we have the following isomorphisms of lcs:*

- $C(K \times L) \simeq C(K \times L \times \alpha\mathbb{N})$.
- $C_p(K \times L) \simeq C_p(K \times L \times \alpha\mathbb{N})$.

We finish this section by mentioning that Theorem 20.4.4 was further generalized in [341] to the case of infinite pseudocompact spaces K and L whose product $K \times L$ is also pseudocompact; the case of the non-pseudocompact product $K \times L$ was investigated in the latter paper, too, but it appeared to be sensitive to additional set-theoretic assumptions.

20.5 Limits of Inverse Systems of Simple Extensions and Efimov Spaces

In this section, we present the theorem of Kąkol, Sobota, and Zdomskyy [354] which asserts that the limit of any inverse system of simple extensions of compact spaces has the finitely supported Josefson–Nissenzweig property (Theorem 20.5.7).

The result has an immediate application to so-called Efimov spaces. Recall that *the Efimov problem* is a long-standing open question asking whether every infinite compact space contains a non-trivial convergent sequence or a homeomorphic copy of $\beta\mathbb{N}$ (see [301] for a general discussion on the problem). An infinite compact space which contains neither of the mentioned subspaces is called *an Efimov space*. Many examples of Efimov spaces have been obtained, but so far each of the constructions has made use of some additional set-theoretic assumption (see [568, Section 6.2]). Several of those examples are obtained as limits of inverse systems consisting of simple extensions, e.g. those of Fedorchuk ([201], under the assumption of Jensen's Diamond Principle $\diamondsuit$), Dow and Pichardo-Mendoza ([176], under the Continuum Hypothesis), or Dow and Shelah ([177], under Martin's axiom).

We first recall what an inverse system of simple extensions is. For general information on limits of inverse systems, see [195, Chapters 2.5 and 3.2].

Definition 20.5.1 An inverse system $\langle K_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \leq \delta \rangle$ of totally disconnected compact spaces is *a system of simple extensions* if:

- it is *continuous*, i.e. for every limit ordinal $\gamma \leq \delta$, the space K_γ is the limit of the inverse system $\langle K_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \leq \gamma \rangle$,
- K_0 is a singleton,
- for every $\alpha < \delta$, the space $K_{\alpha+1}$ is *a simple extension of* K_α, i.e. there is $x_\alpha \in K_\alpha$ such that $\left|(\pi_\alpha^{\alpha+1})^{-1}(x_\alpha)\right| = 2$ and for every $y \in K_\alpha \setminus \{x_\alpha\}$ it holds $\left|(\pi_\alpha^{\alpha+1})^{-1}(y)\right| = 1$.

To prove the main result of this section, Theorem 20.5.7, we also need to provide several definitions concerning complexity of probability measures on compact spaces. First, recall that *the Maharam type* of a probability measure μ on a compact space K is the minimal cardinality of a family $\mathcal{C}$ of Borel subsets of K such that for every Borel subset B of K and $\varepsilon > 0$ there exists $C \in \mathcal{C}$ such that $\mu(B \triangle C) < \varepsilon$ (see Chap. 13). Equivalently, the Maharam type of a probability measure μ is the density of the Banach space $L_1(\mu)$ of all μ-integrable functions. For more information on the topic, see [248, 425], or [501].

A notion closely related to the countable Maharam type is the uniform regularity, introduced by Babiker [47] and later intensively studied in [442, 505], and [376]: a probability measure μ on a compact space K is *uniformly regular* if there exists a countable family $\mathcal{C}$ of zero subsets of K such that for every open subset U of K and every $\varepsilon > 0$ there exists $F \in \mathcal{C}$ such that $F \subseteq U$ and $\mu(U \setminus F) < \varepsilon$. Uniformly regular measures are also called *strongly countably determined* (cf. [505]). Note that

every uniformly regular probability measure μ has necessarily separable support and countable Maharam type.

Recall that a subset Y of a topological space X is *a $\mathbb{G}_\delta$-set* if there exists a countable collection $\mathcal{U}$ of open subsets of X such that $Y = \bigcap \mathcal{U}$. An element $x \in X$ is called *a $\mathbb{G}_\delta$-point* if $\{x\}$ is a $\mathbb{G}_\delta$-set of X. Every zero subset of a topological space is a closed $\mathbb{G}_\delta$-set, and one can easily show that in a normal space every closed $\mathbb{G}_\delta$-set is a zero set; thus, the definition of uniformly regular measures may be stated in terms of closed $\mathbb{G}_\delta$-sets.

Proposition 20.5.2 *Let μ be a uniformly regular measure on a compact space K and $x \in K$ be such that $\mu(\{x\}) > 0$. Then, x is a $\mathbb{G}_\delta$-point.*

Proof Let $\mathcal{C}$ be a countable collection of zero sets (closed $\mathbb{G}_\delta$'s) witnessing that μ is uniformly regular. Put $\mathcal{C}' = \{F \in \mathcal{C} \colon\ x \in F\}$. It follows that $\mathcal{C}' \neq \emptyset$ and $\{x\} = \bigcap \mathcal{C}'$. To see, e.g. the latter, assume that there is $y \in \bigcap \mathcal{C}'$ such that $x \neq y$. Put $\varepsilon = \mu(\{x\})$; so $\varepsilon > 0$. Using the regularity of μ, it is easy to see that there is an open neighbourhood U of x not containing y and such that $\mu(U \setminus \{x\}) < \varepsilon/3$. However, there is no $F \in \mathcal{C}$ such that $F \subseteq U$ and $\mu(U \setminus F) < \varepsilon/3$, which contradicts the choice of $\mathcal{C}$. Indeed, if there was such F, then from the fact that

$$\varepsilon \leq \mu(U) = \mu(U \setminus \{x\}) + \mu(\{x\}) < 4\varepsilon/3$$

we would get that $\mu(F) > 2\varepsilon/3$, so necessarily $x \in F \in \mathcal{C}'$ and hence $y \in F \subseteq U$, whereas $y \notin U$, which would be a contradiction. Since the intersection of a countable collection of $\mathbb{G}_\delta$-sets is $\mathbb{G}_\delta$, x is a $\mathbb{G}_\delta$-point. □

Remark 20.5.3 Note that from Proposition 20.5.2 it immediately follows that if for a uniformly regular measure μ on an infinite compact space K there is a non-isolated point $x \in K$ such that $\mu(\{x\}) > 0$, then K contains a non-trivial sequence converging to x.

Let μ be a probability measure on a compact space K. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ of points in K is *μ-uniformly distributed* if the sequence of sums $(\frac{1}{n}\sum_{i=0}^{n-1} \delta_{x_i})_{n>0}$ converges weak* to μ. Uniformly distributed sequences constitute a useful tool for investigating various properties of probability measures as they allow to treat those measures in a way similar to the classical Jordan measure (content) on the real line; see, e.g. [388, 420, 421], or [442].

Let us say that a sequence $(x_n)_{n\in\mathbb{N}}$ in a space X is *injective* if $x_n \neq x_{n'}$ for every $n \neq n' \in \mathbb{N}$. The following proposition will be crucial for the proof of the main theorem of this section.

Proposition 20.5.4 *If μ is a non-atomic uniformly regular measure on a compact space K, then μ admits a uniformly distributed injective sequence.*

Proof Let μ^∞ be the product Radon measure induced by μ on the product space $K^{\mathbb{N}}$. [442, Corollary 2.8] implies that $\mu^\infty(S) = 1$, where S denotes the subspace of $K^{\mathbb{N}}$ consisting of all μ-uniformly distributed sequences in K. Since by the non-atomicity of μ the subspace T of $K^{\mathbb{N}}$ consisting of all non-injective sequences in K

satisfies $\mu^\infty(T) = 0$ (by, e.g. the Fubini–Tonelli theorem [241, Theorem 7.27]), it follows that there exists a μ-uniformly distributed sequence in S which is injective. □

Proposition 20.5.5 *Let μ be a probability measure on a compact space K and $(x_n)_{n\in\mathbb{N}}$ be a μ-uniformly distributed injective sequence in K. Then, K has the fsJNP.*

Proof Let $\mathbb{N} = \bigcup_{n\in\mathbb{N}} P_n$ be a partition of $\mathbb{N}$ into finite sets such that

$$\max P_n < \min P_{n+1} \quad \text{and} \quad |P_n| \Big/ \max P_n \geq 1/2$$

for every $n \in \mathbb{N}$ (cf. the standard proof that $\sum_{n\in\mathbb{N}} 1/n = \infty$). For every $n \in \mathbb{N}$, let us write

$$\nu_n = \frac{1}{\max P_{n+1}} \sum_{k<\max P_{n+1}} \delta_{x_k} - \frac{1}{\max P_n} \sum_{k<\max P_n} \delta_{x_k}.$$

Then, by the injectivity of $(x_n)_{n\in\mathbb{N}}$

$$\|\nu_n\| \geq \frac{1}{\max P_{n+1}} \cdot |P_{n+1}| \geq 1/2.$$

Since either sum in the definition of ν_n's is weak* convergent to μ, $(\nu_n)_{n\in\mathbb{N}}$ converges weak* to 0. Normalizing the measures ν_n, i.e. setting $\mu_n = \nu_n/\|\nu_n\|$ for each $n \in \mathbb{N}$, we get that $(\mu_n)_{n\in\mathbb{N}}$ is an fsJN-sequence on K. □

The following theorem was proved by Borodulin-Nadzieja [96] in the language of minimally generated Boolean algebras, the notion dual (via the Stone duality) to limits of inverse systems of simple extensions (see [371]).

Theorem 20.5.6 *The following assertions hold for every totally disconnected compact space K:*

(1) [96, Theorem 4.6] K carries either a uniformly regular measure or a measure of uncountable Maharam type.
(2) [96, Theorem 4.9] If K is the limit of an inverse system of simple extensions, then every measure on K has countable Maharam type. In particular, there exists a uniformly regular measure on K.

Let us recall here that Džamonja and Plebanek [193, Lemma 4.1] proved that if an inverse system of simple extensions has length at most ω_1, then every measure on its limit is uniformly regular.

We are in the position to prove the main theorem of this section.

Theorem 20.5.7 (Kąkol–Sobota–Zdomskyy [354, Theorem 6.10]) *Let K be an infinite compact space. If K is the limit of an inverse system of simple extensions, then K has the fsJNP.*

Proof If K is a scattered space, then it contains a non-trivial convergent sequence, and hence, it trivially has the fsJNP. So, let us assume that there exists an infinite closed subspace L of K containing no isolated points. By [195, Proposition 2.5.6], L is also the limit of an inverse system of simple extensions. By Theorem 20.5.6.(2), there exists a uniformly regular measure μ on L. If μ has an atom, then it is non-isolated in L; hence, by Proposition 20.5.2 and Remark 20.5.3, L contains a non-trivial convergent sequence and so L has the fsJNP. If on the other hand μ is non-atomic, then Proposition 20.5.4 implies that μ admits a uniformly distributed injective sequence. Now, Proposition 20.5.5 yields an fsJN-sequence on L. Since the fsJNP is inherited by superspaces, K has the fsJNP, too. □

The following consequence of Theorems 20.1.1 and 20.5.7 generalizes the well-known fact that no minimally generated Boolean algebra has the Grothendieck property.

Corollary 20.5.8 *If $\mathcal{A}$ is a minimally generated Boolean algebra, then $\mathcal{A}$ does not have the ℓ_1-Grothendieck property.*

As we already mentioned at the beginning of this section, as a corollary to Theorem 20.5.7, we also obtain that many Efimov spaces constructed in the literature have the fsJNP.

Corollary 20.5.9 *If K is an Efimov space obtained as the limit of an inverse system of simple extensions, then K has the fsJNP. In particular, the examples of Efimov spaces by Fedorchuk (under $\diamondsuit$; see [201]), Dow and Pichardo-Mendoza (under the Continuum Hypothesis; see [176]), or Dow and Shelah (under Martin's axiom; see [177]) have the fsJNP.*

Let us note that consistently there exist Efimov spaces with the Grothendieck property and hence without the fsJNP; see, e.g. [102, 580], or [568].

Chapter 21
The Nikodym Property of Boolean Algebras

Abstract We study the Nikodym property for Boolean algebras. The property is closely related to the barrelledness of the subspace of $C(St(\mathcal{A}))$, where $St(\mathcal{A})$ denotes the Stone space of a Boolean algebra $\mathcal{A}$, spanned by the set of all characteristic functions corresponding via the Stone duality to elements of $\mathcal{A}$ and hence, to the classical Banach–Steinhaus theorem. We present a proof of Nikodym's uniform boundedness theorem asserting that every σ-complete Boolean algebra has the Nikodym property. Finally, we introduce the strong Nikodym property, a strengthening of the previous property, and show a proof of Valdivia's result asserting that every σ-field has the strong Nikodym property.

21.1 Preliminaries: Space ba($\mathcal{A}$)

Let $\mathcal{A}$ be a Boolean algebra. By $St(\mathcal{A})$ we denote the Stone space of $\mathcal{A}$, that is, the totally disconnected compact Hausdorff space consisting of all ultrafilters on $\mathcal{A}$ (cf. Sect. 19.5). If $A \in \mathcal{A}$, then $[A]_{\mathcal{A}}$ denotes the clopen subset of $St(\mathcal{A})$ corresponding via the Stone duality to A. $0_{\mathcal{A}}$ and $1_{\mathcal{A}}$ denote the zero and unit element of $\mathcal{A}$, respectively.

A function $\mu\colon \mathcal{A} \to \mathbb{R}$ is *a measure on* $\mathcal{A}$ if μ is finitely additive and *bounded* (or *finite*), that is, *the total variation* $\|\mu\|$ of μ, defined as

$$\|\mu\| = \sup\{|\mu(A)| + |\mu(B)|\colon\ A, B \in \mathcal{A}, A \wedge B = 0_{\mathcal{A}}\},$$

is finite (i.e. $\|\mu\| < \infty$); μ may be *signed*, i.e., it may have negative values. The real linear space of all measures on $\mathcal{A}$ is denoted by ba($\mathcal{A}$) and, when endowed with the above defined norm $\|\cdot\|$, called *the total variation norm*, it is a Banach space.

For each measure $\mu \in \mathrm{ba}(\mathcal{A})$ there is a unique (Radon) measure $\widehat{\mu}$ on $St(\mathcal{A})$, i.e. $\widehat{\mu} \in C(St(\mathcal{A}))'$ (so $\widehat{\mu} \in \mathrm{ca}(Bor(St(\mathcal{A})))$), such that $\widehat{\mu}([A]_{\mathcal{A}}) = \mu(A)$ for every $A \in \mathcal{A}$ and $\|\widehat{\mu}\| = \|\mu\|$ (for basic information concerning Radon measures on compact spaces, see Chap. 19). Consequently, ba($\mathcal{A}$) is isometrically isomorphic to the dual space $C(St(\mathcal{A}))'$.

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For a measure $\mu \in \mathrm{ba}(\mathcal{A})$, we define its *variation* $|\mu|\colon \mathcal{A} \to [0, \infty)$ as follows:

$$|\mu|(C) = \sup\{|\mu(A)| + |\mu(B)|\colon\ A, B \in \mathcal{A}, A \wedge B = 0_{\mathcal{A}}, A \vee B \leq C\},$$

for every $C \in \mathcal{A}$. Of course, $\|\mu\| = |\mu|(1_{\mathcal{A}})$ and $\widehat{|\mu|} = |\widehat{\mu}|$. If $A \in \mathcal{A}$ and $\mu \in \mathrm{ba}(\mathcal{A})$, then $\mu \restriction A$ denotes the measure on $\mathcal{A}$ defined by $(\mu \restriction A)(B) = \mu(A \wedge B)$ for every $B \in \mathcal{A}$; we have $\|\mu \restriction A\| = |\mu|(A)$.

For a field $\mathcal{A}$ of subsets of a set Ω, $L(\mathcal{A})$ denotes the real linear space generated by the characteristic functions $e(A)$ of all sets $A \in \mathcal{A}$, endowed with the supremum norm $\|\cdot\|_\infty$. The completion $\widehat{L(\mathcal{A})}$ of $L(\mathcal{A})$ is the space of all bounded $\mathcal{A}$-measurable functions on Ω. The space $\widehat{L(\mathcal{A})}$ is often denoted also by $\ell_\infty(\mathcal{A})$, $\mathcal{B}(\mathcal{A})$, or $B(\mathcal{A})$. Moreover, $\widehat{L(\mathcal{A})}$ is isometrically isomorphic to $C(St(\mathcal{A}))$ and hence $\mathrm{ba}(\mathcal{A})$ is isometrically isomorphic to the dual space $L(\mathcal{A})'$. If $\mathcal{A}$ is a σ-field, then $\mathrm{ca}(\mathcal{A})$ is a closed linear subspace of $\mathrm{ba}(\mathcal{A})$, and hence $\mathrm{ca}(\mathcal{A})$ can be treated as a closed linear subspace of $L(\mathcal{A})'$ (and so of $\widehat{L(\mathcal{A})}'$).

We say that a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathrm{ba}(\mathcal{A})$ is *pointwise bounded* if $\sup_{n\in\mathbb{N}}|\mu_n(A)| < \infty$ for every $A \in \mathcal{A}$, and is *norm bounded* or *uniformly bounded* (or, simply, *bounded*) if $\sup_{n\in\mathbb{N}}\|\mu_n\| < \infty$. Of course, every uniformly bounded sequence is also pointwise bounded, but the converse generally does not hold.

21.2 The Nikodym Property and Its Relation with the Grothendieck Property

The classical Banach–Steinhaus theorem (*the uniform boundedness principle*) asserts that, given a Banach space E and a sequence $(x_n^*)_{n\in\mathbb{N}}$ of functionals in the dual space E', if $\sup_{n\in\mathbb{N}}|x_n^*(x)| < \infty$ for every $x \in E$, then $\sup_{n\in\mathbb{N}}\|x_n^*\| < \infty$. Nikodym [467] proved that, given a σ-field Σ of subsets of a set Ω, every pointwise bounded sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathrm{ca}(\Sigma)$ is also uniformly bounded. Since, as said above, the space $\mathrm{ca}(\Sigma)$ can be treated as a closed linear subspace of $\ell_\infty(\Sigma)'$, Nikodym's result can be considered as a generalization of the Banach–Steinhaus theorem in the case of spaces $\ell_\infty(\Sigma)$ and $\mathrm{ca}(\Sigma)$—Dunford and Schwartz [191, page 309] called this generalization *a striking improvement of the principle of uniform boundedness*. Later on, Andô [9] generalized Nikodym's result and proved that, given a σ-complete Boolean algebra $\mathcal{A}$, actually every pointwise bounded sequence in $\mathrm{ba}(\mathcal{A})$ is uniformly bounded. Nikodym–Andô's result is very often called *Nikodym's uniform boundedness theorem* or shorter *Nikodym's boundedness theorem*.

The Andô result, which we will prove in the next section of this chapter in a slightly more general form, motivates the following definition: a Boolean algebra $\mathcal{A}$ has *the Nikodym property* if every pointwise bounded sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures on $\mathcal{A}$ (that is, in $\mathrm{ba}(\mathcal{A})$) is uniformly bounded. Thus, Andô's theorem

shortly asserts that every σ-complete Boolean algebra has the Nikodym property. Other examples of Boolean algebras with the property include:

- the algebra $\mathcal{J}$ of Jordan measurable subsets of the unit interval $[0, 1]$ (Schachermayer [542], see also Valdivia [622] and Graves and Wheeler [284] for various generalizations of Schachermayer's result),
- algebras having, e.g., Haydon's Subsequential Completeness Property [303] (see the next section) or Freniche's Interpolation Property [253] (see [567] for many other examples of properties).

The class of Boolean algebras without the Nikodym property contains, e.g., the following algebras:

- Boolean algebras $\mathcal{A}$ which can be represented as the union $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ of countably many proper subalgebras $\mathcal{A}_n$ (Schachermayer [542]; see also Sobota [566]), in particular, countable Boolean algebras,
- Boolean algebras $\mathcal{A}$ whose Stone spaces $St(\mathcal{A})$ contain non-trivial convergent sequences (if $St(\mathcal{A})$ contains a non-trivial convergent sequence $(x_n)_{n\in\mathbb{N}}$, then the sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures on $\mathcal{A}$, whose Radon extensions are given for each $n \in \mathbb{N}$ by the formula $\widehat{\mu}_n = n(\delta_{x_{2n}} - \delta_{x_{2n+1}})$, is pointwise bounded but not uniformly bounded),
- Boolean algebras $\mathcal{A}$ whose Stone spaces $St(\mathcal{A})$ can be represented as the product $K \times L$ of two infinite totally disconnected compact spaces K and L (Freniche [253]) or, in other words, Boolean algebras which are free products of two infinite Boolean algebras.

The class of Boolean algebras with the Nikodym property is closed under quotients, but not under passing to subalgebras, e.g., as aforementioned, no countable subalgebra of any Boolean algebra has the Nikodym property. Let us note here that a similar discussion as in Sect. 19.5 can be also conducted for possible cardinalities of infinite Boolean algebras with the Nikodym property, see [567] for a general survey and [566, 568–570] for detailed results.

The following proposition provides several equivalent definitions of the Nikodym property.

Proposition 21.2.1 *For every Boolean algebra $\mathcal{A}$, the following are equivalent:*

(1) $\mathcal{A}$ has the Nikodym property, i.e., every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded,
(2) every pointwise convergent sequence of measures on $\mathcal{A}$ is uniformly bounded,
(3) every pointwise convergent sequence of measures on $\mathcal{A}$ is weak convergent.*

Proof Implication (1)⇒(2) follows from the fact that every pointwise convergent sequence is pointwise bounded. Implication (2)⇒(3) is a consequence of the Stone duality and Lemma 19.1.2 ((1)⇒(2)). So, let us prove implication (3)⇒(1).

Assume for the sake of contradiction that there is a pointwise bounded sequence $(\mu_n)_{n\in\mathbb{N}}$ of non-zero measures on $\mathcal{A}$ such that $\lim_{n\to\infty} \|\mu_n\| = \infty$. For every $n \in \mathbb{N}$ define the measure ν_n on $\mathcal{A}$ as follows:

$$\nu_n = \mu_n \Big/ \sqrt{\|\mu_n\|}.$$

Then, $(\nu_n)_{n\in\mathbb{N}}$ is pointwise convergent to 0. Indeed, for every $A \in \mathcal{A}$ and $n \in \mathbb{N}$ we have:

$$|\nu_n(A)| = \frac{|\mu_n(A)|}{\sqrt{\|\mu_n\|}} \leq \frac{\sup_{m\in\mathbb{N}}|\mu_m(A)|}{\sqrt{\|\mu_n\|}},$$

so, by the pointwise boundedness of $(\mu_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} \nu_n(A) = 0$ for every $A \in \mathcal{A}$. Thus, (3) implies that $(\nu_n)_{n\in\mathbb{N}}$ is weak* convergent, hence, by the Banach–Steinhaus theorem, $\sup_{n\in\mathbb{N}}\|\nu_n\| < \infty$. On the other hand, since $\|\nu_n\| = \sqrt{\|\mu_n\|}$ for every $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \|\nu_n\| = \infty$, a contradiction. □

Recall that a Boolean algebra $\mathcal{A}$ has *the Grothendieck property* if every weak* convergent sequence of measures on $\mathcal{A}$ is weakly convergent (that is, the Banach space $C(St(\mathcal{A}))$ is Grothendieck; see Chap. 19 and Sect. 19.5, in particular). In the view of equivalence (1)⇔(3) of the above proposition, the Nikodym property can be seen as a "pointwise–weak*" analogue of the Grothendieck property. It seems thus natural to ask whether either of the property implies the other one. The answer is however known to be negative (but in one direction only, so far, consistently). Namely, Schachermayer [542] proved (in ZFC) that the Boolean algebra $\mathcal{J}$ of Jordan measurable subsets of [0, 1] has the Nikodym property but it does not have the Grothendieck property or even the ℓ_1-Grothendieck property (cf. also [284]). Assuming Jensen's Diamond Principle $\diamondsuit$, Sobota and Zdomskyy [569] constructed a Boolean algebra with the Nikodym property and having also the additional property of being so-called minimally generated—as shown in Chap. 20 the latter property implies that the Boolean algebra does not have the ℓ_1-Grothendieck property and hence, in particular, the full Grothendieck property.

The question whether the Grothendieck property implies the Nikodym property seems to be much more intricate as so far there is only one significant result in this direction. Namely, assuming the Continuum Hypothesis, Talagrand [583] constructed an example of a Boolean algebra, being a subalgebra of the σ-field of all Borel subsets of the Cantor space $2^{\mathbb{N}}$, with the Grothendieck property but without the Nikodym property. No ZFC example is so far known.

We finish this introductory section making the following remark. If a Boolean algebra $\mathcal{A}$ does not have the Nikodym property, and this is witnessed by a sequence $(\mu_n)_{n\in\mathbb{N}}$ of finitely or countably supported measures (see Chap. 20) on the Stone space $St(\mathcal{A})$, then the sequence $(\mu_n/\|\mu_n\|)_{n\in\mathbb{N}}$ is an fsJN-sequence, hence $\mathcal{A}$ does not have the Grothendieck property either (Theorem 20.1.1). The converse is not true—the lack of the Grothendieck property of the Boolean algebra $\mathcal{J}$ of Jordan measurable subsets of [0, 1] is witnessed by an fsJN-sequence (see [542, Proposition 3.2]) and, as aforementioned, this Boolean algebra has the Nikodym property.

21.3 The Nikodym Property of Boolean Algebras with the Subsequential Completeness Property

The Nikodym–Andô theorem states that every σ-complete Boolean algebra has the Nikodym property [9, 467]. In this section, we prove a bit more general result, first observed by Haydon [303], asserting that every Boolean algebra with the Subsequential Completeness Property has the Nikodym property. Recall that a Boolean algebra $\mathcal{A}$ has *the Subsequential Completeness Property* (*the SCP*) if for every antichain $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ there is an infinite set $M \subseteq \mathbb{N}$ such that the supremum $\bigvee_{n\in M} A_n$ is in $\mathcal{A}$ (cf. Sect. 19.5). Note that every σ-complete Boolean algebra has the SCP, but there are Boolean algebras with the latter property which are not σ-complete (see [303]).

We start with a simple observation.

Lemma 21.3.1 *Let $\mathcal{A}$ be a Boolean algebra. Let $(\mu_n)_{n\in\mathbb{N}}$ be a pointwise bounded sequence of measures on $\mathcal{A}$ such that $\sup_{n\in\mathbb{N}}\|\mu_n \restriction A\| = \infty$ for some $A \in \mathcal{A}$. Then, for every real number $\varrho > 0$, there exist an element $B \in \mathcal{A}$, $B \leq A$, and a number $n \in \mathbb{N}$ such that $|\mu_n(B)| > \varrho$ and $\sup_{m\in\mathbb{N}}\|\mu_m \restriction (A \setminus B)\| = \infty$.*

Proof Since $\sup_{m\in\mathbb{N}}|\mu_m(A)| < \infty$ and $\sup_{m\in\mathbb{N}}\|\mu_m \restriction A\| = \infty$, there exist $C \leq A$ and $n \in \mathbb{N}$ such that

$$|\mu_n(C)| > \sup_{m\in\mathbb{N}} |\mu_m(A)| + \varrho,$$

and hence

$$|\mu_n(A \setminus C)| = |\mu_n(A) - \mu_n(C)| \geq |\mu_n(C)| - |\mu_n(A)| > \varrho.$$

We have $\sup_{m\in\mathbb{N}}\|\mu_m \restriction C\| = \infty$ or $\sup_{m\in\mathbb{N}}\|\mu_m \restriction (A \setminus C)\| = \infty$. If $\sup_{m\in\mathbb{N}}\|\mu_m \restriction C\| = \infty$, then put $B = A \setminus C$, otherwise put $B = C$. In either case, $|\mu_n(B)| > \varrho$ and $\sup_{m\in\mathbb{N}}\|\mu_m \restriction (A \setminus B)\| = \infty$. □

The following lemma is a simple application of Lemma 19.4.3.

Lemma 21.3.2 *Let $\mathcal{A}$ be a Boolean algebra with the SCP. Let $(B_n)_{n\in\mathbb{N}}$ be an antichain in $\mathcal{A}$ and $(\mu_n)_{n\in\mathbb{N}}$ a sequence of measures on $\mathcal{A}$. Then, there exists an almost disjoint family $\{C_\xi : \xi < \omega_1\}$ of infinite subsets of $\mathbb{N}$ such that for every $\xi < \omega_1$ the supremum $\bigvee_{n\in C_\xi} B_n$ exists in $\mathcal{A}$ and the equality*

$$|\mu_k|\Big(\bigvee_{n\in C_\xi} B_n\Big) = \sum_{n\in C_\xi} |\mu_k|(B_n)$$

holds for every $k \in \mathbb{N}$.

Proof Let $\{A_\xi : \xi < \omega_1\}$ be any almost disjoint family of infinite subsets of $\mathbb{N}$. Since $\mathcal{A}$ has the SCP, for each $\xi < \omega_1$ there is an infinite set $C'_\xi \subseteq A_\xi$ such that the supremum $\bigvee_{n\in C'_\xi} B_n$ exists in $\mathcal{A}$. For each $\xi < \omega_1$, since

$$\Big[\bigvee_{n\in C'_\xi} B_n\Big]_{\mathcal{A}} = \overline{\bigcup_{n\in C'_\xi} [B_n]_{\mathcal{A}}},$$

the set $\overline{\bigcup_{n\in C'_\xi}[B_n]_{\mathcal{A}}}$ is open in $St(\mathcal{A})$. Of course, the family $\{C'_\xi : \xi < \omega_1\}$ is also almost disjoint. Looking at the Radon extensions $\widehat{\mu}_k$'s, by Lemma 19.4.3, there exists $\eta < \omega_1$ such that the equality

$$|\widehat{\mu}_k|\Big(\Big[\bigvee_{n\in C'_\xi} B_n\Big]_{\mathcal{A}}\Big) = |\widehat{\mu}_k|\Big(\overline{\bigcup_{n\in C'_\xi}[B_n]_{\mathcal{A}}}\Big) = |\widehat{\mu}_k|\Big(\bigcup_{n\in C'_\xi}[B_n]_{\mathcal{A}}\Big)$$

holds for every $\xi \geq \eta$ and $k \in \mathbb{N}$. Since each $|\widehat{\mu}_k|$ is σ-additive, we get for every $k \in \mathbb{N}$ and $\xi \geq \eta$ that:

$$|\mu_k|\Big(\bigvee_{n\in C'_\xi} B_n\Big) = |\widehat{\mu}_k|\Big(\Big[\bigvee_{n\in C'_\xi} B_n\Big]_{\mathcal{A}}\Big) = \sum_{n\in C'_\xi} |\widehat{\mu}_k|([B_n]_{\mathcal{A}}) = \sum_{n\in C'_\xi} |\mu_k|(B_n).$$

Re-enumerating $\{C'_\xi : \eta \leq \xi < \omega_1\}$ as $\{C_\xi : 0 \leq \xi < \omega_1\}$ finishes the proof. □

We are ready to prove the main theorem of this section.

Theorem 21.3.3 (Nikodym–Andô–Haydon) *If $\mathcal{A}$ is a Boolean algebra with the Subsequential Completeness Property, then $\mathcal{A}$ has the Nikodym property.*

Proof Assume that $\mathcal{A}$ does not have the Nikodym property, so there exists a pointwise bounded sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures on $\mathcal{A}$ such that $\sup_{n\in\mathbb{N}}\|\mu_n\| = \infty$. Using inductively Lemma 21.3.1, we construct a sequence $(B_k)_{k\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{A}$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that for every $k \in \mathbb{N}$ it holds:

$$|\mu_{n_k}(B_k)| > \sum_{i<k} |\mu_{n_k}(B_i)| + k + 1. \qquad (*)$$

(On the k-th step of the induction apply Lemma 21.3.1 to the setting $\varrho = \sum_{i<k} \sup_{m\in\mathbb{N}}|\mu_m(B_i)| + k + 1$ and $A = 1_{\mathcal{A}} \setminus \bigvee_{i<k} B_i$.) Passing to subsequences, if necessary, we may assume that

$$\sum_{j>k} |\mu_{n_k}|(B_j) < 1 \qquad (**)$$

for every $k \in \mathbb{N}$.

By Lemma 21.3.2, we get a strictly increasing sequence $(k_l)_{l\in\mathbb{N}}$ of natural numbers (just enumerate $C_\xi = (k_l)_{l\in\mathbb{N}}$ for some $\xi < \omega_1$, e.g. $\xi = 0$) such that the supremum $B = \bigvee_{j\in\mathbb{N}} B_{k_j}$ exists in $\mathcal{A}$ and for every $l \in \mathbb{N}$ we have:

$$|\mu_{n_{k_l}}|\Big(\bigvee_{j\in\mathbb{N}} B_{k_j}\Big) = \sum_{j\in\mathbb{N}} |\mu_{n_{k_l}}|(B_{k_j}). \qquad (***)$$

For every $l \in \mathbb{N}$ we have:

$$|\mu_{n_{k_l}}(B)| = |\mu_{n_{k_l}}\Big(\bigvee_{j\in\mathbb{N}} B_{k_j}\Big)| =$$

$$= |\mu_{n_{k_l}}\Big(\bigvee_{j<l} B_{k_j}\Big) + \mu_{n_{k_l}}(B_{k_l}) + \mu_{n_{k_l}}\Big(\bigvee_{j>l} B_{k_j}\Big)| \geq$$

$$\geq |\mu_{n_{k_l}}(B_{k_l})| - |\mu_{n_{k_l}}\Big(\bigvee_{j<l} B_{k_j}\Big)| - |\mu_{n_{k_l}}|\Big(\bigvee_{j>l} B_{k_j}\Big) \geq$$

$$\geq |\mu_{n_{k_l}}(B_{k_l})| - \sum_{j<l} |\mu_{n_{k_l}}(B_{k_j})| - \sum_{j>l} |\mu_{n_{k_l}}|(B_{k_j}) > k_l + 1 - 1 = k_l,$$

where the penultimate inequality follows from the triangle inequality and $(***)$ and the last inequality follows from $(*)$ and $(**)$. Since $k_l \geq l$ for every $l \in \mathbb{N}$, we got that $\sup_{n\in\mathbb{N}}|\mu_n(B)| = \infty$, which contradicts the pointwise boundedness of $(\mu_n)_{n\in\mathbb{N}}$. It follows that $\mathcal{A}$ has the Nikodym property. □

21.4 The Strong Nikodym Property of σ-Fields

In this section, we will generalize Theorem 21.3.3 for a σ-field and prove the result due to Valdivia stating that every σ-field Σ has the so-called strong Nikodym property (Theorem 21.4.12). Our arguments will be more functional analytic in nature than in the previous sections and will follow [415].

We first need to recall basic notions concerning polar sets and absolutely convex sets. To the end of the chapter, we will denote by $\mathcal{A}$ a field of subsets of a given set Ω. We will be working mostly with the dual pair $\langle L(\mathcal{A}), \mathrm{ba}(\mathcal{A})\rangle$ (see Sect. 21.1). Of course, for every $A \in \mathcal{A}$ and $\mu \in \mathrm{ba}(\mathcal{A})$, we will simply write $\mu(A)$ for $\mu(e(A))$. *The polar* M° of a subset M of $\mathrm{ba}(\mathcal{A})$ is given by $M^\circ = \{f \in L(\mathcal{A}) : |\mu(f)| \leq 1, \mu \in M\}$ (see [322, Section 8.2]). For each family $\mathcal{B}$ contained in the field $\mathcal{A}$, $\tau_s(\mathcal{B})$ denotes the topology in $\mathrm{ba}(\mathcal{A})$ of pointwise convergence on $\mathcal{B}$ and thus the weak* topology in $\mathrm{ba}(\mathcal{A})$ is $\tau_s(\mathcal{A})$.

The convex (absolutely convex) hull of a subset B of a vector space E is denoted by conv B (absco B) and the seminorm in span B, defined by $\inf\{|\lambda| : x \in \lambda(\text{absco } B)\}$ for each $x \in \text{span } B$, is the gauge of absco B.

The gauge of absco$(\{e(C) : C \in \mathcal{A}\})$ is a norm in $L(\mathcal{A})$ and its dual norm is given by $\sup\{|\mu(C)| : C \in \mathcal{A}\}$ for any $\mu \in \text{ba}(\mathcal{A})$. The norms, supremum $\|\cdot\|_\infty$ and the gauge of absco$(\{e(C) : C \in \mathcal{A}\})$, are equivalent, see [609, Propositions 1 and 2], hence the dual norms, variation ($\|\mu\| = |\mu|(\Omega)$), and $\mathcal{A}$-supremum of modulus ($\sup\{|\mu(C)| : C \in \mathcal{A}\}$), are equivalent. For each $A \in \mathcal{A}$ we have that in ba$(\mathcal{A})$ the seminorms, variation on A ($|\mu|(A)$), and supremum of modulus on $\{C \in \mathcal{A} : C \subset A\}$ ($\sup\{|\mu(C)| : C \in \mathcal{A}, C \subset A\}$), are equivalent.

For the sake of completeness, we give the proof of the following well-known property.

Lemma 21.4.1 *Let B be a subset of a vector space E. Each finite subset C of E contains a subset D such that*

(i) $\text{span}(B \cup C) = \text{span}(B \cup D)$,
(ii) $\text{span } B \cap \text{span } D = \{0\}$,
(iii) and the gauges defined by $\text{absco}(B \cup C)$ *and* $\text{absco}(B \cup D)$ *are equivalent seminorms in* $\text{span}(B \cup C)$.

Proof If $\text{span } B \cap \text{span } C = \{0\}$ then $D := C$ verifies the lemma. If $\text{span } B \cap \text{span } C \neq \{0\}$ then there exists $b_i \in B$, $1 \leq i \leq p$, $q_1 \in C\backslash\{0\}$ and $c_j \in C\backslash\{q_1\}$, $1 \leq j \leq s$, and non-null scalars β_i, δ and γ_j, $1 \leq i \leq p$, $1 \leq j \leq q$, such that

$$\sum_{i=1}^{p} \beta_i b_i = \delta q_1 + \sum_{j=1}^{q} \gamma_j c_j.$$

We may suppose that $\delta = -1$ and we obtain that

$$q_1 = -\sum_{i=1}^{p} \beta_i b_i + \sum_{j=1}^{q} \gamma_j c_j \in C\backslash\{0\}.$$

Then for each $x \in \text{absco}(B \cup C)$, we have

$$x = \sum_{i=1}^{s} \delta_i d_i + \varepsilon q_1 + \sum_{j=1}^{t} \varepsilon_j e_j,$$

with $\sum_{i=1}^{s} |\delta_i| + |\varepsilon| + \sum_{j=1}^{t} |\varepsilon_j| \leq 1$ and $(d_i, e_j) \in B \times (C\backslash\{q_1\})$. After the substitution of q_1, we deduce the equality

$$x = \sum_{i=1}^{s} \delta_i d_i - \sum_{i=1}^{p} \varepsilon\beta_i b_i + \sum_{j=1}^{q} \varepsilon\gamma_j c_j + \sum_{j=1}^{t} \varepsilon_j e_j,$$

implying that $x \in (1+h)\,\mathrm{absco}(B \cup (C\backslash\{q_1\}))$, with $h = \sum_{i=1}^{p} |\beta_i| + \sum_{j=1}^{q} |\gamma_j|$, and then

$$\mathrm{absco}(B \cup C) \subset (1+h)\,\mathrm{absco}(B \cup (C\backslash\{q_1\})). \tag{21.1}$$

From (21.1) and $\mathrm{absco}(B\cup(C\backslash\{q_1\})) \subset \mathrm{absco}(B\cup C)$ it follows that $\mathrm{span}(B\cup C) = \mathrm{span}(B \cup (C\backslash\{q_1\}))$ and that the gauges defined by $\mathrm{absco}(B \cup C)$ and $\mathrm{absco}(B \cup C\backslash\{q_1\})$ are equivalent seminorms in $\mathrm{span}(B \cup C)$.

If $\mathrm{span}\ B \cap \mathrm{span}\ (C\backslash\{q_1\}) \neq \{0\}$ then, as in the previous construction, we determine a non null vector $q_2 \in C\backslash\{q_1\}$ such that $\mathrm{span}(B \cup (C\backslash\{q_1\})) = \mathrm{span}(B \cup (C\backslash\{q_1, q_2\}))$ and the gauges of $\mathrm{absco}(B \cup C\backslash\{q_1\}$ and $\mathrm{absco}(B \cup C\backslash\{q_1, q_2\})$ are equivalent seminorms in $\mathrm{span}(B \cup C)$.

After a finite number of r repetitions of this process, we get a finite subset $D = C\backslash\{q_1, q_2, \cdots, q_r\}$ such that $\mathrm{span}(B \cup C) = \mathrm{span}(B \cup D)$, $\mathrm{span}\, B \cap \mathrm{span}\, D = \{0\}$ and the gauges defined by $\mathrm{absco}(B \cup C\}$ and by $\mathrm{absco}(B \cup D)$ are equivalent in $\mathrm{span}(B \cup C)$. □

Let $\mathcal{A}$ be a field of subsets of a set Ω. For a set $\mathcal{B} \subseteq \mathcal{A}$, we say that a family $M \subseteq \mathrm{ba}(\mathcal{A})$ is *$\mathcal{B}$-pointwise bounded* if $\sup\{|\mu(B)| : \mu \in M\} < \infty$ for every $B \in \mathcal{B}$. Recall that, for a σ-field Σ of subsets of Ω, the Nikodym boundedness theorem (Theorem 21.3.3) implies that a Σ-pointwise bounded subset M of $\mathrm{ba}(\Sigma)$ is (norm) bounded in $\mathrm{ba}(\Sigma)$. This result and the interest in weakening the condition "Σ-pointwise bounded" motivate the following definition (see [542, Definition 2.4] and [622, Definition 1]). A subset $\mathcal{B}$ of a field $\mathcal{A}$ of subsets of a set Ω:

- is *a Nikodym set* (for $\mathrm{ba}(\mathcal{A})$), or has *property* (N), if each $\mathcal{B}$-pointwise bounded subset M of $\mathrm{ba}(\mathcal{A})$ is bounded in $\mathrm{ba}(\mathcal{A})$;
- is *a strong Nikodym set* (for $\mathrm{ba}(\mathcal{A})$), or has *strong property* (N), or *property* (sN) in brief, if for each increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ there exists $p \in \mathbb{N}$ such that $\mathcal{B}_p$ is a Nikodym set for $\mathrm{ba}(\mathcal{A})$.

In the first part of this definition, we may change "each $\mathcal{B}$-pointwise bounded subset M" by "each $\mathcal{B}$-pointwise bounded weak* closed and absolutely convex subset M" or by "each $\mathcal{B}$-pointwise bounded countable subset M". Notice that the Nikodym boundedness theorem says that if Σ is a σ-field then Σ is a Nikodym set for $\mathrm{ba}(\Sigma)$. More generally, if a field $\mathcal{A}$ has the Nikodym property, then $\mathcal{A}$ is a Nikodym set for $\mathrm{ba}(\mathcal{A})$.

Valdivia obtained in [609, Theorem 2] (cf. also [622]) that for a σ-field Σ the set Σ has property (sN) (see Theorem 21.4.12) and in [622, Problem 1] he asked if for every field $\mathcal{A}$ it is true that property (N) of $\mathcal{A}$ implies property (sN) of $\mathcal{A}$. A partial solution to this still open problem is given in [226, Theorem 3.3]. Earlier results, stronger properties, and examples are provided and discussed in [228, 413] and [419].

We now present a proof of aforementioned Valdivia's Theorem 21.4.12. Valdivia's original proof given in [609] uses theory of barrelled spaces (that is, locally convex spaces which satisfy the thesis of the Banach–Steinhaus theorem), whose

main properties may be found in [202, 230] and [491], among others. The proof given below follows [415] and is independent of the theory of barrelled spaces and inspired by López-Alfonso [414].

Firstly, we present a characterization of property (N) for a subset $\mathcal{B}$ of a field $\mathcal{A}$.

Proposition 21.4.2 *Let $\mathcal{B}$ be a subset of a field $\mathcal{A}$. $\mathcal{B}$ has property (N) for* $\mathrm{ba}(\mathcal{A})$ *if and only if for each increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$, there exists $p \in \mathbb{N}$ such that*

$$\overline{\mathrm{absco}\{e(A) : A \in \mathcal{B}_p\}}^{L(\mathcal{A})} \tag{21.2}$$

is a neighbourhood of zero in $L(\mathcal{A})$.

Proof If $\mathcal{B}$ is a not a Nikodym set for $\mathrm{ba}(\mathcal{A})$ there exists an unbounded subset M in $\mathrm{ba}(\mathcal{A})$ which is $\mathcal{B}$-pointwise bounded. By unboundedness, M° is not a neighbourhood of zero in $L(\mathcal{A})$ and the $\mathcal{B}$-pointwise boundedness imply that $\mathcal{B}$ is covered by the increasing sequence formed with the sets

$$\mathcal{B}_n = \{A \in \mathcal{B} : \sup_{\mu \in M} |\mu(A)| \leq n\}, n \in \mathbb{N}.$$

The construction of $\mathcal{B}_n$ and the definition of polar set imply that $\{e(A) : A \in \mathcal{B}_n\} \subset nM^\circ$, for each $n \in \mathbb{N}$, hence

$$\overline{\mathrm{absco}\{e(A) : A \in \mathcal{B}_n\}}^{L(\mathcal{A})} \subset nM^\circ,$$

and this inclusion shows that $\overline{\mathrm{absco}\{e(A) : A \in \mathcal{B}_n\}}^{L(\mathcal{A})}$ is not a neighbourhood of zero in $L(\mathcal{A})$ for each $n \in \mathbb{N}$.

Conversely, if there exists an increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ such that

$$\overline{\mathrm{absco}\{e(A) : A \in \mathcal{B}_n\}}^{L(\mathcal{A})}$$

is not a neighbourhood of zero in $L(\mathcal{A})$ for every $n \in \mathbb{N}$, then the polar sets $\{e(A) : A \in \mathcal{B}_n\}^\circ$ are unbounded, so there exists $\mu_n \in \{e(A) : A \in \mathcal{B}_n\}^\circ$ such that $\|\mu_n\| \geq n$, for each $n \in \mathbb{N}$, hence $M := \{\mu_n : n \in \mathbb{N}\}$ is an unbounded subset of $\mathrm{ba}(\mathcal{A})$. If $A \in \mathcal{B}$ there exists $q_A \in \mathbb{N}$ such that $A \in \mathcal{B}_n$ for each $n \geq q_A$, hence $|\mu_n(A)| \leq 1$ for $n \geq q_A$, and we get that $\{|\mu_n(A)| : n \in \mathbb{N}\}$ is bounded, hence M is $\mathcal{B}$-pointwise bounded. Therefore, we get that $\mathcal{B}$ does not have property (N). □

Notice that the set $\mathcal{B}_p$ of Proposition 21.4.2 satisfies

$$\overline{\mathrm{span}\{e(A) : A \in \mathcal{B}_p\}}^{L(\mathcal{A})} = L(\mathcal{A}). \tag{21.3}$$

Lemma 21.4.3 *Let B and C be two subsets of a normed space E such that B is an absolutely convex closed which is not a neighbourhood of* 0 *in E and C is finite. If* span B *is dense in E then* $\mathrm{absco}(B \cup C)$ *is not a neighbourhood of* 0 *in E.*

Proof By Lemma 21.4.1 there exists a subset D of C such that the $\operatorname{span}(B \cup C) = \operatorname{span}(B \cup D)$ with $\operatorname{span} B \cap \operatorname{span} D = \{0\}$ and the gauges defined by $\operatorname{absco}(B \cup C)$ and $\operatorname{absco}(B \cup D)$ define two equivalent seminorms in $\operatorname{span}(B \cup C)$.

If $\operatorname{absco}(B \cup C)$ is a neighbourhood of 0 in E then $\operatorname{absco}(B \cup D)$ is also a neighbourhood of 0 in E, implying that

$$B = \operatorname{span} B \cap \operatorname{absco}(B \cup D)$$

is a neighbourhood of 0 in $\operatorname{span} B$. Hence $B = \overline{B}$ is a neighbourhood of 0 in $E = \overline{\operatorname{span} B}$. This contradiction proves the lemma. □

Lemma 21.4.4 *Let M be an unbounded, weak*-closed and absolutely convex subset of* $\operatorname{ba}(\mathcal{A})$ *such that* $\overline{\operatorname{span}(M)^\circ}^{L(\mathcal{A})} = L(\mathcal{A})$. *For each finite subset Q of $\mathcal{A}$ we have that $M \cap \{e(A) : A \in Q\}^\circ$ is unbounded in* $\operatorname{ba}(\mathcal{A})$, *i.e.,*

$$\sup_{\mu \in M \cap \{e(A) : A \in Q\}^\circ} |\mu|(\Omega) = \infty. \tag{21.4}$$

Proof The unboundedness of M implies that the polar set M° is not a zero neighbourhood in $L(\mathcal{A})$. The hypothesis, $\overline{\operatorname{span} M^\circ}^{L(\mathcal{A})} = L(\mathcal{A})$ and Lemma 21.4.3, with $E = L(\mathcal{A})$, $B = M^\circ$ and $C = \{e(A) : A \in Q\}$, imply that

$$\operatorname{absco}(M^\circ \cup \{e(A) : A \in Q\})$$

is not a neighbourhood of 0 in E. Hence its polar

$$(\operatorname{absco}(M^\circ \cup \{e(A) : A \in Q\}))^\circ = M^{\circ\circ} \cap \{e(A) : A \in Q\}^\circ$$

is unbounded in $\operatorname{ba}(\mathcal{A})$. As M is weak*-closed and absolutely convex we have $M = M^{\circ\circ}$, hence $M \cap \{e(A) : A \in Q\}^\circ$ is unbounded in $\operatorname{ba}(\mathcal{A})$. □

Proposition 21.4.5 *Let $\{\mathcal{B}_n : n \in \mathbb{N}\}$ be an increasing covering of a subset of $\mathcal{B}$ of a field $\mathcal{A}$ such that $\mathcal{B}$ is a Nikodym set for* $\operatorname{ba}(\mathcal{A})$ *and for every $n \in \mathbb{N}$ the set $\mathcal{B}_n$ is not a Nikodym set for* $\operatorname{ba}(\mathcal{A})$. *Then there exists $p \in \mathbb{N}$ such that for each $n \geq p$ there exists a subset M_n in* $\operatorname{ba}(\mathcal{A})$ *that it is $\mathcal{B}_n$-pointwise bounded, absolutely convex, weak*-closed and such that for each finite subset Q of $\mathcal{A}$ the intersection $M_n \cap \{e(A) : A \in Q\}^\circ$ is unbounded in* $\operatorname{ba}(\mathcal{A})$.

Proof By Proposition 21.4.2 and (21.3), there exists $p \in \mathbb{N}$ such that for each $n \geq p$

$$\overline{\operatorname{span}\{e(A) : A \in \mathcal{B}_n\}}^{L(\mathcal{A})} = L(\mathcal{A}). \tag{21.5}$$

As $\mathcal{B}_n$ is not a Nikodym set for $\operatorname{ba}(\mathcal{A})$, there exists an unbounded, weak*-closed and absolutely convex subset of M_n in $\operatorname{ba}(\mathcal{A})$ which is unbounded in $\operatorname{ba}(\mathcal{A})$ and M_n is pointwise bounded in $\{e(A) : A \in \mathcal{B}_n\}$. The pointwise boundedness imply that $\{e(A) : A \in \mathcal{B}_n\} \subset \operatorname{span} M_n^\circ$, hence for each $n \geq p$ we have by (21.5) that

$$L(\mathcal{A}) = \overline{\text{span}\{e(A) : A \in \mathcal{B}_n\}}^{L(\mathcal{A})} \subset \overline{\text{span}\, M_n^\circ}^{L(\mathcal{A})} \subset L(\mathcal{A}). \tag{21.6}$$

From (21.6) we deduce that $\overline{\text{span}\, M_n^\circ}^{L(\mathcal{A})} = L(\mathcal{A})$, for each $n \geq p$, and the proposition follows from Lemma 21.4.4. □

Claim 21.4.6 Let A be an element of a field $\mathcal{A}$, let M be a subset of $\text{ba}(\mathcal{A})$ such that for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} |\mu|(A) = \infty. \tag{21.7}$$

If $\{A_1, A_2, \ldots, A_q\}$ is a finite partition of A by elements of $\mathcal{A}$ there exist j, $1 \leq j \leq q$, such that for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} |\mu|(A_j) = \infty. \tag{21.8}$$

Proof If this claim were not true, there exists a finite subset Q_i of $\mathcal{A}$, $1 \leq i \leq q$, such that $\sup_{\mu \in M \cap \{e(D):D \in Q_i\}^\circ} |\mu|(A_i) < \infty$. Then for $Q = \cup\{Q_i : 1 \leq i \leq q\}$

$$\sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} |\mu|(A) = \sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} \left\{ \sum_{i=1}^{q} |\mu|(A_i) \right\} \leq$$

$$\leq \sum_{i=1}^{q} \sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} |\mu|(A_i) \leq \sum_{i=1}^{q} \sup_{\mu \in M \cap \{e(D):D \in Q_i\}^\circ} |\mu|(A_i) < \infty,$$

and from this contradiction with (21.7) follows (21.8). □

Lemma 21.4.7 *Let $\mathcal{A}$ be a field of subsets of a set Ω, $A \in \mathcal{A}$ and M a weak*-closed and absolutely convex subset of* $\text{ba}(\mathcal{A})$ *such that for each finite subset Q of $\mathcal{A}$*

$$\sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} |\mu|(A) = \infty.$$

For each $(p, \alpha) \in (\mathbb{N} \backslash \{0, 1\}) \times \mathbb{R}^+$ and each finite subset $\{B_j : 1 \leq j \leq n\}$ of $\mathcal{A}$ there exists a partition $\{A_i : A_i \in \mathcal{A}, 1 \leq i \leq p\}$ of A and a subset $\{\mu_i : 1 \leq i \leq p\}$ of M such that

$$|\mu_i(A_i)| > \alpha \quad \textit{and} \quad \sum_{j=1}^{n} \left|\mu_i(B_j)\right| \leq 1, \quad \textit{for } 1 \leq i \leq p. \tag{21.9}$$

Proof First let us proof that if for each finite subset Q of $\mathcal{A}$ we have that

$$\sup_{\mu\in M\cap\{e(D):D\in Q\}^\circ} |\mu|(A) = \infty, \tag{21.10}$$

then for each $\alpha \in \mathbb{R}^+$ and each subset $\{B_j : 1 \le j \le n\}$ of $\mathcal{A}$ there exists $(\mu_1, A_1) \in M \times \mathcal{A}$, $A_1 \subset A$ such that

$$|\mu_1(A_1)| > \alpha, \quad |\mu_1(A\backslash A_1)| > \alpha, \quad \sum_{j=1}^{n} \left|\mu_1(B_j)\right| \le 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu\in M\cap\{e(D):D\in Q\}^\circ} |\mu|(A\backslash A_1) = \infty.$$

In fact, by (21.10) with $Q = \{A, B_1, \cdots, B_n\}$ there exists $(\nu_1, P_{11}) \in (M \cap \{e(D) : D \in Q\}^\circ) \times \mathcal{A}$, with $P_{11} \subset A$ such that

$$|\nu_1(P_{11})| > n(\alpha + 1), \quad |\nu_1(A)| \le 1 \quad \text{and} \quad \left|\nu_1(B_j)\right| \le 1, \text{ for } 1 \le j \le n.$$

Let $P_{12} := A\backslash P_{11}$ and $\mu_1 = n^{-1}\nu_1$. The measure $\mu_1 \in M$ is such that

$$|\mu_1(P_{11})| > \alpha + 1, \quad |\mu_1(A)| \le 1, \quad \sum_{j=1}^{n} \left|\mu_1(B_j)\right| \le 1,$$

hence

$$|\mu_1(P_{12})| = |\mu_1(A) - \mu_1(P_{11})| \ge |\mu_1(P_{11})| - |\mu_1(A)| > \alpha.$$

By Claim 21.4.6 at least one of the inequalities

$$\sup_{\mu\in M\cap\{e(D):D\in Q\}^\circ} |\mu|(P_{11}) = \infty, \text{ for each finite subset } Q \in \mathcal{A}, \tag{21.11}$$

or

$$\sup_{\mu\in M\cap\{e(D):D\in Q\}^\circ} |\mu|(P_{12}) = \infty, \text{ for each finite subset } Q \in \mathcal{A},$$

holds. If (21.11) is true we define $A_1 := P_{12}$ and otherwise we set $A_1 := P_{11}$, and in both cases we have that there exists a partition $(A_1, A\backslash A_1) \in \mathcal{A} \times \mathcal{A}$ of A and a measure $\mu_1 \in M$ such that

$$|\mu_1(A_1)| > \alpha, \quad |\mu_1(A\backslash A_1)| > \alpha, \quad \sum_{j=1}^{n} \left|\mu_1(B_j)\right| \le 1,$$

and, for each finite subset Q of $\mathcal{A}$,

$$\sup_{\mu\in M\cap\{e(D):D\in Q\}^\circ} |\mu|(A\backslash A_1) = \infty.$$

If we repeat the preceding argument with $A\backslash A_1$, then we get a partition $(A_2, A\backslash(A_1\cup A_2)) \in \mathcal{A}\times\mathcal{A}$ of $A\backslash A_1$ and a measure $\mu_2 \in M$ such that

$$|\mu_2(A_2)| > \alpha, \quad |\mu_2(A\backslash(A_1\cup A_2))| > \alpha, \quad \sum_{j=1}^{n} \left|\mu_2(B_j)\right| \leq 1,$$

and such that, for each finite subset Q of $\mathcal{A}$,

$$\sup_{\mu\in M\cap\{e(D):D\in Q\}^\circ} |\mu|(A\backslash(A_1\cup A_2)) = \infty.$$

In the $(p-1)$-th repetition of this method, we get a partition $(A_{p-1}, (A\backslash(A_1\cup A_2\cup\cdots\cup A_{p-2}\cup A_{p-1}))) \in \mathcal{A}\times\mathcal{A}$ of $A\backslash(A_1\cup A_2\cup\cdots\cup A_{p-2})$ and a measure $\mu_{p-1}\in M$ such that $\left|\mu_{p-1}(A_{p-1})\right| > \alpha$, $\left|\mu_{p-1}(A\backslash(A_1\cup\cdots\cup A_{p-1}))\right| > \alpha$ and $\sum_{j=1}^{n}\left|\mu_{p-1}(B_j)\right| \leq 1$.

To finish the proof, we define $A_p := A\backslash(A_1\cup A_2\cup\cdots\cup A_{p-2}\cup A_{p-1})$ and $\mu_p := \mu_{p-1}$. □

The property given in Lemma 21.4.7 yields the following technical corollary, which together with Remark 21.4.9 and Lemma 21.4.10 will enable us to develop the inductive process described in the proof of Proposition 21.4.11.

Corollary 21.4.8 *Fix $p \geq 1$ natural numbers $n_1, \ldots, n_p$. Let $\mathcal{A}$ be a field of subsets of a set Ω, $A \in \mathcal{A}$ and $M_n, n \in \mathbb{N}$, a weak*-closed and absolutely convex subset of* $\mathrm{ba}(\mathcal{A})$ *such that for each finite subset Q of $\mathcal{A}$ the equality*

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|(A) = \infty$$

holds for the natural numbers $n = n_i$, $1 \leq i \leq p$, and for infinitely many natural numbers $n \in \mathbb{N}\backslash\{n_1, \ldots, n_p\}$. For each $\alpha \in \mathbb{R}^+$ and each finite subset $\{B_j : 1 \leq j \leq n\}$ of $\mathcal{A}$ there exists a partition $(A_1, A\backslash A_1) \in \mathcal{A}\times\mathcal{A}$ of A and a measure $\mu_1 \in M_{n_1}$ such that

$$|\mu_1(A_1)| > \alpha \quad \textit{and} \quad \sum_{j=1}^{n} \left|\mu_1(B_j)\right| \leq 1$$

and for each finite subset Q of $\mathcal{A}$ the equality

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|(A\backslash A_1) = \infty$$

holds for the natural numbers $n = n_i$, $1 \leq i \leq p$, and for infinitely many natural numbers $n \in \mathbb{N}\backslash\{n_1, \ldots, n_p\}$.

Proof By Lemma 21.4.7 for each $(p+2, \alpha) \in \mathbb{N} \times \mathbb{R}^+$ and for the subset $\{B_j : 1 \leq j \leq n\}$ of $\mathcal{A}$ there exists a partition $\{D_i : D_i \in \mathcal{A}, 1 \leq i \leq p+2\}$ of A and a subset $\{\upsilon_i : 1 \leq i \leq p+2\}$ of M_{n_1} such that

$$|\upsilon_i(D_i)| > \alpha \quad \text{and} \quad \sum_{j=1}^{n} \left|\upsilon_i(B_j)\right| \leq 1, \quad \text{for } 1 \leq i \leq p+2.$$

From Claim 21.4.6, for each $1 \leq j \leq p$ there exists $i_j \in \{1, 2, \cdots, p+2\}$ such that for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_{n_j} \cap \{e(D) : D \in Q\}^\circ} |\mu|\left(D_{n_{i_j}}\right) = \infty \tag{21.12}$$

and also there exists $i_0 \in \{1, 2, \cdots, p+2\}$ such that for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_n \cap \{e(D) : D \in Q\}^\circ} |\mu|\left(D_{n_{i_0}}\right) = \infty \tag{21.13}$$

for infinitely many natural numbers n. Let us suppose that $i^* \in \{1, 2, \cdots, p+2\}\backslash\{i_m : m = 0, 1, \cdots, p\}$. To finish this proof, let $\mu_1 := \upsilon_{i^*}$ and $A_1 := D_{i^*}$. Then

$$|\mu_1(A_1)| = |\upsilon_{i^*}(D_{i^*})| > \alpha \text{ and } \sum_{j=1}^{n} \left|\mu_1(B_j)\right| = \sum_{j=1}^{n} \left|\upsilon_{i^*}(B_j)\right| \leq 1$$

and for each finite subset Q of $\mathcal{A}$ the equality

$$\sup_{\mu \in M_n \cap \{e(D) : D \in Q\}^\circ} |\mu|\left(A\backslash A_1\right) = \infty$$

holds for the natural numbers $n = n_i$, $1 \leq i \leq p$, and for infinitely many natural numbers $n \in \mathbb{N}\backslash\{n_1, \ldots, n_p\}$, because $A\backslash A_1 = A\backslash D_{i^*}$ contains $\cup\{D_{n_{i_j}} : 0 \leq j \leq p\}$ and then from (21.12) and (21.13) it follows that for $1 \leq j \leq p$

$$\sup_{\mu \in M_{n_j} \cap \{e(D) : D \in Q\}^\circ} |\mu|\left(A\backslash A_1\right) \geq \sup_{\mu \in M_{n_j} \cap \{e(D) : D \in Q\}^\circ} |\mu|\left(D_{n_{i_j}}\right) = \infty$$

and for infinitely many natural numbers $n \in \mathbb{N}\backslash\{n_1, \ldots, n_p\}$

$$\sup_{\mu \in M_n \cap \{e(D) : D \in Q\}^\circ} |\mu|\left(A\backslash A_1\right) \geq \sup_{\mu \in M_n \cap \{e(D) : D \in Q\}^\circ} |\mu|\left(D_{n_{i_0}}\right) = \infty.$$

□

Remark 21.4.9 Corollary 21.4.8 may be applied without the finite subset $\{B_i : 1 \leq i \leq n\}$ of $\mathcal{A}$. Then we get that $|\mu_1(A_1)| > \alpha$ and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_n \cap \{e(D):D \in Q\}^\circ} |\mu|(A \backslash A_1) = \infty$$

holds for the natural numbers $n = n_i$, $1 \leq i \leq p$, and for infinitely many natural numbers $n \in \mathbb{N} \backslash \{n_1, \ldots, n_p\}$.

Lemma 21.4.10 *Fix $p \geq 1$ natural numbers $n_1, \ldots, n_p$. Let $\mathcal{A}$ be a field of subsets of a set Ω, $A \in \mathcal{A}$ and $M_n, n \in \mathbb{N}$ a weak*-closed and absolutely convex subset of* $\mathrm{ba}(\mathcal{A})$ *such that for each finite subset Q of $\mathcal{A}$*

$$\sup_{\mu \in M_n \cap \{e(D):D \in Q\}^\circ} |\mu|(A) = \infty$$

holds for the natural numbers $n = n_i$, $1 \leq i \leq p$, and for infinitely many natural numbers $n \in \mathbb{N} \backslash \{n_1, \ldots, n_p\}$. For each $(p, \alpha) \in \mathbb{N} \times \mathbb{R}^+$ and each subset $\{B_j : 1 \leq j \leq n\}$ of $\mathcal{A}$ there exists a partition $\{A_i : A_i \in \mathcal{A}, 1 \leq i \leq p+1\}$ of A and $\mu_i \in M_{n_i}$, $1 \leq i \leq p$, such that

$$|\mu_i(A_i)| > \alpha, \quad \sum_{j=1}^{n} \left|\mu_i(B_j)\right| \leq 1, \quad \textit{for } 1 \leq i \leq p$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_n \cap \{e(D):D \in Q\}^\circ} |\mu|(A_{p+1}) = \infty$$

for the natural numbers $n = n_i$, $1 \leq i \leq p$, and for infinitely many natural numbers $n \in \mathbb{N} \backslash \{n_1, \ldots, n_p\}$.

Proof Corollary 21.4.8 provides in A a subset $A_1 \in \mathcal{A}$ and $\mu_1 \in M_{n_1}$ such that

$$|\mu_1(A_1)| > \alpha \quad \text{and} \quad \sum_{j=1}^{n} \left|\mu_1(B_j)\right| \leq 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_n \cap \{e(D):D \in Q\}^\circ} |\mu|(A \backslash A_1) = \infty$$

for the natural numbers $n = n_i$, $1 \leq i \leq p$, and for infinitely many natural numbers $n \in \mathbb{N} \backslash \{n_1, \ldots, n_p\}$. If we apply again Corollary 21.4.8 to $A \backslash A_1$ we get $A_2 \in \mathcal{A}$, with $A_2 \subset A \backslash A_1$, and $\mu_2 \in M_{n_2}$ such that

$$|\mu_2(A_2)| > \alpha, \quad \sum_{j=1}^{n} \left|\mu_2(B_j)\right| \leq 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_n \cap \{e(D): D \in Q\}^\circ} |\mu|\left(A \backslash (A_1 \cup A_2)\right) = \infty$$

for the natural numbers $n = n_i$, $1 \leq i \leq p$, and for infinitely many natural numbers $n \in \mathbb{N} \backslash \{n_1, \ldots, n_p\}$.

Repeating this method, for each $1 \leq i \leq p-1$ we get in A the pairwise disjoint subsets $A_i \in \mathcal{A}$ and in $\mathrm{ba}(\mathcal{A})$ the measures $\mu_i \in M_{n_i}$, $1 \leq i \leq p-1$, such that

$$|\mu_i(A_i)| > \alpha, \quad \sum_{j=1}^{n} \left|\mu_i(B_j)\right| \leq 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_n \cap \{e(D): D \in Q\}^\circ} |\mu|\left(A \backslash (A_1 \cup A_2 \cup \cdots \cup A_{p-1})\right) = \infty$$

for the natural numbers $n = n_i$, $1 \leq i \leq p$, and for infinitely many natural numbers $n \in \mathbb{N} \backslash \{n_1, \ldots, n_p\}$. Corollary 21.4.8 applied to $A \backslash (A_1 \cup A_2 \cup \cdots \cup A_{p-1})$ provides $A_p \in \mathcal{A}$, $A_p \subset A \backslash (A_1 \cup A_2 \cup \cdots \cup A_{p-1})$, and $\mu_p \in M_{n_p}$ such that

$$\left|\mu_i(A_p)\right| > \alpha, \quad \sum_{j=1}^{n} \left|\mu_i(B_j)\right| \leq 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_n \cap \{e(D): D \in Q\}^\circ} |\mu|\left(A \backslash (A_1 \cup A_2 \cup \cdots \cup A_{p-1} \cup A_p)\right) = \infty$$

for the natural numbers $n = n_i$, $1 \leq i \leq p$, and for infinitely many natural numbers $n \in \mathbb{N} \backslash \{n_1, \ldots, n_p\}$. With $A_{p+1} := A \backslash (A_1 \cup A_2 \cup \cdots \cup A_{p-1} \cup A_p)$ the proof is done. □

Proposition 21.4.11 *Let $\{\mathcal{B}_n : n \in \mathbb{N}\}$ be an increasing covering of a subset of $\mathcal{B}$ of a field $\mathcal{A}$ of subsets of a set Ω such that $\mathcal{B}$ is a Nikodym set for* $\mathrm{ba}(\mathcal{A})$ *and for every $n \in \mathbb{N}$ the set $\mathcal{B}_n$ is not a Nikodym set for* $\mathrm{ba}(\mathcal{A})$. *Then there exists an increasing sequence $(n_i : i \in \mathbb{N})$ such that for each $(i, j) \in \mathbb{N}^2$ with $1 \leq i \leq j$, there exists $A_{ij} \in \mathcal{A}$ and $\mu_{ij} \in \mathrm{ba}(\mathcal{A})$ such that the sets A_{ij} are pairwise disjoint, for each $i \in \mathbb{N}$ the set of measures $\{\mu_{ij} : j \in \mathbb{N},\ j \geq i\}$ is pointwise bounded in $\mathcal{B}_{n_i}$ and*

$$\left|\mu_{ij}(A_{ij})\right| > j, \quad \sum_{1\le k\le m<j} \left|\mu_{ij}(A_{km})\right| \le 1.$$

Proof By Proposition 21.4.5, there exists $p \in \mathbb{N}$ such that for each $n \ge p$ there exists in $\mathrm{ba}(\mathcal{A})$ an absolutely convex and weak*-closed subset M_n which is pointwise bounded in $\mathcal{B}_n$ and such that for each finite subset Q of $\mathcal{A}$ we have

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|(\Omega) = \infty.$$

We may suppose that $p = 1$, deleting the sets $\mathcal{B}_n$, $n \le p-1$ and renumbering the subindex n. The proof will be obtained by induction on j.

For $j = 1$, Corollary 21.4.8 in the case considered in Remark 21.4.9 with $A = \Omega$, $n = n_1 = 1$ and $\alpha = 1$ provides a measure $\mu_{11} \in M_{n_1}$ and $A_{11} \in \mathcal{A}$ such that

$$|\mu_{11}(A_{11})| > 1$$

and for each finite subset Q of $\mathcal{A}$ we have that

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|(\Omega\backslash A_{11}) = \infty,$$

for $n = n_1$, and for each natural number n that belongs to an infinite subset N_1 of $\mathbb{N}\backslash\{n_1\}$. Then let $n_2 = \min\{n : n \in N_1\}$.

By Lemma 21.4.10 with $A = \Omega\backslash A_{11}$, $n \in \{n_1, n_2\} \cup (N_1\backslash\{n_2\})$, $p = \alpha = 2$ and with $\{B_1, \cdots, B_n\}$ equal to $\{A_{11}\}$ we obtain two measures $\mu_{i2} \in M_{n_i}$, $i = 1, 2$, and two disjoints elements of $\mathcal{A}$, A_{12} and A_{22}, contained in $\Omega\backslash A_{11}$ such that

$$|\mu_{i2}(A_{i2})| > 2, \quad |\mu_{i2}(A_{11})| \le 1, \quad \text{for } 1 \le i \le 2$$

and for each finite subset Q of $\mathcal{A}$ we have that

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|(\Omega\backslash(A_{11}\cup A_{12}\cup A_{22})) = \infty,$$

for $n \in \{n_1, n_2\} \cup N_2$, where N_2 is an infinite subset of $N_1\backslash\{n_2\}$. Then we define $n_3 = \min\{n : n \in N_2\}$.

Let us suppose that the step j produces the measures $\mu_{ij} \in M_{n_i}$ and the pairwise disjoints elements A_{ij}, $1 \le i \le j$, contained in $\Omega\backslash(\cup\{A_{km} : 1 \le k \le m < j\})$ with $A_{ij} \in \mathcal{A}$ such that

$$\left|\mu_{ij}(A_{ij})\right| > j, \quad \sum_{1\le k\le m<j} \left|\mu_{ij}(A_{km})\right| \le 1, \quad \text{for } 1 \le i \le j$$

and for each finite subset Q of $\mathcal{A}$ we have that

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|\left(\Omega\backslash(\cup\{A_{km}:1\le k\le m\le j\})\right)=\infty$$

for $n=\{n_1,n_2,\cdots,n_j\}\cup N_j$, with N_j an infinite subset of $N_{j-1}\backslash\{n_j\}$.

Then we define $n_{j+1}=\min\{n:n\in N_j\}$ and from Lemma 21.4.10 with $A=\Omega\backslash(\cup\{A_{km}:1\le k\le m\le j\})$, $n\in\{n_1,n_2,\cdots,n_j,n_{j+1}\}\cup\{n\in N_j:n>n_{j+1}\}$, $p=\alpha=j+1$ and with $\{B_1,\cdots,B_n\}$ equal to $\{A_{km}:1\le k\le m\le j\}$ we obtain the measures $\mu_{i,j+1}\in M_{n_i}$ and the pairwise disjoints elements $A_{i,j+1}$ of $\mathcal{A}$, $1\le i\le j+1$, such that each $A_{i,j+1}$ is contained in $\Omega\backslash(\cup\{A_{km}:1\le k\le m\le j\})$, and for $1\le i\le j+1$

$$\left|\mu_{i,j+1}(A_{i,j+1})\right|>j+1,\qquad \sum_{1\le k\le m<j+1}\left|\mu_{i,j+1}(A_{km})\right|\le 1$$

and for each finite subset Q of $\mathcal{A}$ we have that

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|\left(\Omega\backslash(\cup\{A_{km}:1\le k\le m\le j+1\})\right)=\infty,$$

for $n=\{n_1,n_2,\cdots,n_j,n_{j+1}\}\cup N_{j+1}$, where N_{j+1} is an infinite subset of $N_j\backslash\{n_{j+1}\}$. To finish the induction we define $n_{j+2}=\min\{n:n\in N_{j+1}\}$. □

We are in the position to prove the main theorem of this section.

Theorem 21.4.12 *Let $\{\mathcal{B}_n:n\in\mathbb{N}\}$ be an increasing covering of a σ-field Σ of subsets of a set Ω. There exists $q\in\mathbb{N}$ such that $\mathcal{B}_n$ is a Nikodym set for* ba(Σ) *for each $n\ge q$.*

Proof Let us proceed by contradiction and suppose that every $\mathcal{B}_n$, $n\in\mathbb{N}$, is not a Nikodym set for ba(Σ). By the Nikodym theorem Σ is a Nikodym set for ba(Σ), hence by Proposition 21.4.11 with $\mathcal{A}=\Sigma$ and $\mathcal{B}=\Sigma$ there exists an increasing sequence $(n_i:i\in\mathbb{N})$ such that for each $(i,j)\in\mathbb{N}^2$, with $1\le i\le j$, there exists $A_{ij}\in\Sigma$ and $\mu_{ij}\in$ ba(Σ) which satisfy the inequalities

$$\left|\mu_{ij}(A_{ij})\right|>j,\qquad \sum_{1\le k\le m<j}\left|\mu_{ij}(A_{km})\right|\le 1,$$

the sets A_{ij} are pairwise disjoint and the set of measures $\{\mu_{ij}:j\in\mathbb{N},\ j\ge i\}$ is pointwise bounded in $\mathcal{B}_{n_i}$, for each $i\in\mathbb{N}$.

We claim that there exists a sequence $(i_m,j_m)_{m\in\mathbb{N}}$ such that $(i_m)_{m\in\mathbb{N}}$ is the sequence of the first components of the sequence obtained when the elements of $\mathbb{N}^2$ are ordered by the diagonal order, i.e.,

$$(i_1,i_2,i_3,i_4,i_5,i_6,i_7,\cdots)=(1,1,2,1,2,3,1,\cdots),$$

and $(j_m)_{m\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers such that for each $m \in \mathbb{N}$

$$\left|\mu_{i_m,j_m}\right|(\cup\{A_{i_p,j_p} : p > m\}) \leq 1.$$

Let $(i_1, j_1) := (1, 1)$. Suppose that $\left\|\mu_{i_1,j_1}\right\| \leq k_1$ and split the set $\{j \in \mathbb{N} : j > j_1\}$ in k_1 infinite subsets $N_{11}, \cdots, N_{1k_1}$. At least one of these subsets, named N_1, satisfies

$$\left|\mu_{i_1,j_1}\right|(\cup\{A_{i,j} : i \leq j,\, j \in N_1\}) \leq 1,$$

because

$$k_1 \geq \left\|\mu_{i_1,j_1}\right\| \geq \sum_{1\leq r\leq k_1} \left|\mu_{i_1,j_1}\right|(\cup\{A_{i,j} : i \leq j,\, j \in N_{1r}\}).$$

Then we define $j_2 := \inf\{j : j \in N_1\}$. Suppose that we have obtained the natural number $j_m := \inf\{j : j \in N_{m-1}\}$ and the infinite subset N_m of $\{n \in N_{m-1}\backslash\{j_m\}\}$ such that

$$\left|\mu_{i_m,j_m}\right|(\cup\{A_{i,j} : i \leq j,\, j \in N_m\}) \leq 1.$$

Then we define $j_{m+1} = \inf\{j : j \in N_m\}$ and if $\left\|\mu_{i_{m+1},j_{m+1}}\right\| \leq k_{m+1}$ we split the set $N_m\backslash\{j_{m+1}\}$ in k_{m+1} infinite subsets $N_{m+1,1}, \cdots, N_{m+1,k_{m+1}}$. At least one of these subsets, named N_{m+1}, satisfies

$$\left|\mu_{i_{m+1},j_{m+1}}\right|(\cup\{A_{i,j} : i \leq j,\, j \in N_{m+1}\}) \leq 1$$

because

$$k_{m+1} \geq \left\|\mu_{i_{m+1},j_{m+1}}\right\| \geq \sum_{1\leq r\leq k_{m+1}} \left|\mu_{i_{m+1},j_{m+1}}\right|(\cup\{A_{i,j} : i \leq j,\, j \in N_{m+1,r}\}).$$

As $A = \cup\{A_{i_m,j_m} : m \in \mathbb{N}\} \in \Sigma$ there exists $r \in \mathbb{N}$ such that $A \in \mathcal{B}_{n_r}$. By construction there exists and increasing sequence $(m_s : s \in \mathbb{N})$ such that each $i_{m_s} = n_r$, $s \in \mathbb{N}$. Therefore the set of measures $\{\mu_{i_{m_s},j_{m_s}} : s \in \mathbb{N}\} = \{\mu_{n_r,j_{m_s}} : s \in \mathbb{N}\}$ is pointwise bounded in $\mathcal{B}_{n_r}$. Then the relation $A \in \mathcal{B}_{n_r}$ implies that

$$\sup\{\left|\mu_{i_{m_s},j_{m_s}}(A)\right| : s \in \mathbb{N}\} = \sup\{\left|\mu_{n_r,j_{m_s}}(A)\right| : s \in \mathbb{N}\} < \infty. \tag{21.14}$$

Finally, from

$$\begin{aligned}\left|\mu_{n_r,j_{m_s}}(A)\right| &= \left|\mu_{i_{m_s},j_{m_s}}\left(\bigcup_{m\in\mathbb{N}} A_{i_m,j_m}\right)\right| \\ &\geq \left|\mu_{i_{m_s},j_{m_s}}(A_{i_{m_s},j_{m_s}})\right| - \sum_{1\leq k\leq m<j_{m_s}} \left|\mu_{i_{m_s},j_{m_s}}(A_{km})\right| \\ &\quad - \left|\mu_{i_{m_s},j_{m_s}}\right|\left(\bigcup_{m>j_{m_s}} A_{i_m,j_m}\right) > j_{m_s} - 2\end{aligned}$$

we get that $\lim_{s\to\infty}\left|\mu_{i_{m_s},j_{m_s}}(A)\right| = \infty$, in contradiction with (21.14). From this contradiction, it follows that there exists $\mathcal{B}_n$, $n \in \mathbb{N}$, which is a Nikodym set for $\mathrm{ba}(\Sigma)$. □

By the closed graph theorem, it follows that if μ is a bounded finitely additive vector measure defined in a σ-field Σ and with values in the inductive limit of a sequence of Fréchet spaces $(F_n)_n$, then there exists a natural number m such that μ is a bounded vector measure defined in the σ-field Σ and with values in F_m. More applications concerning localization of ranges of bounded finitely additive measures are provided in [609, Section 3].

21.5 Strong Properties (G) and (VHS)

Recall that a field of sets $\mathcal{A}$ has *the Grothendieck property* if the space $\widehat{L(\mathcal{A})}$ is a Grothendieck Banach space, i.e., if in the dual pair $\langle\widehat{L(\mathcal{A})}, \mathrm{ba}(\mathcal{A})\rangle$ the sequential weak* convergence in $\mathrm{ba}(\mathcal{A})$ implies sequential weak convergence in $\mathrm{ba}(\mathcal{A})$. Grothendieck's theorem (Theorem 19.4.10) implies that each σ-field has the Grothendieck property.

Notice that a field $\mathcal{A}$ of subsets of a set Ω has the Grothendieck property if and only if for each bounded sequence $(\mu_n)_{n=1}^{\infty}$ and each μ, both in $\mathrm{ba}(\mathcal{A})$, the equalities $\lim_{n\to\infty}\mu_n(A) = \mu(A)$, for each $A \in \mathcal{A}$, imply that the sequence $(\mu_n)_{n=1}^{\infty}$ converges weakly to μ. One side of this equivalence follows easily from the Banach–Steinhaus theorem, which says that the condition $\lim_{n\to\infty}\mu_n(f) = \mu(f)$, for each $f \in \widehat{L(\mathcal{A})}$, implies that the sequence $(\mu_n)_{n=1}^{\infty}$ is bounded in $\mathrm{ba}(\mathcal{A})$, and the other side follows directly from Claim 21.5.1, which enables us to say that if $(\mu_n)_{n=1}^{\infty}$ is a bounded sequence in $\mathrm{ba}(\mathcal{A})$ such that $\lim_{n\to\infty}\mu_n(A) = \mu(A)$, for every $A \in \mathcal{A}$ and some $\mu \in \mathrm{ba}(\mathcal{A})$, then $\lim_{n\to\infty}\mu_n(v) = \mu(v)$, for every $v \in \widehat{L(\mathcal{A})}$ (cf. Lemma 19.1.2).

Claim 21.5.1 Let F be a subset of a Banach space E, $\overline{F}$ the closure of F and let $(\mu_n)_{n=1}^{\infty}$ a bounded sequence in its dual E^* endowed with the polar norm. If $\mu \in E^*$ and $\lim_n \mu_n(f) = \mu(f)$, for each $f \in F$, then $\lim_n \mu_n(v) = \mu(v)$, for every $v \in \overline{F}$.

Proof By density, for $\varepsilon > 0$ and $v \in \overline{F}$ there exists $f \in F$ such that $\|v - f\| < \varepsilon(2(1 + \|\mu\| + \sup_n \|\mu_n\|)^{-1}$. By the hypothesis that $\lim_n \mu_n(f) = \mu(f)$ there exists n_ε such that $|(\mu_n - \mu)(f)| < 2^{-1}\varepsilon$, for every $n > n_\varepsilon$. Hence, for $n > n_\varepsilon$ we have that

$$|(\mu_n - \mu)(v - f)| + |(\mu_n - \mu)(f)| < \frac{\varepsilon(\|\mu\| + \sup_n \|\mu_n\|)}{2(1 + \|\mu\| + \sup_n \|\mu_n\|)} + \frac{\varepsilon}{2} \leq \varepsilon,$$

so $|(\mu_n - \mu)(v)| \leq |(\mu_n - \mu)(v - f)| + |(\mu_n - \mu)(f)| < \varepsilon$. This proves that $\lim_n \mu_n(v) = \mu(v)$. □

The above described equivalence and the definition of property (sN) motivate the following definition. A subset $\mathcal{B}$ of a field $\mathcal{A}$ of subsets of Ω has:

- *property* (G) (for $\mathrm{ba}(\mathcal{A})$), if for each bounded sequence $(\mu_n)_{n=1}^{\infty}$ and μ, both in $\mathrm{ba}(\mathcal{A})$ and such that $\lim_{n\to\infty} \mu_n(B) = \mu(B)$ for each $B \in \mathcal{B}$, we have that $(\mu_n)_{n=1}^{\infty}$ converges weakly to μ,
- *strong property* (G) (for $\mathrm{ba}(\mathcal{A})$), or *property* (sG) in brief, if for each increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ there exists $p \in \mathbb{N}$ such that $\mathcal{B}_p$ has property (G) for $\mathrm{ba}(\mathcal{A})$.

Of course, if a field $\mathcal{A}$ has the Grothendieck property, then the set $\mathcal{A}$ has property (G).

Theorem 21.5.2 *Let $\mathcal{B}$ be a subset of a field $\mathcal{A}$ of subsets of a set Ω such that $\mathcal{B}$ is a Nikodym set for* $\mathrm{ba}(\mathcal{A})$ *and $\mathcal{A}$ has property (G). Then $\mathcal{B}$ has strong property (G) for* $\mathrm{ba}(\mathcal{A})$.

Proof Let $\{\mathcal{B}_n : n \in \mathbb{N}\}$ be an increasing covering of $\mathcal{B}$. From Proposition 21.4.2, we deduce that there exists a natural number p such that

$$\overline{\mathrm{span}\{e(A) : A \in \mathcal{B}_p\}}^{L(\mathcal{A})} = L(\mathcal{A}). \tag{21.15}$$

Let $(\mu_n)_n$ be a bounded sequence in $\mathrm{ba}(\mathcal{A})$ and $\mu \in \mathrm{ba}(\mathcal{A})$ such that

$$\lim_{n\to\infty} \mu_n(A) = \mu(A),$$

for every $A \in \mathcal{B}_p$. By Claim 21.5.1 and (21.15) we have that $\lim_{n\to\infty} \mu_n(A) = \mu(A)$, for every $A \in \mathcal{A}$. Then, as $\mathcal{A}$ has property (G) we have that $(\mu_n)_n$ converges weakly to μ, hence $\mathcal{B}_p$ has property (G) for $\mathrm{ba}(\mathcal{A})$ and we get that $\mathcal{B}$ has strong property (G) for $\mathrm{ba}(\mathcal{A})$. □

In the particular case of a σ-field Σ, Grothendieck's Theorem 19.4.10 says that the set Σ has property (G), so the preceding theorem refines Grothendieck's result

because it states that if $\{\mathcal{B}_n : n \in \mathbb{N}\}$ is an increasing covering of Σ, then there exists $p \in \mathbb{N}$ such that $\mathcal{B}_p$ has property (G) for $\mathrm{ba}(\mathcal{A})$. Notice that this result is the exact counterpart for property (G) of Valdivia's Theorem 21.4.12 asserting that if a σ-field Σ is covered by an increasing sequence $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of subsets, there is $p \in \mathbb{N}$ such that $\mathcal{B}_p$ has property (N) for $\mathrm{ba}(\mathcal{A})$.

In [542, Introduction] it is stated that each σ-field Σ satisfies the conclusion of the Vitali–Hahn–Saks theorem, that is, that every sequence $(\mu_n)_{n=1}^{\infty}$ of $\mathrm{ba}(\Sigma)$ such that there exists $\mu \in \mathrm{ba}(\Sigma)$ which satisfies the equality

$$\lim_{n\to\infty} \mu_n(B) = \mu(B), \text{ for every } B \in \Sigma,$$

is *uniformly exhaustive*, i.e., for each sequence $(B_j)_{j=1}^{\infty}$ of pairwise disjoint subsets of Σ

$$\lim_{j\to\infty} \mu_n(B_j) = 0$$

uniformly in $n \in \mathbb{N}$. We say that a field $\mathcal{A}$ of subsets of a set Ω has *property* (VHS) if it satisfies the thesis of the Vitali–Hahn–Saks theorem. Equivalently, $\mathcal{A}$ has property (VHS) if $\mathcal{A}$ has both properties (N) and (G) (see [542, Theorem 2.5] or [231, Theorem 4.2]).

Notice that if a field $\mathcal{A}$ has (VHS) and a sequence $(\mu_n)_{n=1}^{\infty}$ in $\mathrm{ba}(\mathcal{A})$ is such that there exists $\mu \in \mathrm{ba}(\mathcal{A})$ with $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for every $A \in \mathcal{A}$, then $(\mu_n)_{n=1}^{\infty}$ is a bounded subset of $\mathrm{ba}(\mathcal{A})$, by property (N) of $\mathcal{A}$. Consequently, property (G) of $\mathcal{A}$ implies that $(\mu_n)_{n=1}^{\infty}$ converges weakly to μ. Thus, a field $\mathcal{A}$ has property (VHS) if and only if $\mathcal{A}$ has property (N) and for every sequence $(\mu_n)_{n=1}^{\infty}$ and μ, both in $\mathrm{ba}(\mathcal{A})$ and such that $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for every $A \in \mathcal{A}$, $(\mu_n)_{n=1}^{\infty}$ converges weakly to μ. This last equivalence and the definitions of properties (sN) and (sG) suggest the two parts of the following definition. A subset $\mathcal{B}$ of a field $\mathcal{A}$ of subsets of a set Ω has:

- *property* (VHS) (for $\mathrm{ba}(\mathcal{A})$), if $\mathcal{B}$ is a Nikodym set for $\mathrm{ba}(\mathcal{A})$ and for each sequence $(\mu_n)_{n=1}^{\infty}$ and μ, both in $\mathrm{ba}(\mathcal{A})$ and such that $\lim_{n\to\infty} \mu_n(B) = \mu(B)$ for each $B \in \mathcal{B}$, $(\mu_n)_{n=1}^{\infty}$ converges weakly to μ,
- *strong property* (VHS) for $\mathrm{ba}(\mathcal{A})$, or *property* $(sVHS)$ in brief, if for each increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ there exists $p \in \mathbb{N}$ such that $\mathcal{B}_p$ has property (VHS) for $\mathrm{ba}(\mathcal{A})$.

A subset $\mathcal{B}$ of a field $\mathcal{A}$ has property (VHS) if and only if $\mathcal{B}$ has properties (N) and (G). In particular, a field $\mathcal{A}$ has strong property (VHS) if and only if $\mathcal{A}$ has strong property (N) and strong property (G).

From the following corollary, we deduce immediately that if a σ-field Σ is covered by an increasing sequence $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of subsets, there is $p \in \mathbb{N}$ such that $\mathcal{B}_p$ has property (VHS) for $\mathrm{ba}(\mathcal{A})$, providing the counterpart for property (VHS) of Valdivia's result for Nikodym sets.

Corollary 21.5.3 *Let $\mathcal{B}$ be a subset of a field $\mathcal{A}$ of subsets of a set Ω such that $\mathcal{B}$ has strong property (N) for* ba$(\mathcal{A})$ *and $\mathcal{A}$ has property (G). Then $\mathcal{B}$ has strong property (VHS) for* ba$(\mathcal{A})$.

Proof Let $\{\mathcal{B}_n : n \in \mathbb{N}\}$ be an increasing covering of $\mathcal{B}$. From the preceding theorem and the hypothesis that $\mathcal{B}$ has strong property (N) we deduce that there exists a natural number p such that $\mathcal{B}_p$ has properties (N) and (G) for ba$(\mathcal{A})$. Hence $\mathcal{B}_p$ is a set with property (VHS) for ba$(\mathcal{A})$. □

In particular, when $\mathcal{B} = \mathcal{A}$, we have the following characterization which follows directly from the above results.

Corollary 21.5.4 *Let $\mathcal{A}$ be a field of subsets of a set Ω which has strong property (N). Then, $\mathcal{A}$ has strong property (VHS) if and only if $\mathcal{A}$ has property (VHS).*

Therefore, an affirmative answer to Valdivia's open question, asking if it is true that for every field $\mathcal{A}$ property (N) of $\mathcal{A}$ implies property (sN) of $\mathcal{A}$ [622, Problem 1], provides an affirmative answer also to the following corresponding problem with property (VHS).

Problem 21.5.5 Let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be an increasing covering of a field $\mathcal{A}$ with property (VHS). Is it true that there exists $p \in \mathbb{N}$ such that $\mathcal{A}_p$ has property (VHS) for ba$(\mathcal{A})$?

21.6 Web Properties

Let us recall that by definition a family of subsets $\{\mathcal{B}_{n_1 n_2 \cdots n_m} : n_i \in \mathbb{N}, 1 \leq i \leq m, m \in \mathbb{N}\}$ of a set $\mathcal{B}$ is *an increasing web* if $\{\mathcal{B}_{n_1} : n_1 \in \mathbb{N}\}$ is an increasing covering of $\mathcal{B}$ and for each $m \in \mathbb{N}$ and each $(n_1 n_2 \cdots n_m) \in \mathbb{N}^m$ the countable family $\{\mathcal{B}_{n_1 n_2 \cdots n_m n_{m+1}} : n_{m+1} \in \mathbb{N}\}$ is an increasing covering of $\mathcal{B}_{n_1 n_2 \cdots n_m}$.

A subset $\mathcal{B}$ of a field $\mathcal{A}$ of subsets of a set Ω is a *web Nikodym set* (for ba$(\mathcal{A})$), or has *property* (wN) in brief, if for each increasing web $\{\mathcal{B}_{n_1 n_2 \cdots n_m} : n_i \in \mathbb{N}, 1 \leq i \leq m, m \in \mathbb{N}\}$ of $\mathcal{B}$ there is a sequence $(p_m)_{m=1}^{\infty}$ such that $\mathcal{B}_{p_1 p_2 \cdots p_m}$ has property (N) for ba$(\mathcal{A})$ for every $m \in \mathbb{N}$. It is obvious that $(wN) \Rightarrow (sN) \Rightarrow (N)$, and that $\mathcal{B}$ has property (wN) if and only if each increasing web of $\mathcal{B}$ contains an increasing web formed by elements that have property (wN). It is direct to prove that properties (wN), $(s(wN))$, $(w(sN))$, and $(w(wN))$ (defined in a natural way) are equivalent, see [416, Proposition 1].

Properties (wG) and $(wVHS)$ are defined as in the case of property (wN), changing N by G or by VHS. A subset $\mathcal{B}$ of a field $\mathcal{A}$ of subsets of a set Ω has *property* (wG) (*property* $(wVHS)$) for ba$(\mathcal{A})$ if for each increasing web $\{\mathcal{B}_{n_1 n_2 \cdots n_m} : n_i \in \mathbb{N}, 1 \leq i \leq m, m \in \mathbb{N}\}$ of $\mathcal{B}$ there exists a sequence $(p_m)_{m=1}^{\infty}$ of natural numbers such that $\mathcal{B}_{p_1 p_2 \cdots p_m}$ has property (G) (property (VHS)) for ba$(\mathcal{A})$, for every $m \in \mathbb{N}$. Clearly $(wG) \Rightarrow (sG) \Rightarrow (G)$, and $(wVHS) \Rightarrow (sVHS) \Rightarrow (VHS)$.

Recently it has been proved that every σ-field Σ has properties (wN) (see [337] and [419]) and (wG) (see [227]). Every σ-field Σ has also property $(wVHS)$, because a set $\mathcal{B}$ has property $(wVHS)$ if and only if $\mathcal{B}$ has properties (wN) and (wG) [415, Corollary 2]. The case of arbitrary fields is more delicate. For instance, Schachermayer's result [542], asserting that the field $\mathcal{J}([0,1])$ of all Jordan measurable subsets of $[0,1]$ has property (N) but fails property (G), was complemented in [622] by getting that, for $p \in \mathbb{N}$, the field $\mathcal{J}([0,1]^p)$ of all Jordan measurable subsets of $[0,1]^p$ has property (sN) and, more recently, it was proved in [413] that $\mathcal{J}([0,1]^p)$ has property (wN). From Schachermayer's result it follows that the field $\mathcal{J}([0,1]^p)$ does not have property (G). It was also observed in [183] that the ring $\mathcal{Z}$ of subsets of $\mathbb{N}$ of asymptotic density zero has property (N) but it does not have property (G), and in [226, Remark 3.4] it is proved that $\mathcal{Z}$ has property (wN) (cf. also [226, Theorem 3.3]).

As in the preceding section we prove the following result (cf. Corollary 21.5.3).

Proposition 21.6.1 *Let $\mathcal{B}$ be a subset of a field $\mathcal{A}$ of subsets of a set Ω such that $\mathcal{B}$ is a web Nikodym set for* $\mathrm{ba}(\mathcal{A})$ *and $\mathcal{A}$ has property (G). Then $\mathcal{B}$ has property $(wVHS)$ for* $\mathrm{ba}(\mathcal{A})$.

Chapter 22
Banach Spaces with Many Projections

Abstract In this chapter, we discuss Banach spaces that have a rich family of norm-one projections onto separable subspaces. One of the tools, coming from logic, is the concept of an elementary submodel.

22.1 Preliminaries, Model-Theoretic Tools

In this chapter, we discuss Banach spaces which have a rich family of norm one projections onto separable subspaces.

Recall that a *projection* in a Banach space E is a bounded linear operator $P\colon E \to E$ such that $P^2 = P$. A subspace Y of a Banach space E is *complemented* if there exists a projection $P\colon E \to E$ such that $Y = P[E]$. Equivalently, there exists a closed subspace Z of E such that $Y \cap Z = 0$ and $Y + Z = E$. As usual, we write $E = Y \oplus Z$.

One of the important tools for investigating non-separable Banach spaces and, in particular, for constructing projections is the method of elementary substructures. Roughly speaking, given a Banach space X, we shall use a countable structure M such that $X_M := \overline{X \cap M}$ is a separable subspace of X which shares some properties of X or the embedding $X_M \subset X$ has certain useful property.

Elementary substructures are used successfully in set theory and topology, replacing various closing-off arguments. We shall now explain this concept.

Let N be a fixed set. The pair $\langle N, \in\rangle$, where $\in$ is restricted to $N \times N$, is a structure of the language of set theory. Given a formula $\varphi(x_1, \ldots, x_n)$ with all free variables shown and given $a_1, \ldots, a_n \in N$ one defines the relation "$\langle N, \in\rangle$ satisfies $\varphi(a_1, \ldots, a_n)$" (briefly "$\langle N, \in\rangle \models \varphi(a_1, \ldots, a_n)$" or just "$N \models \varphi(a_1, \ldots, a_n)$") in the usual way, by induction on the length of the formula. Namely, $N \models a_1 \in a_2$ if and only if $a_1 \in a_2$ and $N \models a_1 = a_2$ if and only if $a_1 = a_2$. It is clear how "satisfaction" is defined for conjunction, disjunction and negation. Finally, if φ is of the form $(\exists\ y)\psi(x_1, \ldots, x_n, y)$ then $N \models \varphi(a_1, \ldots, a_n)$ if and only if there exists $b \in N$ such that $N \models \psi(a_1, \ldots, a_n, b)$. For example, if $s = \{a, b, c\}$ and $s, a, b \in N$ while $c \notin N$, then N satisfies "s has at most two elements", because for every $x \in N$ if $x \in s$ then either $x = a$ or $x = b$.

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Instead of the above definition, some authors use relativization, see e.g. Kunen's book [389]. Given a formula φ, the *relativization* of φ to N is a formula φ^N which is built from φ by replacing each quantifier of the form "$\forall\ x$" by "$\forall\ x \in N$" and each quantifier of the form "$\exists\ x$" by "$\exists\ x \in N$". By this way, $N \models \varphi(a_1, \ldots, a_n)$ if and only if $\varphi^N(a_1, \ldots, a_n)$ holds (of course, $a_1, \ldots, a_n$ must be elements of N).

Given a set x, we define the *transitive closure* of x to be $\mathrm{tc}(x) = \bigcup_n \mathrm{tc}_n(x)$, where $\mathrm{tc}_n(x) = x \cup \bigcup x$ and $\mathrm{tc}_{n+1}(x) = \mathrm{tc}_1(\mathrm{tc}_n(x))$.

In other words: $y \in \mathrm{tc}(x)$ if and only if there are $x_0 \in x_1 \in \cdots \in x_k$ such that $y \in x_0$ and $x_k \in x$. Thanks to the Axiom of Regularity, these two definitions of transitive closure are equivalent and every set of the form $\mathrm{tc}(x)$ is *transitive*, i.e., $y \in \mathrm{tc}(x)$ implies $y \subseteq \mathrm{tc}(x)$.

Given a cardinal θ, we denote by $H(\theta)$ the class of all sets whose transitive closure has cardinality $< \theta$. It is well known that $H(\theta)$ is a set, not a proper class. It is clear that $H(\theta)$ is transitive. We shall consider elementary substructures of $\langle H(\theta), \in\rangle$. It is well known that for a regular uncountable cardinal θ, the structure $\langle H(\theta), \in\rangle$ satisfies all the axioms of set theory, except possibly the Power Set Axiom, see [389, IV.3].

Recall that a substructure M of $\langle H(\theta), \in\rangle$ is called *elementary* if for every formula $\varphi(x_1, \ldots, x_n)$ with all free variables shown, for every $a_1, \ldots, a_n \in M$, we have that

$$M \models \varphi(a_1, \ldots, a_n) \iff H(\theta) \models \varphi(a_1, \ldots, a_n).$$

The fact that M is an elementary submodel of $\langle H(\theta), \in\rangle$ is denoted by $M \preceq \langle H(\theta), \in\rangle$.

The reason for using elementary submodels of $H(\theta)$ is that these structures satisfy most of the axioms of set theory: if $\theta > \aleph_0$ is regular then $H(\theta)$ satisfies all the axioms except possibly the power-set, because it may happen that $2^\lambda > \theta$ for some $\lambda < \theta$. Moreover, in practice it is usually easy to point out a cardinal θ such that $H(\theta)$ satisfies given finitely many formulas with parameters, needed for applications.

Another useful feature of $H(\theta)$ with θ regular is the fact for every formula $\varphi(x_1, \ldots, x_n)$ in which all quantifiers are bounded (i.e., of the form "$\forall\ x \in y$" or "$\exists\ x \in y$") and for every $a_1, \ldots, a_n \in H(\theta)$, $\varphi(a_1, \ldots, a_n)$ holds if and only if

$$H(\theta) \models \varphi(a_1, \ldots, a_n).$$

For more information, see [389, IV.3]. Since in most cases we indeed use formulas with bounded quantifiers, one can simply "check" their validity by looking at a sufficiently large $H(\theta)$.

One can also use *Reflection Principle*, which says that given a formula of set theory $\varphi(x_1, \ldots, x_n)$ and given sets $a_1, \ldots, a_n$ such that $\varphi(a_1, \ldots, a_n)$ holds, there exists θ such that the structure $\langle H(\theta), \in\rangle$ satisfies $\varphi(a_1, \ldots, a_n)$. In some cases, θ may not be regular, although it may be arbitrarily big and it may have arbitrarily big

cofinality. More precisely: the class of cardinals θ with the above property is closed and unbounded. Thus, when considering finitely many formulas and parameters, we can "check" their validity by restricting attention to $H(\theta)$, where θ is a "big enough" cardinal, meaning that the cofinality of θ is greater than a prescribed cardinal and all relevant formulas are satisfied in $\langle H(\theta), \in\rangle$.

Summarizing: assume we would like to use in our arguments formulas $\varphi_1, \ldots, \varphi_n$ and parameters from a finite set S. We then find a cardinal θ so that $S \subseteq H(\theta)$ and, by Reflection Principle, all valid formulas $\varphi_1, \ldots, \varphi_n$ with suitable parameters are satisfied in $H(\theta)$. Finally, we shall use elementary substructures of $H(\theta)$ which contain S. If the formulas $\varphi_1, \ldots, \varphi_n$ have only bounded quantifiers, then we do not need to use Reflection Principle, since the formulas will be satisfied in every $H(\theta)$ with θ big enough, i.e., every regular θ greater than some fixed cardinal θ_0.

A particular case of the Löwenheim–Skolem theorem (for the language of set theory) says that for every infinite set $S \subseteq H(\theta)$ there exists $M \preceq \langle H(\theta), \in\rangle$ such that $|M| = |S|$. This theorem can be viewed as the "ultimate" closing-off argument and its typical proof indeed proceeds by "closing-off" the given set S, by adding elements which witness "satisfaction" of all suitable formulas of the form $(\exists\ x)\ \psi$. Important for applications is the fact that, thanks to the Löwenheim–Skolem theorem, we may consider *countable* elementary substructures of an arbitrarily large $H(\theta)$.

Proposition 22.1.1 *Let θ be an uncountable regular cardinal, and let $M \preceq \langle H(\theta), \in\rangle$.*

(a) Assume $u \in H(\theta)$, $a_1, \ldots, a_n \in M$ and $\varphi(y, x_1, \ldots, x_n)$ is a formula such that u is the unique element of $H(\theta)$ for which we have $H(\theta) \models \varphi(u, a_1, \ldots, a_n)$. Then $u \in M$.
(b) Let $s \subseteq M$ be a finite set. Then $s \in M$.
(c) If $S \in M$ is a countable set then $S \subseteq M$.

Proof (a) By elementarity there exists $v \in M$ such that $M \models \varphi(v, a_1, \ldots, a_n)$. Using elementarity again, we see that $H(\theta) \models \varphi(v, a_1, \ldots, a_n)$. Thus $u = v$.

(b) Let $s = \{a_1, \ldots, a_n\}$. Then s is the unique set satisfying the formula $\varphi(s, a_1, \ldots, a_n)$, where $\varphi(x, y_1, \ldots, y_n)$ is

$$(\forall\ t)\ t \in x \iff t = y_1 \vee t = y_2 \vee \cdots \vee t = y_n.$$

Applying (a), we see that $s \in M$.

(c) By induction and by (a), we see that all natural numbers are in M. Also by (a), the set of natural numbers $\mathbb{N}$ is an element of M, being uniquely defined as the minimal infinite ordinal. Notice that $H(\theta)$ satisfies "there exists a surjection from $\mathbb{N}$ onto S".

By elementarity, there exists $f \in M$ such that M satisfies "f is a surjection from $\mathbb{N}$ onto S". Again using (a), we see that $f(n) \in M$ for each $n \in \mathbb{N}$. Finally, it

suffices to observe that f is indeed a surjection, i.e., for every $x \in S$ there is n such that $x = f(n)$. This follows from elementarity, because assuming $f[\mathbb{N}] \neq S$, the formula "$(\exists\, x \in S)(\forall\, n \in \omega)\ \ x \neq f(n)$" would be satisfied in M, contradicting that f is a surjection. □

Now fix a Banach space X and choose a "big enough" regular cardinal θ, so that $X \in H(\theta)$. Take an elementary substructure M of $\langle H(\theta), \in\rangle$ such that $X \in M$. Note that M may be countable, by the Löwenheim–Skolem theorem.

What can we say about the set $X \cap M$? By elementarity, it is closed under addition. By Proposition 22.1.1(a), the field of rationals is contained in M, therefore $X \cap M$ is a $\mathbb{Q}$-linear subspace of X. Consequently, the norm closure of $X \cap M$ is a Banach subspace of X. In particular, the weak closure of $X \cap M$ equals the norm closure of $X \cap M$. We shall write X_M instead of $\overline{(X \cap M)}$ and we shall call X_M the *subspace induced by* M.

In case of some typical Banach spaces, we can describe the subspace X_M. For instance, let $X = \ell_p(\Gamma)$, where $1 \leqslant p < \infty$ and Γ is an uncountable set. Then X_M can be identified with $\ell_p(\Gamma \cap M)$. Indeed, identify $x \in \ell_p(\Gamma \cap M)$ with its extension $x' \in \ell_p(\Gamma)$ defined by $x'(\alpha) = 0$ for $\alpha \in \Gamma \setminus M$. Let $x \in X \cap M$. Then

$$\operatorname{supp}(x) = \{\alpha \in \Gamma \colon x(\alpha) \neq 0\}$$

is a countable set, and hence, by elementarity, it belongs to M. By Proposition 22.1.1(c) we have $\operatorname{supp}(x) \subseteq M$. Thus $x \in \ell_p(\Gamma \cap M)$. On the other hand, if

$$x \in \ell_p(\Gamma \cap M),$$

then arbitrarily close to x we can find $y \in \ell_p(\Gamma \cap M)$ such that $s = supp(y)$ is finite. Moreover, we may assume that $y(\alpha) \in \mathbb{Q}$ for $\alpha \in s$. By Proposition 22.1.1(b), $y \restriction s \in M$ and consequently also $y \in M$. Hence

$$x \in \overline{(X \cap M)} = X_M.$$

Given a compact space $K \in H(\theta)$ and $M \preceq \langle H(\theta), \in\rangle$, define the following equivalence relation $\sim_M$ on K:

$$x \sim_M y \iff (\forall\, f \in C(K) \cap M)\ f(x) = f(y).$$

We shall write K/M instead of $K/{\sim_M}$ and we shall denote by q^M (or, more precisely, q_K^M) the canonical quotient map. It is not hard to check that K/M is a compact Hausdorff space of weight not exceeding the cardinality of M. This construction has been used by Bandlow [70, 71] for characterizing Corson compact spaces in terms of elementary substructures.

Lemma 22.1.2 *Let K be a compact space, let θ be a big enough regular cardinal and let $M \preceq \langle H(\theta), \in\rangle$ be such that $K \in M$. Then*

$$\overline{C(K) \cap M} = \{\varphi \circ q^M : \varphi \in C(K/M)\},$$

where the closure is the norm closure in the above formula.

Proof Let Y denote the set on the right-hand side. Then Y is a closed linear subspace of $C(K)$. Given $\psi \in C(K) \cap M$, by the definition of $\sim_M$, there exists a (necessarily continuous) function ψ' such that $\psi = \psi' \circ q^M$. Thus $C(K) \cap M \subseteq Y$. Let

$$R = \{\varphi \in C(K/M) : \varphi \circ q^M \in M\}.$$

Then R is a subring of $C(K/M)$ which separates points and which contains all rational constants. By the Stone–Weierstrass theorem, R is dense in $C(K/M)$, which implies that $C(K) \cap M$ is dense in Y. □

Observe that, under the assumptions of the above lemma, the norm closure of $C(K \cap M)$ is pointwise closed. Indeed, if

$$f \in C(K) \setminus \overline{(K \cap M)},$$

then there are $x, y \in K$ such that $x \sim_M y$ while $f(x) \neq f(y)$. Consequently,

$$V = \{g : g(x) \neq g(y)\}$$

is a neighbourhood of f in the pointwise convergence topology, disjoint from $\overline{(K \cap M)}$.

Given a map of compact spaces $f : K \to L$ and given $M \preceq H(\theta)$ with $f \in M$, observe that the relation $\sim_M$ is preserved by M, i.e.,

$$x \sim_M y \implies f(x) \sim_M f(y).$$

Indeed, if $f(x) \not\sim_M f(y)$ and $\varphi \in C(L) \cap M$ witnesses it, then $\varphi \circ f \in C(K) \cap M$ separates x and y. As a consequence of this observation, for every map $f : K \to L$ in M there exists a (necessarily continuous) map $f^M : K/M \to L/M$, for which the diagram

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ {\scriptstyle q_K^M}\downarrow & & \downarrow{\scriptstyle q_L^M} \\ K/M & \xrightarrow[f^M]{} & L/M \end{array}$$

commutes. The continuity of f^M follows from the continuity of $q_L^M \circ f$ and from the fact that q_K^M is a quotient map. It is an easy exercise to show that $(f \circ g)^M = f^M \circ g^M$ whenever $f, g \in M$ are compatible. Clearly, $(\mathrm{id}_K)^M = \mathrm{id}_{K/M}$, so $f \mapsto f^M$ is a functor from the category of compact spaces and continuous maps in M into the category of compact spaces of weight $\leqslant |M|$.

Further, we show that this functor preserves finite products.

Lemma 22.1.3 *Let K, L be compact spaces, let θ be a big enough regular cardinal and let $M \preceq \langle H(\theta), \in\rangle$ be such that $K, L \in M$. Then $(K \times L)/M$ is naturally homeomorphic to $(K/M) \times (L/M)$. The homeomorphism is given by the formula*

$$[\langle x, y\rangle]_{\sim_M} \mapsto \langle [x]_{\sim_M}, [y]_{\sim_M}\rangle. \tag{22.1}$$

Proof Let pr_K, pr_L denote the respective projections. Then $\mathrm{pr}_K^M([\langle x, y\rangle]_{\sim_M}) = [x]_{\sim_M}$ and $\mathrm{pr}_L^M([\langle x, y\rangle]_{\sim_M}) = [y]_{\sim_M}$. This shows that (22.1) well defines a continuous map which is clearly a surjection. It remains to show that it is one-to-one. For fix $x \sim_M x'$ and $y \sim_M y'$. We must show that $\langle x, y\rangle \sim_M \langle x', y'\rangle$.

Suppose otherwise and choose $\varphi \in M \cap C(K \times L)$ so that $\varphi(x, y) < \varphi(x', y')$. Find rational numbers s, t so that $\varphi(x, y) < s < t < \varphi(x', y')$. Then $A = \varphi^{-1}[(\leftarrow, s]]$ and $B = \varphi^{-1}[[t, \rightarrow)]$ are disjoint closed subsets of $K \times L$ such that $\langle x, y\rangle \in A$ and $\langle x', y'\rangle \in B$. Further, $A, B \in M$. Using compactness and elementarity, we may find finite families $\mathcal{U}, \mathcal{V} \in M$ if basic open subsets of $K \times L$ such that $\mathcal{U}$ covers A, $\mathcal{V}$ covers B and

$$\overline{\bigcup \mathcal{U}} \cap \overline{\bigcup \mathcal{V}} = \emptyset.$$

Find $U \in \mathcal{U}$ and $V \in \mathcal{V}$ so that $\langle x, y\rangle \in U$ and $\langle x', y'\rangle \in V$. Then $U = U_0 \times U_1$, $V = V_0 \times V_1$ and $U_0, U_1, V_0, V_1 \in M$. Find $i < 2$ such that $\overline{U_i} \cap \overline{V_i} = \emptyset$. By symmetry, we may assume that $i = 0$. Let $A_0 = \overline{U_0}$ and $B_0 = \overline{V_0}$. Then $A_0, B_0 \in M$ are closed, disjoint and $x \in A_0$, $x' \in B_0$. Using Urysohn's lemma and elementarity, there is $g \in M \cap C(K)$ such that $g \restriction A_0 = 0$ and $g \restriction B_0 = 1$. Finally, $g(x) = 0$ and $g(x') = 1$, which shows that $x \not\sim_M x'$. □

The next statement shows that typical algebraic structures on a compact space are preserved under canonical quotients.

Corollary 22.1.4 *Let K be a compact space, $n \in \omega$ and let $a\colon K^n \to K$ be a continuous map. Further, let $M \preceq \langle H(\theta), \in\rangle$, where θ is big enough and $a \in M$. Then M induces a continuous map $a^M\colon (K/M)^n \to K/M$ such that the canonical quotient q_K^M becomes a homomorphism of the structures $\langle K, a\rangle$ and $\langle K/M, a^M\rangle$.*

Proof By Lemma 22.1.3, $(K/M)^n$ is homeomorphic to K^n/M and therefore a^M is defined as the unique map satisfying $q_K^M \circ a = a^M \circ q_{K^n}^M$. This equation actually shows that a^M is a homomorphism. □

Corollary 22.1.5 *Let $\langle K, <\rangle$ be a compact line, let $M \preceq H(\theta)$ be such that θ is big enough and $\langle K, <\rangle \in M$. Then K/M is a compact line, and the canonical quotient $q_M\colon K \to K/M$ is increasing.*

Proof One can look at K as a meet semilattice $\langle K, \wedge\rangle$, where $x \wedge y = \min\{x, y\}$. By Corollary 22.1.4, K/M has a binary operator $\wedge^M$ and q^M is a homomorphism. Since it is surjective, it is clear that $\wedge^M$ is a meet semilattice which defines a linear order by $p \leqslant q \iff p = p \wedge^M q$. □

Lemma 22.1.6 *Let K be a totally disconnected compact space, let θ be a big enough regular cardinal, and let $M \preceq H(\theta)$ be such that $K \in M$. Then K/M is totally disconnected.*

Proof Fix $x, y \in K$ so that $x \not\sim_M y$ and choose $f\colon K \to \mathbb{I}$ in M such that $f(x) < f(y)$. Find disjoint closed rational intervals I, J such that $f(x) \in I$, $f(y) \in J$. Then $A = f^{-1}[I]$ and $B = f^{-1}[J]$ are disjoint closed subsets of K and $A, B \in M$. Using the 0-dimensionality of K and elementarity, find a clopen set $U \in M$ such that $B \subseteq U$ and $A \cap U =$. Then the characteristic function of U shows that $[x]_{\sim_M}$ and $[y]_{\sim_M}$ are separated by a continuous 2-valued map. □

Proposition 22.1.7 *There exists a continuous surjection $\varphi\colon 2^\omega \to \mathbb{I}$ such that for every 0-dimensional compact K the map $T\colon C(K, 2^\omega) \to C(K, \mathbb{I})$ defined by*

$$Tf = f \circ \varphi$$

is a surjection. In other words, for every $g\colon K \to \mathbb{I}$ there is $f\colon K \to 2^\omega$ such that $g = f \circ \varphi$.

Note that the map φ does not depend on K.

Proof First, observe that, given a 0-dimensional compact K, every map $f\colon K \to \mathbb{I}$ factorizes through the Cantor set. More precisely, there are maps $g\colon K \to 2^\omega$ and and $h\colon 2^\omega \to \mathbb{I}$ such that $f = h \circ g$. In order to see it, fix a countable elementary submodel M of a big enough $H(\theta)$ such that $f \in M$. Then $f = f^M \circ q_K^M$ and K/M is a 0-dimensional metric compact, by Lemma 22.1.6. Thus K/M embeds into the Cantor set. Taking a retraction $r\colon 2^\omega \to K/M$, we see that $f = f^M \circ r \circ i \circ q_K^M$, where i is the left-inverse to r.

Thus, it suffices to prove our statement for $K = 2^\omega$. Let Z be the Čech–Stone compactification of the disjoint topological sum $\bigoplus_{f \in C(2^\omega, \mathbb{I})} 2^\omega \times \{f\}$. Note that Z is totally disconnected. Let $F\colon Z \to \mathbb{I}$ be the (uniquely defined) map satisfying $F(x, f) = f(x)$ for $\langle x, f\rangle \in 2^\omega \times \{f\}$.

Using the same factorization property as before, we obtain that $F = G \circ \varphi$ for some map $G\colon Z \to 2^\omega$. It is clear that φ is the desired map. □

22.2 Projections from Elementary Submodels

We show how to use elementary submodels for constructing projections in some Banach spaces.

Let $(X, \|\cdot\|)$ be a Banach space. Recall that a set $D \subseteq X'$ is *norming* if the formula $(*)$

$$|x| = \sup\{|\varphi(x)|/\|\varphi\| : \varphi \in D \setminus \{0\}\} \tag{22.2}$$

defines an equivalent norm on X. More precisely, we say that D is *r-norming* if $\|x\| \leqslant r|x|$ for every $x \in X$. D is 1-*norming* if $|\cdot| = \|\cdot\|$.

The following fact is rather well known.

Proposition 22.2.1 *Let D be a norming subset of X'. Then D is 1-norming with respect to the norm $|\cdot|$ defined by Eq. (22.2).*

Proof Let D' be the linear span of D and let

$$D_1 = \{\varphi \in D' : \|\varphi\| \leqslant 1\}.$$

Then $|x| = \sup_{\varphi\in D_1} |\varphi(x)|$. Thus, it remains to notice that

$$D_1 = \{\varphi \in D' \colon |\varphi| \leqslant 1\}.$$

Indeed, if $\varphi \in D_1$ then $|\varphi(x)| \leqslant 1$ whenever $|x| \leqslant 1$, so $|\varphi| \leqslant 1$. Conversely, if $\varphi \in D \setminus D_1$ then there is $x_0 \in X$ such that $|x_0| = 1$ and $|\varphi(x_0)| > 1$, so $|\varphi| > 1$. □

The following lemma, in case of (WCG) Banach spaces, was proved by Koszmider [372].

Lemma 22.2.2 *Assume X is a Banach space, $D \subseteq X'$ is r-norming and M is an elementary substructure of a big enough $\langle H(\theta), \in\rangle$ such that $X, D \in M$. Then*

(a) $X_M \cap {}^{\perp}(D \cap M) = \{0\}$;
(b) the canonical projection $P\colon X_M \oplus {}^{\perp}(D \cap M) \to X_M$ has norm $\leqslant r$.

Proof Fix $x \in X \cap M$, $y \in {}^{\perp}(D \cap M)$ and fix $\varepsilon > 0$. Since D is r-norming, there exists $d \in D$ such that

$$r|d(x)|/\|d\| \geqslant \|x\| - \varepsilon.$$

Since $x \in M$, by elementarity we may assume that $d \in M$. Thus $d \in D \cap M$ and $d(y) = 0$. It follows that

$$\|x\| \leqslant r|d(x)|/\|d\| + \varepsilon = r|d(x + y)|/\|d\| + \varepsilon \leqslant r\|x + y\| + \varepsilon.$$

By continuity, we see that $\|x\| \leqslant r\|x + y\|$ whenever $x \in X_M$ and $y \in {}^{\perp}(D \cap M)$. In particular,

$$X_M \cap {}^{\perp}(D \cap M) = \{0\},$$

because if $x \in X_M \cap {}^{\perp}(D \cap M)$, then we have $-x \in {}^{\perp}(D \cap M)$ and $\|x\| \leqslant r\|x - x\| = 0$. □

Note that, in the above lemma, the subspace $X_M \oplus {}^{\perp}(D \cap M)$ is closed in X.

It may happen that ${}^{\perp}(D \cap M) = \{0\}$ (consider $X = \ell_\infty$), and in that case, the above lemma is meaningless. We are going to discuss Banach spaces for which Lemma 22.2.2 provides a way to construct full projections.

We demonstrate the use of elementary submodels for finding projections in (WCG) Banach spaces.

Proposition 22.2.3 *Let X be a (WCG) Banach space and let θ be a big enough regular cardinal. Further, let $M \prec \langle H(\theta), \in\rangle$ be such that $X \in M$. Then there exists a norm one projection $P_M : X \to X_M$ such that* $\ker(P_M) = {}^{\perp}\big(X' \cap M\big)$.

Proof Let K be a linearly dense weakly compact subset of X. By Lemma 22.2.2, it suffices to check that $X_M \cup {}^{\perp}\big(X' \cap M\big)$ is linearly dense in X.

Suppose $\varphi \in X' \setminus \{0\}$ is such that $(X \cap M) \subseteq \ker(\varphi)$ and ${}^{\perp}\big(X' \cap M\big) \subseteq \ker(\varphi)$. The latter inclusion implies that $\varphi \in \overline{(X' \cap M)}$, (the closure in $\sigma(X', X)$) because $X' \cap M$ is $\mathbb{Q}$-linear. Fix $p \in K$ such that $\varphi(p) \neq 0$. Let $U_0, U_1 \subseteq \mathbb{R}$ be disjoint open rational intervals such that $0 \in U_0$ and $\varphi(p) \in U_1$. Let K_0 be the weak closure of $K \cap M$. Note that $\varphi \restriction K_0 = 0$, because φ is weakly continuous. Using the fact that

$$\varphi \in \overline{(X' \cap M)},$$

for each $x \in K_0$ choose $\psi_x \in X' \cap M$ such that $\psi_x(x) \in U_0$ and $\psi_x(p) \in U_1$. By compactness, there are

$$x_0, x_1, \ldots, x_{n-1} \in K_0$$

such that

$$K_0 \subseteq \bigcup_{i<n} \psi_{x_i}^{-1}[U_0]. \tag{$*$}$$

Note that $U_0, U_1 \in M$. Let $\Psi = \{\psi_{x_i} : i < n\}$. By Proposition 22.1.1(b), we note that also $\Psi \in M$. Further, $p \in K$ witnesses the validity of

$$(\exists\, x \in K)(\forall\, \psi \in \Psi)\ \ \psi(x) \in U_1.$$

All parameters in the above formula are in M, therefore by elementarity there exists $x \in K \cap M \subseteq K_0$ such that $\psi_{x_i}(x) \in U_1$ for each $i < n$. This contradicts $(*)$. □

Fix a Banach space X and a norming set $D \subseteq X'$. We shall say that D *generates projections in* X if for a big enough regular cardinal θ, for every countable elementary substructure $M \prec \langle H(\theta), \in\rangle$ with $D \in M$ it holds that

$$X = X_M \oplus {}^{\perp}(D \cap M).$$

We shall say that the projection $P : X \to X$ such that $im\, P = X_M$ and $\ker P = {}^{\perp}(D \cap M)$ is *induced by* the triple $\langle X, D, M\rangle$.

We finish this section with a short discussion of projectional generators.

Let X be a Banach space. A pair $\langle D, \Phi\rangle$ is a *projectional generator* in X if

(1) $D \subseteq X'$ is a norming $\mathbb{Q}$-linear subspace,
(2) $\Phi\colon D \to [X]^{\leqslant \aleph_0}$,
(3) $(\bigcup \Phi[B])^{\perp} \cap \overline{B}^{*} = 0$, whenever $B \subseteq D$ is $\mathbb{Q}$-linear.

Projectional generators were introduced by Orihuela and Valdivia [484] as a tool for showing that certain Banach spaces have a projectional resolution of the identity. First projectional generators were implicitely constructed by Lindenstrauss [407, 408]. We refer to Chapter 6 of Fabian's book [196] for more information.

Further, we show how projectional generators together with elementary submodels induce projections.

Proposition 22.2.4 *Let X be a Banach space which has a projectional generator with domain $D \subseteq X'$. Then D generates projections in X.*

Proof Fix $M \prec \langle H(\theta), \in\rangle$ with $D \in M$, where θ is big enough so that $H(\theta)$ satisfies "there exists a projectional generator on X with domain D". By elementarity, there is $\Phi \in M$ such that M satisfies "$\langle D, \Phi\rangle$ is a projectional generator on X". It suffices to check that the only $\psi \in X'$ which vanishes on $X_M \cup {}^{\perp}(D \cap M)$ is the zero functional.

Let $B := D \cap M$. By elementarity, B is $\mathbb{Q}$-linear because D is assumed to be $\mathbb{Q}$-linear. By the definition of a projectional generator,

$$(\bigcup \Phi[B])^{\perp} \cap \overline{B}^{*} = 0.$$

Thus, if $\psi \in X'$ is such that $X_M \cup {}^{\perp}(D \cap M) \subseteq \ker \psi$ then $\psi \in (\bigcup \Phi[B])^{\perp}$. This is because $\Phi(b) \subseteq M$ whenever $b \in B$ (recall that Φ has countable values). It follows that $\psi = 0$. □

22.3 The Lindelöf Property of Weak Topologies

It has already been mentioned that every (WCG) Banach space is weakly Lindelöf (even K-analytic), see Sect. 12.4. We shall prove a general result on Lindelöf property, using projections induced by elementary submodels.

Theorem 22.3.1 *Let X be a Banach space and let $D \subseteq X'$ be a norming space that generates projections in X. Then $\langle X, \sigma(X, D)\rangle$ is Lindelöf.*

Proof Recall that a standard basis of the topology $\sigma(X, D)$ consists of finite intersections of sets of the form

$$W = \{x \in X\colon \varphi(x) < r\},$$

where $\varphi \in D$ and r is a rational number. Fix a cover $\mathcal{V}$ of X consisting of basic sets in $\sigma(X, D)$.

Let θ be a big enough regular cardinal and let $M \preceq \langle H(\theta), \in\rangle$ be countable such that $X, D \in M$ and $\mathcal{V} \in M$. We claim that $\mathcal{V} \cap M$ is a cover of X, which will of course complete the proof.

Fix $x \in X$. Let $P\colon X \to X$ be the projection induced by $\langle X, D, M\rangle$. Find a rational $r > 0$ such that the ball $B(Px, r)$ is contained in some element of $\mathcal{V}$. Find $z \in X \cap M$ so that $\|Px - z\| < r/2$. Let $B = B(z, r/2)$. Then $B \in M$, because $z, r \in M$. Further, $Px \in B$ and $B \subseteq B(Px, r)$, therefore B is contained in some element of the cover $\mathcal{V}$. By elementarity, there exists $V \in \mathcal{V} \cap M$ such that $B \subseteq V$. It remains to show that $x \in V$.

Recall that $V = \bigcap_{i<n} W_i$, where $W_i = \{x\colon \varphi_i(x) < r_i\}$, $\varphi_i \in D$ and $r_i \in \mathbb{Q}$. By elementarity, $\varphi_i \in M$. Now recall that $x = Px + y$, where $y \in {}^{\perp}(D \cap M)$. In particular, $\varphi_i(y) = 0$ and hence $\varphi_i(x) = \varphi_i(Px)$. This holds for every $i < n$, showing that $x \in V$. □

In case where X' generates projections in X (equivalently, in case X is (WLD), as we shall see later), the above result says that X is weakly Lindelöf.

22.4 The Separable Complementation Property

We present selected results concerning complementation property in general Banach spaces, in order to motivate the study of projectional skeletons and projectional resolutions. We start with the following classical result due to Sobczyk.

Theorem 22.4.1 (Sobczyk [564]) *Every isomorphic copy of the sequence space c_0 in a separable Banach space is complemented.*

Proof Fix a separable Banach space X and fix an embedding $i\colon c_0 \to X$ so that $\|i\| \leqslant r$ and $\|i^{-1}\| \leqslant r$. Let $\{e_n\}_{n\in\omega}$ be the canonical basis of c_0 and let e_n^* be the n-th coordinate functional. Let K denote the closed unit ball of X'. In what follows, we shall have in mind the $weak^*$ topology in X'. Define

$$F_n = \{y \in X'\colon i'(y) = e_n\}.$$

Observe that $F_n \cap rK \neq \emptyset$, by the Hahn–Banach theorem. We prove the following

Claim. Every neighbourhood of $0 \in X'$ intersects all but finitely many of the sets $F_n \cap 2rK$.

Suppose otherwise and fix a sequence $\{k_n\}_{n\in\omega} \subseteq \omega$ and a neighbourhood W of 0 so that $W \cap F_{k_n} = \emptyset$ for $n \in \omega$. Choose a sequence $\{y_n\}_{n\in\omega}$ such that

$$y_n \in F_{k_n} \cap rK \setminus W.$$

Taking a subsequence if necessary, we may assume that $y_n \to y$ for some $y \in rK$ (recall that we consider the $weak^*$ topology on rK, which is compact and

metrizable). By continuity, $i'(y) = 0$. Hence $z_n := y_n - y$ is an element of $F_n \cap 2rK$. Finally,

$$\lim_{n\to\infty} z_n = 0,$$

therefore

$$z_n \in F_{k_n} \cap 2rK \cap W$$

for all but finitely many $n \in \omega$, a contradiction.

Fix a metric ϱ on $2rK$ which induces the $weak^*$ topology. Given $n \in \omega$, choose $\varphi_n \in 2rK$ so that

$$\varrho(\varphi_n, 0) < \operatorname{dist}(0, 2rK) + 1/n.$$

By the above arguments, $\{\varphi_n\}_{n\in\omega}$ is $weak^*$ convergent to 0. Finally, define $P\colon X \to c_0$ by setting $P(x)(n) = \varphi_n(x)$. Clearly, P is a projection onto c_0 and $\|P\| \leqslant 2r$. □

Recall that a Banach space E has the *separable complementation property* if for every separable $A \subseteq E$ there exists a projection $P : E \to E$ such that $A \subseteq im\, P$.

Corollary 22.4.2 *Assume E is a Banach space having the separable complementation property. Then every copy of c_0 is complemented in E.*

Proof Assume $c_0 \subseteq E$ and fix a projection $P : E \to E$ such that $X = \operatorname{im} P$ contains c_0. By Sobczyk's theorem, there exists a projection $Q\colon X \to X$ with $\operatorname{im} Q = c_0$. The composition $Q \circ P$ is a projection of E onto c_0. □

Let K denote the one-point compactification of the natural numbers, i.e., $K = \mathbb{N} \cup \{\infty\}$ with the obvious topology. The space c_0 is naturally identified with the subspace of $C(K)$ consisting of all functions vanishing at ∞. More generally, if K is any compactification of $\mathbb{N}$ then the subspace of $C(K)$ consisting of functions which are constantly zero on $K \setminus \mathbb{N}$ is naturally isometric to c_0. We say that this is a *canonical copy* of c_0.

The next theorem is a straight generalization of a result of Whitley [640]. The special case saying that c_0 is not complemented in ℓ_∞ has already been proved by Sobczyk [564].

Theorem 22.4.3 *Let K be a compactification of the natural numbers $\mathbb{N}$. If the canonical copy of c_0 is complemented in $C(K)$ then $K \setminus \mathbb{N}$ admits a strictly positive regular Borel measure.*

Proof Set

$$E = \{f \in C(K)\colon f \restriction (K \setminus \mathbb{N}) = 0\}$$

and assume $P\colon C(K) \to E$ is a bounded projection. Define $\varphi_n(f) = P(f)(n)$ and let μ_n be a real-valued measure on K which induces φ_n, that is

$$\varphi_n(f) = \int_K f\, d\mu_n.$$

Define

$$\nu = \sum_{n\in\omega} 2^{-n}|\mu_n|,$$

where $|\mu_n|$ denotes the variation of μ_n. Then ν is a positive regular Borel measure on K. We claim that ν is strictly positive on $K \setminus \mathbb{N}$.

For fix a non-empty open set $U \subseteq K$ such that $\nu(U \setminus \mathbb{N}) = 0$ and fix $f \in C(K)$ with $supp\,(f) \subseteq U$. Observe that $|\mu_n|(U \setminus \{n\}) = 0$, which implies that

$$P(f)(n) = \int_X f\, d\mu_n = f(n)$$

for every $n \in \mathbb{N}$. Thus $f \in E$, i.e., $f(x) = 0$ for every $x \in K \setminus \mathbb{N}$. As f was chosen arbitrarily, it follows that $U \setminus \mathbb{N} = \emptyset$. □

This yields the following

Corollary 22.4.4 *(a) The space c_0 is not complemented in ℓ_∞.*
(b) There exists a Banach space of density $\aleph_1$ which has uncomplemented copy of c_0.

Proof (a) Recall that $\ell_\infty \approx C(\beta\mathbb{N})$ and $\beta\mathbb{N}\setminus\mathbb{N}$ does not satisfy the countable chain condition (and hence does not admit a strictly positive measure).

(b) Let K be the one-point compactification of a locally compact space of the form $\mathbb{N}\cup\mathcal{A}$, where $\mathcal{A}$ is an almost disjoint family of subsets of $\mathbb{N}$ and $|\mathcal{A}| = \aleph_1$. More precisely: $\mathcal{A}$ consists of infinite subsets of $\mathbb{N}$, $a \cap b$ is finite whenever $a, b \in \mathcal{A}$ are distinct, elements of $\mathbb{N}$ are isolated and a basic neighbourhood of $a \in \mathcal{A}$ is of the form

$$\{a\} \cup (a \setminus s),$$

where s is a finite set. Then K is a compactification of $\mathbb{N}$ whose remainder is homeomorphic to the one-point compactification of an uncountable discrete space. There is no strictly positive Borel measure on $K \setminus \mathbb{N}$, therefore the natural copy of c_0 in $C(K)$ is not complemented by Theorem 22.4.3. □

Every quotient map $f : K \to L$ between compact spaces induces an isometric embedding $f'\colon C(L) \to C(K)$, where $f'(\varphi) = \varphi \circ f$. A bounded linear operator

$$P\colon C(K) \to C(L)$$

satisfying $P(\varphi \circ f) = \varphi$ for $\varphi \in C(L)$ is called an *averaging operator* for f. It is clear that its existence is equivalent to the fact that $C(L)$ is complemented in $C(K)$, when embedded via f'. This happens, for example, when f is a topological retraction, i.e., if there is a continuous map $j: L \to K$ such that $f \circ j = \mathrm{id}_L$.

A classical result of Milyutin says that the standard quotient map $f: 2^\omega \to \mathbb{I}$, defined by "gluing" all jumps in the Cantor set, does not admit any averaging operator. This gives an example of an uncomplemented copy of $C(\mathbb{I})$ in a separable Banach space, $C(2^\omega)$—actually isomorphic to $C(\mathbb{I})$. A more abstract result on non-existence of averaging operators for Cantor's type maps was proved by Pełczyński [489]. We present a similar result for increasing maps of compact lines. Its particular case is Milyutin's theorem.

A point x in a linearly ordered set X is *internal* if it is not isolated from either side, i.e., $x = \inf(x, \to)$ and $x = \sup(\leftarrow, x)$. A linearly ordered compact space will be called shortly a *compact line*.

Theorem 22.4.5 (Kubiś [380]) *Let $\theta: K \to L$ be an increasing quotient of compact lines. If θ admits an averaging operator then the set*

$$Q = \{y \in L : |\theta^{-1}(y)| > 1 \text{ and } y \text{ is internal in } L\}$$

is nowhere dense.

Proof For each $x \in L$ define $x^- = \min \theta^{-1}(x)$ and $x^+ = \max \theta^{-1}(x)$. Then each fibre of θ is of the form $[x^-, x^+]$, where $x \in L$. Recall that θ identifies $C(L)$ with the set of all $f \in C(K)$ which are constant on every interval $[x^-, x^+]$, where $x \in L$. Suppose

$$P: C(K) \to C(K)$$

is a bounded linear projection onto $C(L)$ embedded via θ in $C(K)$. Fix $N \in \omega$ such that

$$-1 + N/3 \geqslant \|P\|.$$

Given $p \in Q$, choose an increasing function $\chi_p \in C(K)$ such that $\chi_p(t) = 0$ for $t \leqslant p^-$ and $\chi_p(t) = 1$ for $t \geqslant p^+$. Let $h_p = P\chi_p$. There exists a (unique) function $\overline{h}_p \in C(L)$ such that $h_p = \overline{h}_p \theta$. Slightly abusing notation, we shall write h_p instead of $\overline{h}_p$, i.e., we shall treat h_p as a function on L. Define

$$Q^- = \{q \in Q : \overline{h}_q(q) < 2/3\} \quad , \qquad Q^+ = \{q \in Q : \overline{h}_q(q) > 1/3\}.$$

Then at least one of the above sets is somewhere dense. Further, define

$$U_p^- = (\overline{h}_p)^{-1}(-\infty, 2/3) \quad , \qquad U_p^+ = (\overline{h}_p)^{-1}(1/3, +\infty).$$

Suppose that the set Q^- is dense in the interval (a, b). Choose $p_0 < p_1 < \cdots < p_{N-1}$ in $Q^- \cap (a, b)$ so that $p_i \in U^-_{p_0} \cap \cdots \cap U^-_{p_{i-1}}$ for every $i < N$. This is possible, because each p_i is internal in L. Choose $f \in C(K)$ such that $0 \leqslant f \leqslant 1$ and

$$f \restriction \theta^{-1}(p_i) = \chi_{p_i} \restriction \theta^{-1}(p_i) \quad , \quad i < N$$

and f is constant on $[p^-, p^+]$ for every $p \in L \setminus \{p_0, \ldots, p_{N-1}\}$. The function f can be constructed as follows. For each $i < N - 1$ choose a continuous function $\varphi_i : [p_i, p_{i+1}] \to \mathbb{I}$ such that $\varphi_i(p_i) = 1$ and $\varphi_i(p_{i+1}) = 0$. Define

$$f(t) = \begin{cases} 0 & t < p_0, \\ \chi_i(t) & t \in [p_i^-, p_i^+],\ i < N, \\ \varphi_i\theta(t) & t \in [p_i^+, p_{i+1}^-],\ i < N-1, \\ 1 & t > p_{N-1} \end{cases}$$

Let $g = f - \sum_{i<N} \chi_{p_i}$. Then g is constant on each interval of the form $\theta^{-1}(p)$, where $p \in L$. Indeed, if $p \notin \{p_0, \ldots, p_{N-1}\}$, then all the functions $f, \chi_{p_0}, \ldots, \chi_{p_{N-1}}$ are constant on $\theta^{-1}(p)$. If

$$t \in [p_j^-, p_j^+],$$

then

$$g(t) = f(t) - \sum_{i \leqslant j} \chi_{p_i}(t) = f(t) - (j-1) - \chi_{p_j}(t) = j - 1,$$

because $f(t) = \chi_{p_j}(t)$. It follows that $g \in C(L)$, i.e., $Pg = g$. Hence

$$Pf = Pg + P\Big(\sum_{i<N} \chi_{p_i}\Big) = g + \sum_{i<N} h_{p_i}.$$

Now choose

$$t \in U^-_{p_0} \cap \cdots \cap U^-_{p_{N-1}}$$

such that $t > p_{N-1}$. Note that $|(Pf)(t)| \leqslant \|P\|$, because $0 \leqslant f \leqslant 1$. On the other hand, $h_{p_i}(t) < 2/3$ for $i < N$ and consequently

$$-\|P\| \leqslant (Pf)(t) = g(t) + \sum_{i<N} h_{p_i}(t) = f(t) - \sum_{i<N} \chi_{p_i}(t) + \sum_{i<N} h_{p_i}(t) =$$

$$f(t) - N + \sum_{i<N} h_{p_i}(t) < 1 - N + \sum_{i<N} (2/3) = 1 - N/3 \leqslant -\|P\|,$$

which is a contradiction.

In case where the set Q^- is nowhere dense, we use the fact that Q^+ must be somewhere dense, and we choose a decreasing sequence $p_0 > p_1 > \cdots > p_{N-1}$ in Q^+ so that $p_i \in U^+_{p_0} \cap \cdots \cap U^+_{p_{i-1}}$ for $i < N$. Taking $t \in U^+_{p_0} \cap \cdots \cap U^+_{p_{N-1}}$ with $t < p_{N-1}$ and considering a similar function f, we obtain

$$\|P\| \geqslant (Pf)(t) = f(t) - \sum_{i<N} \chi_{p_i}(t) + \sum_{i<N} h_{p_i}(t) =$$

$$f(t) + \sum_{i<N} h_{p_i}(t) \ > -1 + \sum_{i<N} (1/3) = -1 + N/3 \geqslant \|P\|,$$

which again is a contradiction. □

A deeper analysis of the above proof can give a more precise result. This is done in [361]. In particular, the assumption that Q be nowhere dense is not sufficient for the existence of an averaging operator. We refer to [361] for the details.

22.5 Projectional Skeletons

In this section we discuss Banach spaces with projectional skeletons, defined further.

Recall that a partially ordered set (briefly: poset) Γ is *directed* if for every $s_0, s_1 \in \Gamma$ there is $t \in \Gamma$ such that $s_0 \leqslant t$ and $s_1 \leqslant t$.

Γ is *σ-complete* if every sequence $s_0 < s_1 < \dots$ has the least upper bound in Γ.

A subset A of Γ is *closed* if $\sup_{n\in\omega} s_n \in A$ whenever $\{s_n : n \in \omega\} \subseteq A$ is such that $s_0 < s_1 < \dots$.

A set $A \subseteq \Gamma$ is *cofinal* if for every $s \in \Gamma$ there exists $t \in A$ with $s \leqslant t$.

Let E be a Banach space. A *projectional skeleton* in E is a family $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ satisfying the following conditions.

(1) Each P_s is a projection of E with a separable image.
(2) Γ is an up-directed partially ordered set, i.e., for each $s, t \in \Gamma$ there is $r \in \Gamma$ with $s \leqslant r$ and $t \leqslant r$.
(3) $P_s \circ P_t = P_t \circ P_s = P_s$ whenever $s \leqslant t$.
(4) $E = \bigcup_{s\in\Gamma} \operatorname{im} P_s$.
(5) If $s_0 \leqslant s_1 \leqslant \dots$ in Γ then there exists $t = \sup_{n\in\omega} s_n$ and

$$\operatorname{im} P_t = \overline{\Big(\bigcup_{n\in\omega} \operatorname{im} P_{s_n}\Big)}.$$

So far we have not required that the projections in the skeleton $\mathfrak{s}$ are uniformly bounded. We shall do so, in view of the following fact.

Proposition 22.5.1 *Let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ be a projectional skeleton in a Banach space E. Then there exists a closed cofinal set $\Pi \subseteq \Gamma$ such that*

$$\sup_{t\in\Pi} \|P_t\| < +\infty.$$

Proof For each $n \geqslant 1$ define

$$\Pi_n = \{s \in \Gamma : \|P_s\| \leqslant n\}.$$

We first show that each Π_n is closed in Γ.

Indeed, fix $k \geqslant 1$, fix $s_0 < s_1 < \ldots$ in Π_k and let $t = \sup_{n\in\omega} s_n$. We need to show that $\|P_t\| \leqslant k$.

Suppose this is not true and fix $x \in E$ with $\|x\| = 1$ and $\|P_t(x)\| = r > k$. Let $\varepsilon = (r-k)/2$. Using the second part of (5), find $m \in \omega$ and $y \in \operatorname{im} P_{s_m}$ such that

$$\|P_t(x-y)\| < \varepsilon/k.$$

Note that $P_{s_m} = P_{s_m} \circ P_t$. Using the fact that $\|P_{s_m}\| \leqslant k$, we get $\|P_{s_m}(x)\| \leqslant k$ and

$$\|y - P_{s_m}(x)\| = \|P_{s_m}(y - P_t(x))\| \leqslant k\|y - P_t(x)\| < \varepsilon.$$

Thus,

$$\|P_t x\| \leqslant \|P_t x - y| + \|y - P_{s_m} x\| + \|P_{s_m} x\| < \varepsilon/k + \varepsilon + k \leqslant 2\varepsilon + k = r = \|P_t x|.$$

This contradiction shows that $\|P_t\| \leqslant k$.

Finally, we claim that one of the sets Π_n is cofinal in Γ. Suppose otherwise and for each $n \in \omega$ choose t_n such that $\|P_s\| > n$ whenever $t_n \leqslant s$. Using the directedness of Γ we construct a sequence $s_1 < s_2 < \ldots$ such that $t_n \leqslant s_n$ for $n \in \omega$. Let $s_\infty = \sup_{n\in\omega} s_n$. Then $\|P_{s_\infty}\| = +\infty$, a contradiction. □

From now on, we shall require that a projectional skeleton $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ satisfies

(6) $\sup_{s\in\Gamma} \|P_s\| < +\infty$.

We shall say that $\mathfrak{s}$ is a *k-projectional skeleton* if $\|P_s\| \leqslant k$ for each $s \in \Gamma$.

Proposition 22.5.2 *Let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ be a projectional skeleton in a Banach space E and let $x \in E$. Then*

$$P_t x = \lim_{n\to\infty} P_{s_n} x$$

whenever $s_0 \leqslant s_1 \leqslant \ldots$ in Γ is such that $t = \sup_{n\in\omega} s_n$.

Proof Let $r = \sup_{s\in\Gamma} \|P_s\|$ and fix $\varepsilon > 0$. Using the second part of (5) find $y \in \bigcup_{n\in\omega} \operatorname{im} P_{s_n}$ such that

$$\|P_t x - y\| < \varepsilon/(1+r).$$

Choose k such that $y \in im\, P_{s_k}$. Note that $P_t y = y$ and $P_{s_n} y = y$ for $n \geqslant k$. Thus, given $n \geqslant k$, we have

$$\|P_t x - P_{s_n} x\| \leqslant \|P_t x - y\| + \|y - P_{s_n} x\| < \varepsilon/(1+r) + \|P_{s_n}(y - P_t x)\|$$

$$\leqslant \varepsilon/(1+r) + r\|y - P_t x\| < \varepsilon/(1+r) + r\varepsilon/(1+r) = \varepsilon.$$

This shows that $\lim_{n\to\infty} \|P_t x - P_{s_n} x\| = 0$. □

The above fact provides a simple way to construct projections onto non-separable subspaces.

Proposition 22.5.3 *Let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ be a projectional skeleton in a Banach space E and let $T \subseteq \Gamma$ be a directed set. Then the formula*

$$P_T x = \lim_{s\in T} P_s x$$

well defines a bounded projection of E onto $\overline{(\bigcup_{s\in T} \operatorname{im} P_s)}$.

Proof It is enough to show that $\{P_s x\}_{s\in T}$ is a Cauchy net for every $x \in E$. Suppose this is not the case and fix $x \in E$ and $\varepsilon > 0$ such that for each $s \in T$ there are $t_1, t_2 \in T$, $s \leqslant t_1$, $s \leqslant t_2$ with $\|P_{t_1} x - P_{t_2} x\| > 2\varepsilon$. As T is directed, we have that for each $s \in T$ there are $t_1, t_2 \in T$ with $s \leqslant t_1 \leqslant t_2$ and

$$\|P_{t_1} x - P_{t_2} x\| > \varepsilon.$$

Hence, by induction we can construct a sequence

$$t_1 \leqslant t_2 \leqslant t_3 \leqslant t_4 \leqslant \dots$$

in T such that

$$\|P_{t_{2k-1}} x - P_{t_{2k}} x\| > \varepsilon$$

for $k \in \mathbb{N}$. This contradicts Proposition 22.5.2. □

Further, we recall the notion of a projectional resolution of the identity, a useful tool for investigating some classes of non-separable Banach spaces. The idea of "long sequences of projections" goes back to Lindenstrauss, see [407, 408]. For more information and historical comments, we refer to the books [160] and [196].

Let X be a Banach space and let λ be a limit ordinal. A *projectional sequence* of length λ in X is a sequence of projections $\{P_\xi\}_{\xi<\lambda}$ satisfying the following conditions:

(i) $\xi < \eta \Rightarrow P_\xi = P_\eta \circ P_\xi = P_\xi \circ P_\eta$,
(ii) $P_\delta X = \overline{(\bigcup_{\xi<\delta} P_\xi X)}$ for every limit ordinal $\delta < \lambda$,
(iii) $X = \overline{(\bigcup_{\xi<\lambda} P_\xi X)}$.

A special case is a *projectional resolution of the identity* (*PRI*): this is a projectional sequence $\{P_\xi\}_{\xi<\lambda}$ such that $\|P_\xi\| = 1$ and $\operatorname{dens}(P_\xi X) \leqslant |\xi| + \aleph_0$ for each $\xi < \lambda$, where $|\xi|$ denotes the cardinality of ξ.

The existence of a reasonable projectional sequence or PRI in a given Banach space allows proving certain properties by transfinite induction. On the other hand, some Banach spaces can be built from smaller spaces using projections of norm 1. This is described in the following:

Theorem 22.5.4 (Kubiś [380]) *Assume E is a Banach space which is the union of a continuous chain $\{E_\alpha\}_{\alpha<\kappa}$ of closed subspaces such that for every $\alpha < \kappa$, E_α is 1-complemented in $E_{\alpha+1}$. Then there exist projections $S_\alpha\colon E \to E$, $\alpha < \kappa$, such that $\|S_\alpha\| = 1$, $im\, S_\alpha = E_\alpha$ and $S_\alpha \circ S_\beta = S_\alpha = S_\beta \circ S_\alpha$ whenever $\alpha \leqslant \beta < \kappa$.*

Proof We construct inductively projections $\{S_\alpha^\beta\colon \alpha \leqslant \beta \leqslant \kappa\}$ with the following properties:

(a) $S_\alpha^\beta : E_\beta \to E_\alpha$ has norm 1 for every $\alpha \leqslant \beta$,
(b) $S_\alpha^\beta \circ S_\beta^\gamma = S_\alpha^\gamma$ whenever $\alpha \leqslant \beta \leqslant \gamma$.

We start with $S_0^0 = \mathrm{id}_{E_0}$. Fix $0 < \delta \leqslant \kappa$ and assume S_α^β have been constructed for each $\alpha \leqslant \beta < \delta$ and they satisfy conditions (a), (b). There are two cases:

Case 1: $\delta = \gamma + 1$. Using the assumption, fix a projection

$$T\colon E_{\gamma+1} \to E_\gamma$$

with $\|T\| = 1$ and define $S_\alpha^\delta = S_\alpha^\gamma \circ T$ for every $\alpha < \delta$. Clearly both (a) and (b) are satisfied.

Case 2: δ is a limit ordinal. Let

$$G = \bigcup_{\alpha<\delta} E_\alpha.$$

Then G is a dense linear subspace of E_δ. Define $h_\alpha\colon G \to E_\alpha$ by setting

$$h_\alpha(x) = S_\alpha^\beta x, \qquad \beta = \min\{\xi \in [\alpha, \delta)\colon x \in E_\xi\}.$$

Note that if $\alpha < \beta$, $x \in E_\beta$ and $\gamma \in [\beta, \delta)$ then

$$S_\alpha^\gamma x = S_\alpha^\beta(S_\beta^\gamma x) = S_\alpha^\beta x, \qquad (*)$$

because of (b) and by the fact that $S_\beta^\gamma \restriction E_\beta = \mathrm{id}_{E_\beta}$. Using $(*)$, it is easy to see that h_α is a linear operator. Clearly $\|h_\alpha\| = 1$, thus it can be uniquely extended to a linear operator

$$S_\alpha^\delta \colon E_\delta \to E_\alpha.$$

Finally,

$$\|S_\alpha^\delta\| = \|h_\alpha\| = 1$$

and S_α^δ is a projection onto E_α. Thus (a) holds.

It remains to show (b). Fix $\alpha < \beta < \delta$. By continuity, it suffices to check that $S_\alpha^\delta x = S_\alpha^\beta S_\beta^\delta x$ holds for every $x \in G$. Fix $x \in G$ and find $\gamma \in [\beta, \gamma)$ such that $x \in E_\gamma$. We have

$$S_\alpha^\beta S_\beta^\delta x = S_\alpha^\beta S_\beta^\gamma x = S_\alpha^\gamma x = S_\alpha^\delta x.$$

Thus both conditions (a) and (b) hold. It follows that the construction can be carried out.

Finally, define $S_\alpha := S_\alpha^\kappa$. Clearly, S_α is a projection of E onto E_α and $\|S_\alpha\| = 1$. If $\alpha < \beta < \kappa$ then

$$S_\alpha S_\beta = S_\alpha^\kappa \circ S_\beta^\kappa = S_\alpha^\beta \circ S_\beta^\kappa \circ S_\beta^\kappa = S_\alpha^\beta \circ S_\beta^\kappa = S_\alpha^\kappa = S_\alpha$$

and of course $S_\beta S_\alpha = S_\alpha$, because $E_\alpha \subseteq E_\beta$. This completes the proof. □

Theorem 22.5.5 (Kubiś [382]) *Every Banach space with* 1*-projectional skeleton has a projectional resolution of the identity onto subspaces with the same property.*

Proof Fix a Banach space E with $\kappa = \operatorname{dens} E$ and let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ be a projectional skeleton in E such that $\|P_s\| = 1$ for every $s \in \Gamma$. Further, fix a continuous chain $\{T_\alpha\}_{\alpha<\kappa}$ of up-directed subsets of Γ satisfying $|T_\alpha| \leqslant \alpha + \aleph_0$ for each α and such that

$$\bigcup\{P_s E \colon s \in T_\alpha,\ \alpha < \kappa\}$$

is dense in E. Continuity of the chain means that $T_\delta = \bigcup_{\xi<\delta} T_\xi$ whenever δ is a limit ordinal. Let $X_\alpha = \overline{(\bigcup_{s\in T_\alpha} P_s E)}$. Then $\{X_\alpha\}_{\alpha<\kappa}$ is a chain of closed subspaces of E, the density of X_α does not exceed $|\alpha| + \aleph_0$ and

$$X_\delta = \overline{(\bigcup_{\xi<\delta} X_\xi)}$$

for every limit ordinal δ. By Proposition 22.5.3, formula

$$P_\alpha x = \lim_{s\in T_\alpha} P_s x$$

defines a projection of E onto X_α. Clearly, $\|P_\alpha\| = 1$, because $\|P_s\| = 1$ for $s \in \Gamma$. Fix $\alpha < \beta$. Then $T_\alpha \subseteq T_\beta$, therefore $P_\beta \circ P_\alpha x = P_\alpha$. Observe that $P_s \circ P_\beta = P_s$ for every $s \in T_\alpha$. Indeed, given $s \in T_\alpha$, the set

$$A = \{t \in T_\beta : t \geqslant s\} \subseteq T_\beta$$

is cofinal in T_β, so

$$P_s P_\beta x = \lim_{t\in T_\beta} P_s P_t x = \lim_{t\in A} P_s P_t x = P_s x.$$

It follows that

$$P_\alpha P_\beta x = \lim_{s\in T_\alpha} P_s P_\beta x = \lim_{s\in T_\alpha} P_s x = P_\alpha x.$$

Thus, $\{P_\alpha\}_{\alpha<\kappa}$ is a PRI on E.

Finally, for each α let Γ_α be the smallest σ-closed subset of Γ which contains T_α. Then $\{P_s \restriction X_\alpha\}_{s\in\Gamma_\alpha}$ is a projectional skeleton in X_α. □

Corollary 22.5.6 *Let E be a Banach space which has a projectional skeleton. Then E has a locally uniformly convex equivalent norm and E admits a bounded injective linear operator into $c_0(S)$ for some set S.*

Proof The first statement is a well-known result of Zizler [645]; we omit its rather technical proof. The second statement is also well known and can be proved by induction. We sketch the argument here.

Assume $\{P_\alpha\}_{\alpha<\kappa}$ is a PRI on E such that $P_0 = 0$ and each space $X_\alpha = im\, P_\alpha$ has an injective operator

$$Q_\alpha : X_{\alpha+1} \to c_0(S_\alpha)$$

with $\|Q_\alpha\| \leqslant 1$. Let Y be the c_0-sum of the family

$$\{c_0(S_\alpha)\}_{\alpha<\kappa}.$$

Then $Y = c_0(S)$, where $S = \bigcup_{\alpha<\kappa} S_\alpha$, assuming that the sets S_α are pairwise disjoint. Define $Q : E \to Y$ by

$$(Qx)(\alpha) = Q_\alpha P_{\alpha+1} x - Q_\alpha P_\alpha x, \qquad x \in E.$$

We need to check that Q is well defined. For fix $x \in E$ and $\varepsilon > 0$. We claim that

$$\|P_{\alpha+1}x - P_\alpha x\| < \varepsilon$$

for all but finitely many $\alpha < \kappa$. Once we prove it, using the fact that $\|Q_\alpha\| \leqslant 1$ we conclude that $Qx \in Y$.

Suppose that

$$\|P_{\alpha_n+1}x - P_{\alpha_n}x\| \geqslant \varepsilon$$

for some sequence $\{\alpha_n\}_{n\in\omega}$. We may assume that this sequence is strictly increasing. Let $\beta = \sup_{n\in\omega}\alpha_n$. Then

$$P_\beta x = \lim_{\xi<\beta} P_\xi x$$

and hence there is $\eta < \beta$ such that $\|P_\xi x - P_\alpha x\| < \varepsilon$ for every $\eta < \xi, \alpha < \beta$. This is a contradiction.

It is clear that $\|Q\| \leqslant 1$. To see that $\ker Q = 0$, fix $x \in E \setminus \{0\}$ and let ϱ be the minimal ordinal $\xi < \kappa$ for which $P_\xi x \neq 0$. Then $\varrho = \alpha + 1$ and by assumption $Q_\alpha P_{\alpha+1}x \neq 0$, so $(Qx)(\alpha) \neq 0$. □

22.6 Norming Subspaces Induced by a Projectional Skeleton

We shall now look at the dual of a space with a projectional skeleton. Let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ be a projectional skeleton in a Banach space X. The set

$$D = \bigcup_{s\in\Gamma} P_s^* X'$$

is clearly a linear subspace of X. Notice that $P_s^* X' \cap B_{X'}$ endowed with the topology $\sigma(X', X)$ is second countable, because $P_s^* X'$ is linearly homeomorphic to the dual of $P_s X$. Let

$$r = \sup_{s\in\Gamma} \|P_s\|.$$

Given $x \in S_X$, there is $s \in \Gamma$ such that $x = P_s x$; choose $\varphi \in X'$ such that $\varphi(x) = 1 = \|\varphi\|$. Then

$$(P_s^*\varphi)x = \varphi(P_s x) = \varphi(x) = 1$$

and

$$\|P_s^*\varphi\| \leqslant r.$$

This shows that the space D is r-norming. We shall say that D is the dual norming subspace *induced by* $\mathfrak{s}$, and we shall denote it by $D(\mathfrak{s})$.

Let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ and $\mathfrak{t} = \{Q_t\}_{t\in\Pi}$ be projectional skeletons in the same Banach space X. We say that $\mathfrak{s}$ and $\mathfrak{t}$ are *equivalent* if they induce the same norming subspace, i.e., $\bigcup_{s\in\Gamma} \operatorname{im} P_s^* = \bigcup_{t\in\Pi} \operatorname{im} Q_t^*$. It turns out that, with help of elementary submodels, a projectional skeleton can be recovered (up to equivalence) from the norming space.

Lemma 22.6.1 *Let $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ be a projectional skeleton in a Banach space X and let $D \subseteq D(\mathfrak{s})$ be norming for X. Further, let θ be a big enough regular cardinal and let $M \preceq \langle H(\theta), \in\rangle$ be countable and such that $\mathfrak{s} \in M$. Then the projection induced by $\langle X, D, M\rangle$ equals P_t, where $t = \sup(\Gamma \cap M)$.*

Proof Observe that

$$\bigcup_{s\in\Gamma\cap M} \operatorname{im} P_s \subseteq \overline{(X \cap M)}.$$

This is because, given $s \in \Gamma \cap M$, by elementarity there exists a countable set $A \in M$ which is dense in $\operatorname{im} P_s$. By Proposition 22.1.1(c), $A \subseteq X \cap M$, therefore $\operatorname{im} P_s \subseteq \overline{(X \cap M)}$. It follows that

$$\operatorname{im} P_t = \overline{(\bigcup_{s\in\Gamma\cap M} \operatorname{im} P_s)} \subseteq \overline{(X \cap M)}.$$

On the other hand, given $x \in X \cap M$, by elementarity there is $s \in \Gamma \cap M$ such that $x \in \operatorname{im} P_s$; thus $x \in \operatorname{im} P_t$. Hence $\operatorname{im} P_t = X_M$. Notice that, again by elementarity,

$$P_t'\varphi = \varphi$$

whenever $\varphi \in D \cap M$. Thus

$$\ker P_t \subseteq {}^{\perp}(D \cap M),$$

because if $P_t x = 0$ and $\varphi \in D \cap M$ then

$$\varphi(x) = (P_t'\varphi)x = \varphi(P_t x) = 0.$$

It follows that $\ker P_t = {}^{\perp}(D \cap M)$, because $X = X_M \oplus \ker P_t$ and

$$X_M \cap {}^{\perp}(D \cap M) = \{0\}$$

(Lemma 22.2.2(a)). □

Theorem 22.6.2 (Kubiś [382]) *Let X be a Banach space and let $D \subseteq X'$ be r-norming ($r \geqslant 1$). The following properties are equivalent.*

(a) X has an r-projectional skeleton $\mathfrak{s}$ such that $D \subseteq D(\mathfrak{s})$.
(b) D generates projections in X.

Proof Implication (a) $\Rightarrow$ (b) follows from Lemma 22.6.1.

(b) $\Rightarrow$ (a) Fix a big enough regular cardinal θ and let Γ be the family of all countable elementary substructures M of $\langle H(\theta), \in\rangle$ such that $D \in M$. Endow Γ with inclusion. Clearly, Γ is a σ-directed poset. Fix $M \in \Gamma$ and let P_M be the projection onto $\overline{(X \cap M)}$ with

$$\ker(P_M) = {}^{\perp}(D \cap M).$$

By definition, P_M is defined on the entire space. Further, $\|P_M\| \leqslant r$, by Lemma 22.2.2. Given a sequence $M_0 \subseteq M_1 \subseteq \ldots$ in Γ, the union $M = \bigcup_{n\in\omega} M_n$ is an elementary substructure of $H(\theta)$ such that $\overline{(X \cap M)}$ is the closure of

$$\bigcup_{n\in\omega} \overline{(X \cap M_n)}.$$

It follows that

$$\mathfrak{s} = \{P_M\}_{M\in\Gamma}$$

is a projectional skeleton in X. It is clear that $D \subseteq D(\mathfrak{s})$, because $D \cap M \subseteq \operatorname{im} P'_M$. □

We note the following:

Corollary 22.6.3 *Let X be a Banach space with a projectional skeleton. Then there exists a renorming of X under which X has an equivalent 1-projectional skeleton.*

Proof Let D be the dual norming subspace induced by a fixed projectional skeleton in X. By Lemma 22.6.1, D generates projections in X. Consider a renorming of X after which D becomes 1-norming (see Proposition 22.2.1). By Theorem 22.6.2, X has a 1-projectional skeleton. □

As an application of Theorem 22.6.2, we prove that the class of Banach spaces with a 1-projectional skeleton is stable under arbitrary c_0- and ℓ_p-sums.

Theorem 22.6.4 (Kubiś [382]) *Let $\{X_\alpha\}_{\alpha<\kappa}$ be a collection of Banach spaces and let $X = \bigoplus_{\alpha<\kappa} X_\alpha$ be endowed either with the c_0-norm or with ℓ_p-norm ($1 \leqslant p < \infty$). Further, assume that for each $\alpha < \kappa$, $D_\alpha \subseteq X_\alpha$ is 1-norming and generates projections in X_α. Then the set*

$$D = \{\varphi \in X' (\forall\, \alpha)\ \varphi \restriction X_\alpha \in D_\alpha \ \text{ and } \ |\alpha : \varphi \restriction X_\alpha \neq 0| \leqslant \aleph_0\}$$

is 1-norming and generates projections in X.

Proof The fact that D is 1-norming follows from the properties of the c_0-sum and the ℓ_p-sum. Define

$$\operatorname{supp}(\varphi) = \{\alpha : \varphi \restriction X_\alpha \neq 0\}.$$

Fix a suitable $M \preceq \langle H(\theta), \in \rangle$ so that $D \in M$. Let $S = \kappa \cap M$. Note that $\operatorname{supp}(\varphi) \subseteq S$ whenever $\varphi \in D \cap M$. Suppose

$$X \neq X_M \oplus {}^{\perp}(D \cap M)$$

and fix $\psi \in X'$ satisfying $X \cap M \subseteq \ker \psi$ and

$${}^{\perp}(D \cap M) \subseteq \ker \psi.$$

Then ψ is in the $weak^*$ closure of the linear hull of $D \cap M$, therefore $\operatorname{supp}(\psi) \subseteq S$. Assuming $\psi \neq 0$, there is $\alpha \in S$ such that

$$\psi_\alpha := \psi \restriction X_\alpha \neq 0.$$

Note that $X_\alpha \cap M \subseteq \ker \psi_\alpha$. If $x \in {}^{\perp}(D_\alpha \cap M)$ and $\varphi \in D \cap M$ then

$$\varphi \restriction X_\alpha \in D_\alpha \cap M$$

so $\varphi(x) = 0$. It follows that

$${}^{\perp}(D \cap M) \cap X_\alpha = {}^{\perp}(D_\alpha \cap M).$$

Thus ${}^{\perp}(D_\alpha \cap M) \subseteq \ker \psi$. On the other hand,

$$X_\alpha = \overline{(X_\alpha \cap M)} \oplus {}^{\perp}(D_\alpha \cap M),$$

because $D_\alpha \in M$. This is a contradiction. □

Given a Banach space E, it is natural to ask when E generates projections in E', as a subspace of E''. It turns out that this happens exactly in case where E is Asplund. This is a result of Fabian and Godefroy [197]. Recall that a Banach space E is *Asplund* if the dual of every separable subspace of E is separable. Namely:

Theorem 22.6.5 *Let E be a Banach space. The following properties are equivalent.*

(a) E is an Asplund space.
(b) $E \subseteq E''$ generates projections in E'.

That (a) $\Longrightarrow$ (b) is a consequence of the existence of a projectional generator in E' with domain E. This is a highly non-trivial result of [197]. For details we refer either to [197] or to Fabian's book [196]. Further, we justify the implication (b) $\Longrightarrow$ (a).

Assume that E generates projections in E' and fix a separable subspace Y of E. Fix a countable $M \preceq H(\theta)$ such that $E \in M$ and $Y \cap M$ is dense in Y and let $P : E' \to E'$ be the projection with $\operatorname{im} P = \overline{(E' \cap M)}$ and $\ker P = (E \cap M)^{\perp}$. Then $P'y = y$ for every $y \in Y$, because Y is contained in the $weak^*$ closure of

$E \cap M$. Fix $\varphi \in Y'$ and let $\psi \in E'$ be an extension of φ. Then

$$(P\psi)y = \psi(P'y) = \psi(y) = \varphi(y)$$

for every $y \in Y$. Thus, $\varphi = (P\psi) \restriction Y$. It follows that Y' is separable because so is im P.

Next we exhibit a topological property of norming spaces induced by a projectional skeleton, see [382].

Theorem 22.6.6 (Kubiś) *Let X be a Banach space with a projectional skeleton $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ and let $D \subseteq X'$ be the norming space induced by $\mathfrak{s}$, endowed with the weak* topology. Then:*

(a) The weak closure in X' of the convex hull of every countable bounded subset of D is metrizable and contained in D.*
(b) D has countable tightness.

Proof Part (a) is trivial: every countable subset of D is contained in $Y_s = P_s'X'$ for some $s \in \Gamma$ and every bounded subset of Y_s with the *weak** topology is second countable.

(b) Let $A \subseteq D$ and $p \in \overline{(A)} \cap D$ be given, where the closure is taken in $\sigma(X', X)$. Replacing A by $A - p$, we may assume that $p = 0$. Fix a big enough regular cardinal θ and a countable elementary substructure M of $\langle H(\theta), \in\rangle$ such that $X, \mathfrak{s}, A \in M$. We claim that $0 \in \overline{(A \cap M)}$.

Let $t = \sup(\Gamma \cap M)$ and let $Y_t = P_t'X'$. Fix a *weak** neighbourhood U of p. We may assume that

$$U = \bigcap_{i<n} U(x_i, \varepsilon),$$

where

$$U(x, \varepsilon) := \{y \in X' \colon |y(x)| < \varepsilon\},$$

and

$$x_0, \ldots, x_{n-1} \in X$$

and $\varepsilon > 0$ is rational. By Lemma 22.6.1, $P_tX = \overline{(X \cap M)}$. By Banach's open mapping principle, ${P_t}^{-1}[X \cap M]$ is dense in X. Hence, without loss of generality, we may assume that $P_t x_i \in M$ for each $i < n$.

Thus,

$$W := \bigcap_{i<n} U(P_t x_i, \varepsilon)$$

is a $weak^*$ neighbourhood of 0 and $W \in M$, because $P_t x_i \in M$ and $\varepsilon \in M$. By elementarity, there is $a \in A \cap M \cap W$. It follows that $a \in Y_t$, i.e., $P_t' a = a$. Given $i < n$, we have

$$a(x_i) = (P_t' a)x_i = a(P_t x_i).$$

Thus $a \in U(x_i, \varepsilon)$. Finally, $a \in A \cap M \cap U$. □

Corollary 22.6.7 *Let X be a Banach space and let $D, E \subseteq X'$ be norming spaces induced by projectional skeletons. If $D \cap E$ is total, then $D = E$.*

Proof Note that $D \cap E$ is a linear space. Assuming it is total, it must be $weak^*$ dense. Thus, given $p \in D$, we have that $p \in \overline{(D \cap E)}$ so $p \in \overline{(A)}$ for some countable $A \subseteq D \cap E$ (Theorem 22.6.6(b)). By Theorem 22.6.6(a), we have

$$\overline{(A)} \subseteq D \cap E.$$

This shows that $D \subseteq E$. By symmetry, $D = E$. □

Corollary 22.6.8 *Let X be a Banach space and let $S \subseteq X'$ generate projections in X. Further, let $D \supseteq S$ be the smallest linear subspace of X' such that $\overline{(A)} \subseteq D$ for every countable set $A \subseteq D$, where the closure is taken in $\sigma(X', X)$. Then D is induced by a projectional skeleton in X.*

Proof By Theorem 22.6.2, there exists a projectional skeleton $\mathfrak{s}$ in X such that $S \subseteq D(\mathfrak{s})$. By Theorem 22.6.6(a), we have

$$D \subseteq D(\mathfrak{s}).$$

On the other hand, D is $weak^*$ dense, because it is a norming (and hence total) linear space. Hence, given $p \in D(\mathfrak{s})$, we have that $p \in \overline{(D)}$ so, by Theorem 22.6.6(b), one gets

$$p \in \overline{(A)}$$

for some countable $A \subseteq D$. Hence $p \in D$. This shows that $D = D(\mathfrak{s})$. □

We finish this section characterizing the weak Lindelöf property for Banach spaces with projectional skeletons. A version involving projectional generators (instead of projectional skeletons) was proved by Cascales, Orihuela and Namioka [123].

Theorem 22.6.9 *Let X be a Banach space with a projectional skeleton. Then the following properties are equivalent.*

(a) X is weakly Lindelöf.
(b) X has property (C) of Corson.
(c) X' generates projections in X.

Proof (a) $\Longrightarrow$ (b) is trivial.

(b) $\Longrightarrow$ (c) Assume $D \subseteq X'$ is a norming subspace induced by a fixed projectional skeleton in X. We need to show that $D = X'$.

By Theorem 22.6.6(a), the *weak** closed linear span of every countable subset of D is contained in D. Since X has property (C), by Theorem 13.1.6 X' has countable *convex tightness*, i.e., given $y \in \overline{D}^*$ there exists a countable $A \subseteq D$ such that y belongs to the *weak** closure of the convex hull of A. It follows that $D = X'$ and hence X' generates projections in X.

(c) $\Longrightarrow$ (a) If X' generates projections in X, then X is $\sigma(X, X')$-Lindelöf by Theorem 22.3.1. □

We have already mentioned that a Banach space is (WLD) if and only if its dual generates projections. Since this is not an original definition, we shall explain the details later in Sect. 24.12.

22.7 Sigma-Products

Further, we present several properties of Σ(A)-products which will be needed later. Some of them are interesting on their own. The following notion is a special case of the one from (9.2).

Recall again that a $\Sigma(A)$*-product* is a topological space of the form

$$\Sigma(A) = \{x \in \mathbb{R}^A : |\operatorname{supp}(x)| \leqslant \aleph_0\},$$

where $\operatorname{supp}(x) = \{\alpha : x(\alpha) \neq 0\}$ denotes the support of $x \in \mathbb{R}^A$. There is a more general notion of a Σ-product of general topological spaces; however, we shall concentrate on Σ-products of real lines or even closed real intervals, since this is sufficient for our purposes. The following property is well known.

Proposition 22.7.1 $\Sigma(\kappa)$ *has countable tightness for every* κ.

Proof Fix

$$p \in \overline{A} \subset \Sigma(\kappa)$$

and fix a countable $M \preceq H(\chi)$ so that $p, A \in M$ and χ is big enough. Let $B = A \cap M$. We claim that $p \in \overline{B}$. For fix a standard neighbourhood V of p, i.e.,

$$V = \{x : x(i) \in I_i \ i \in s\},$$

where $s \subset \kappa$ is finite and each I_i is an open rational interval. Let $S = \kappa \cap M$ and let

$$W = \{x : x(i) \in I_i \ i \in s \cap S\}.$$

Then W is a neighbourhood of p and $W \in M$. Observe that $\mathrm{supp}(p) \subseteq S$, so $p(i) = 0$ for $i \in S \setminus s$. Since

$$M \models W \cap A \neq \emptyset,$$

by elementarity, there exists $a \in A \cap M = B$ such that $a \in W$. Then

$$\mathrm{supp}(a) \subseteq M \cap \kappa = S$$

and hence $a(i) = 0 = p(i) \in I_i$ for $i \in S \setminus s$. It follows that $a \in V$. Thus, finally $a \in B \cap V$. □

The next, rather technical, statement will be needed later. It implies in particular that Σ-products are normal topological spaces and that $[0, 1]^\kappa$ is the Čech–Stone compactification of $\Sigma(\kappa) \cap [0, 1]^\kappa$. These facts were proved by Corson [146] and Glicksberg [276] respectively.

We given $x \in \mathbb{R}^\kappa$ and $S \subseteq \kappa$, we shall denote by $x \mid S$ the function $x \cdot \chi_S$, i.e., $(x \mid S)(\alpha) = x(\alpha)$ for $\alpha \in S$ and $(x \mid S)(\alpha) = 0$ for $\alpha \notin S$.

Lemma 22.7.2 *Let $X \subseteq [0, 1]^\kappa$ be such that $X \cap \Sigma(\kappa)$ is dense in X and let $f : X \to Y$ be a continuous map. Let $\mathcal{B}$ be an open base for $f[X]$. Assume χ is big enough and $M \preceq H(\chi)$ is such that $f, \kappa \in M$ and $\mathcal{B} \subseteq M$. Let $S = M \cap \kappa$. Then*

(a) $x \mid S \in \overline{(X)}$ for every $x \in X$, where the closure is taken in $[0, 1]^\kappa$.
(b) If $x \in X$ and $x \mid S \in X$ then $f(x) = f(x \mid S)$.

Proof (a) Fix $x \in X$ and fix a standard basic neighbourhood u of $x \mid S$. That is, u is defined by a finite function, say ψ, whose values are open rational intervals. Let u' denote the neighbourhood defined by the function ψ restricted to S. Then $u' \in M$. Now

$$M \models u' \cap X \neq \emptyset,$$

so there exists $y \in M$ with

$$y \in \Sigma(\kappa) \cap u' \cap X.$$

Thus

$$\mathrm{supp}(y) \subseteq M$$

(because it is countable) and therefore $\mathrm{supp}(y) \subseteq S$. It follows that $y \in u$, so $u \cap X \neq \emptyset$. As u is arbitrary, we get $x \mid S \in \overline{X}$.

(b) Suppose $x, x \mid S \in X$ and $f(x) \neq f(x \mid S)$. Let $v, w \in \mathcal{B}$ be disjoint and such that $f(x) \in v$, $f(x \mid S) \in w$. There exist basic neighbourhoods u_1, u_2 of x and $x \mid S$ respectively, such that

$$f[u_1] \subseteq v$$

and $f[u_2] \subseteq w$. Assume each u_i is defined by a finite function ψ_i whose values are open rational intervals. Shrinking both neighbourhoods if necessary, we may assume that $\psi = \psi_1 \restriction S = \psi_2 \restriction S$. Let u denote the basic open set defined by ψ. Then $u \in M$ and $M \models f^{-1}[v] \cap u \neq \emptyset$. Thus, there exists $y \in M$ such that

$$y \in \Sigma(\kappa) \cap X \cap u$$

and $f(y) \in v$. But $\operatorname{supp}(y) \subseteq M$; hence $y(i) = 0$ for $i \notin S$ and therefore $y \in u_2$, because

$$(x \mid S)(i) = 0 \in \psi_2(i)$$

for $i \notin S$. It follows that $f(y) \in w$, a contradiction. □

Corollary 22.7.3 *Let A, B be disjoint closed subsets of $\Sigma(\kappa)$. Then $\overline{(A)} \cap \overline{(B)} = \emptyset$, where the closure is taken in $\mathbb{R}^\kappa$. In particular, $\Sigma(\kappa)$ is a normal topological space.*

Proof First, replace $\mathbb{R}$ by the open interval $(0, 1)$, so that

$$A, B \subseteq [0, 1]^\kappa.$$

Suppose there is

$$p \in \overline{(A)} \cap \overline{(B)}$$

in $[0, 1]^\kappa$ and fix a countable $M \preceq H(\chi)$ so that $A, B, p \in M$. Let $S = \kappa \cap M$ and observe that $\operatorname{supp}(p) \subseteq S$, because $p \in M$. By Lemma 22.7.2(a), we have that

$$p = p \mid S \in \overline{(A)} \cap \overline{(B)}.$$

Hence $p \in A \cap B$, a contradiction. □

22.8 Markushevich Bases, Plichko Spaces, and Plichko Pairs

A *biorthogonal system* in a Banach space E is a sequence of pairs

$$\{\langle x_\alpha, y_\alpha \rangle\}_{\alpha \in S}$$

such that

$$\{x_\alpha : \alpha \in S\} \subseteq E,$$

$$\{y_\alpha : \alpha \in S\} \subseteq E'$$

and $y_\alpha(x_\alpha) = 1$ and $y_\alpha(x_\beta) = 0$ whenever $\alpha \neq \beta$.

If moreover the set $\{x_\alpha : \alpha \in S\}$ is linearly dense in E, the system is called *complete*.

A *Markushevich basis* in E is a complete biorthogonal system

$$\{\langle x_\alpha, y_\alpha\rangle\}_{\alpha\in S}$$

such that the set $\{y_\alpha : \alpha \in S\}$ is total for E, i.e., $\bigcap_{\alpha\in S} \ker y_\alpha = \{0\}$. A Markushevich basis

$$\{\langle x_\alpha, y_\alpha\rangle\}_{\alpha\in S}$$

is *norming* if the space

$$D_{<\omega} = \{y \in E' : |\alpha : y(x_\alpha) \neq 0| < \aleph_0\}$$

is norming for E. If the space

$$D_\omega = \{y \in E' : |\alpha : y(x_\alpha) \neq 0| \leqslant \aleph_0\}$$

is norming for E, we say that $\{\langle x_\alpha, y_\alpha\rangle\}_{\alpha\in S}$ is a *countably norming Markushevich basis*.

Note that the space $D_{<\omega}$ is the linear span of $\{y_\alpha : \alpha \in S\}$, while the space D_ω is the countably $weak^*$ closed linear span of $\{y_\alpha : \alpha \in S\}$, i.e., D_ω is the smallest linear space $V \subseteq E'$ that contains $\{y_\alpha : \alpha \in S\}$ and has the property $\overline{A} \subseteq V$ whenever $A \subseteq V$ is countable and the closure is taken in $\sigma(E', E)$. In the above definitions, we shall often say "k-norming" or "countably k-norming" if the respective spaces are k-norming.

The following result is due to Markushevich. We refer the readers to a recent book of Hajek et al. [299] for more information on biorthogonal systems in Banach spaces and related subjects.

Theorem 22.8.1 *Every separable Banach space has a* 1*-norming Markushevich basis.*

Proof Let X be a separable Banach space. Let $\{x_n\}_{n\in\omega}$ enumerate a fixed dense subset of X and let $\{y_n\}_{n\in\omega}$ be a sequence of norm one functionals such that $y_n(x_n) = \|x_n\|$. Then the linear span of $\{y_n\}_{n\in\omega}$ is 1-norming for X. We construct by induction a biorthogonal system

$$\{\langle e_n, f_n\rangle\}_{n\in\omega}$$

so that

$$lin\,\{x_i\}_{i<n} \subseteq lin\,\{e_i\}_{i<2n}$$

and

$$lin\,\{y_i\}_{i<n} \subseteq lin\,\{f_i\}_{i<2n}.$$

Start with $e_0 = x_0$ and $f_0 = y_0$. Now find the smallest index ℓ_1 such that $y_{\ell_1} \notin lin\,(f_0)$ and define

$$f_1 = y_{\ell_1} - y_{\ell_1}(e_0) f_0.$$

Next, find an index k_1 so that $f_1(x_{k_1}) \neq 0$ and define

$$e_1 = (x_{k_1} - f_0(x_{k_1})e_0)/f_1(x_{k_1}).$$

Notice that $f_i(e_i) = \delta_{i,j}$ for $i, j < 2$.

Continue in this way: at step $2n$ choose $x_{k_{2n}} \notin lin\,\{e_i\}_{i<2n}$ and construct e_{2n} first, while at step $2n+1$ choose

$$y_{\ell_{2n+1}} \notin lin\,\{f_i\}_{i<2n+1}$$

and construct f_{2n+1} first.

Finally we obtain a Markushevich basis

$$\{\langle e_n, f_n\rangle\}_{n\in\omega}$$

which is 1-norming because the linear span of $\{f_n\}_{n\in\omega}$ contains $\{y_n\}_{n\in\omega}$. □

We shall see later that many of the well-known non-separable Banach spaces have a countably norming Markushevich basis. Further, we give an example which distinguishes the properties of having norming and countably norming Markushevich bases. The next statement follows from other results; however, we give a direct proof here.

Proposition 22.8.2 *The space $C(\omega_1 + 1)$ has a countably 1-norming Markushevich basis.*

Proof For each $\alpha \leqslant \omega_1$ let x_α be the characteristic function of the interval $[0, \alpha]$. Given $\alpha < \omega_1$, let y_α be the functional corresponding to the measure

$$\delta_\alpha - \delta_{\alpha+1},$$

where δ_x denotes the Dirac measure of $x \in \omega_1 + 1$. Finally, let y_{ω_1} be the functional corresponding to the Dirac measure of $0 \in \omega_1 + 1$. It is clear that

$$\{\langle x_\alpha, y_\alpha\rangle\}_{\alpha\in\omega_1+1}$$

is a biorthogonal system in $C(\omega_1 + 1)$.

Fix

$$\mu \in C(\omega_1 + 1)' \setminus \{0\}$$

and let K be the support of μ. Let $\alpha = \min K$. If $\alpha = \omega_1$ then $\mu = r \cdot \delta_{\omega_1}$, where $r \neq 0$; hence

$$\mu(x_{\omega_1}) = r \neq 0.$$

If $\alpha < \omega_1$ then $\mu(x_\alpha) = \mu(\{\alpha\}) \neq 0$. It follows that

$$\{x_\alpha\}_{\alpha \leqslant \omega_1}$$

is linearly dense.

Let D denote the countable closure of the linear span of $\{y_\alpha\}_{\alpha \leqslant \omega_1}$. We claim that D contains the 1-norming set

$$S = \{\delta_\xi : \xi < \omega_1\}.$$

Suppose $S \not\subseteq D$ and let α be minimal such that $\delta_\alpha \notin D$. Then $\alpha > 0$, because

$$\delta_0 = y_{\omega_1} \in D.$$

Further, α cannot be a limit ordinal, because given $\xi_0 < \xi_1 < \xi_2 < \dots$ with $\alpha = \sup_n \xi_n$, we have that

$$\delta_\alpha = \lim_{n \to \infty} \delta_{\xi_n}$$

in the $weak^*$ topology. Thus $\alpha = \eta + 1$. But

$$\delta_\alpha = \delta_\eta - (\delta_\eta - \delta_{\eta+1}) = \delta_\eta - y_\eta \in D,$$

a contradiction. □

It is remarkable that the space $C(\omega_1 + 1)$ does not have a norming Markushevich basis. This is a result of Alexandrov and Plichko [4]. Another interesting result, due to Kalenda, says that $C(\omega_2 + 1)$ does not have a countably norming Markushevich basis, see [359]. In fact, Kalenda's result is valid for every regular cardinal $\kappa > \omega_1$. On the other hand, it is unknown whether it holds for arbitrary ordinals, namely whether the existence of a countably norming Markushevich basis on $C(\eta + 1)$ implies that $\eta < \omega_2$.

We now turn to the crucial notion of this section.

A Banach space E is called a *Plichko space* if there are a linearly dense set $A \subseteq E$ and a norming set $B \subseteq E'$ such that

$$|\{x \in A : y(x) \neq 0\}| \leqslant \aleph_0$$

for every $y \in B$. In that case, $\langle E, B\rangle$ will be called a *Plichko pair* and A will be called a *witness* for this property.

We shall add a prefix "k-" in the above definitions, whenever $k \geqslant 1$ is such that B is k-norming. Note that if $\langle E, B\rangle$ is a Plichko pair witnessed by $A \subseteq E$, then the formula

$$T(y)(a) = y(a)$$

defines a one-to-one *weak** continuous linear operator

$$T : E' \to \mathbb{R}^A$$

such that $B \subseteq T^{-1}[\Sigma(A)]$. Conversely, given a one-to-one *weak** continuous linear operator like above, then $\langle E, T^{-1}[\Sigma(A)]\rangle$ is a Plichko pair. The subspace $T^{-1}[\Sigma(A)]$ will be sometimes called a Σ-*space* or, more precisely, a Σ-*subspace* of E'.

Proposition 22.8.3 *Let E be a Banach space and let $k \geqslant 1$. The following properties are equivalent.*

(a) E is a k-Plichko space.
(b) There exists a weak continuous injective linear operator $T : E' \to \mathbb{R}^\Gamma$ such that $T^{-1}[\Sigma(\Gamma)]$ is k-norming.*

Proof (a) $\Rightarrow$ (b): Let $\langle E, B\rangle$ be a k-Plichko pair with witness A and define

$$T : E' \to \mathbb{R}^A$$

by $Tf = f \restriction A$. By definition, $Tf \in \Sigma(A)$ whenever $f \in B$.

(b) $\Rightarrow$ (a): Since T is *weak** continuous, for each $\gamma \in \Gamma$ there is $x_\gamma \in E$ such that $pr_\gamma T = x_\gamma$, where $pr_\gamma : \mathbb{R}^\Gamma \to \mathbb{R}$ is the projection onto γ-th coordinate. Let

$$A = \{x_\gamma : \gamma \in \Gamma\}$$

and let

$$B = T^{-1}[\Sigma(\Gamma)].$$

Then A witnesses that $\langle E, B\rangle$ is a k-Plichko pair. □

Using PRIs we shall prove that Plichko spaces are precisely those Banach spaces which have a countably norming Markushevich basis.

Theorem 22.8.4 *Let $\langle E, D\rangle$ be a Plichko pair. Then D generates projections in E.*

Proof Fix

$$M \preceq \langle H(\theta), \in \rangle$$

such that $D \in M$. Suppose $\varphi \in E'$ is such that $E \cap M \subseteq \ker \varphi$ and

$$^{\perp}(D \cap M) \subseteq \ker \varphi.$$

The latter fact means that

$$\varphi \in \overline{(D \cap M)},$$

where the closure is taken in $\sigma(E', E)$. Let $G \subseteq E$ be a linearly dense set such that $\mathrm{supp}(y, G)$ is countable for each $y \in D$. By elementarity, we may assume that $G \in M$.

Suppose $\varphi \neq 0$ and fix $u \in G$ such that $\varphi(u) \neq 0$. Since φ is in the $weak^*$ closure of $D \cap M$, we may find $\psi \in D \cap M$ such that $\psi(u) \neq 0$. Thus

$$u \in \mathrm{supp}(\psi, G).$$

On the other hand, $\mathrm{supp}(\psi, G) \in M$, because $\psi, G \in M$. By Proposition 22.1.1(c) one gets

$$\mathrm{supp}(\psi, G) \subseteq M.$$

In particular $u \in E \cap M$ and hence $\varphi(u) = 0$, a contradiction. □

A similar statement was proved by Koszmider [372, Lemma 4.1], using the existence of a countably 1-norming Markushevich basis.

The next result provides non-trivial examples of Plichko spaces of continuous functions. One can find it in the paper of Kalenda [360].

Theorem 22.8.5 *Let K be a compact Abelian group. Then $C(K)$ is 1-Plichko.*

Proof We first show that the complex $C(K)$ is 1-Plichko. After that, it is easy to deduce that the real $C(K)$ is 1-Plichko too (the converse implication is an open problem).

Let G be the Pontryagin dual of K. Since the circle group is a subset of the complex field, G may be regarded as a subspace of $C(K)$. Given $g \in G$, define

$$g^*(f) = \int_K f \cdot \overline{g}\, d\mu,$$

where μ is the Haar probability measure on K. Then $g^* \in C(K)'$. Note that, given $\chi \in G$, either $\int_K \chi\, d\mu = 1$ when $\chi = 1$, or else $\int_K \chi\, d\mu = 0$. Indeed, if $\chi \neq 1$ then choose $a \in K$ so that $\chi(a) \neq 1$ and observe that

$$\int_K \chi(x)\,d\mu = \int_K \chi(a+x)\,d\mu = \int_K \chi(a)\chi(x)\,d\mu = \chi(a)\cdot\int_K \chi(x)\,d\mu.$$

The first equality holds because μ is translation invariant. Thus $\int_K \chi\,d\mu = 0$.

It follows that if $f, g \in G$ and $f \neq g$ then $g^*(f) = 0$ and $g^*(g) = 1$. Observe that, by the Stone–Weierstrass theorem, G is linearly dense in $C(K)$, since it separates points, contains the constant one function and is closed under multiplication and complex conjugation. Thus, $\{\langle g, g^*\rangle\}_{g\in G}$ is a Markushevich basis. It remains to show that it is countably 1-norming.

Let $\Phi\colon L^1(\mu) \to C(K)'$ be the linear operator defined by

$$\Phi(h)f = \int_K f\overline{h}\,d\mu, \qquad f \in C(K).$$

In particular, $\Phi(g) = g^*$ for $g \in G$. Note that Φ is an isometric embedding. Since μ is strictly positive, $\operatorname{im}\Phi$ is 1-norming for $C(K)$ (note that $C(K)$ is canonically embedded into $L^\infty(\mu)$ and $L^1(\mu)$ is 1-norming for that space).

Note that $C(K)$ is a formally a subspace of $L^1(\mu)$ and it is dense with respect to the L^1 norm. Let H be the linear span of G. As we have already noticed, H is dense in $C(K)$ with respect to the maximum norm. Thus, H is also dense in $C(K)$ with respect to the L^1 norm inheritted from $L^1(\mu)$. It follows that H is norm dense in $L^1(\mu)$. Consequently ΦH is norm dense in $L^1(\mu)$ and therefore it is 1-norming for $C(K)$. Finally, observe that ΦH is the linear span of $\{g^*\colon g \in G\}$ and the space

$$\{\varphi \in C(K)' : |\{g \in G\colon \varphi(g) \neq 0\}| \leqslant \aleph_0\}$$

is norm closed in $C(K)'$. This completes the proof. □

22.9 Preservation of Plichko Spaces

Following is the announced preservation result involving the notion of a projectional sequence. We shall use it for characterizing Plichko spaces in terms of projectional skeletons. Finally, we show that every Plichko space has a countably norming Markushevich basis.

Theorem 22.9.1 *Let $\{P_\alpha\}_{\alpha<\kappa}$ be a projectional sequence in a Banach space E and let $D \subseteq E'$ be a norming space such that*

$$D = \bigcup_{\alpha<\kappa} P'_\alpha D$$

and $\langle P_\alpha E, P'_\alpha D\rangle$ is a Plichko pair for each $\alpha < \kappa$. Then $\langle E, D\rangle$ is a Plichko pair.

Note that we do not assume that the above projections are uniformly bounded.

Before the proof, we need an auxiliary fact.

Lemma 22.9.2 *Let $\langle E, D\rangle$ be a Plichko pair, and let $P : E \to E$ be a bounded projection such that $P'D \subseteq D$. Then $\langle \ker P, D \cap \ker P'\rangle$ is a Plichko pair.*

Proof Let A be a linearly dense subset of E such that

$$|\operatorname{supp}(y, A)| \leqslant \aleph_0$$

for every $y \in D$. Let $B = \{a - Pa : a \in A\}$. Then B is linearly dense in

$$\ker P = \operatorname{im}(\mathrm{id}_E - P).$$

Let $H = D \cap \ker P'$. Fix $y \in D$ with $\|y\| = 1$. Let $z = y - P'y$. Then $z \in D$, because P' preserves D. Further, $z \in \ker P'$ and

$$\|z\| \leqslant 1 + \|P\|.$$

Given $x \in \ker P$, we have

$$|z(x)| = |y(x) - (P'y)x| = |y(x) - y(Px)| = |y(x)|.$$

This shows that H is norming for $\ker P$. Finally, given $y \in H$, we have

$$y(a - Pa) = y(a) - y(Pa) = y(a) - (P'y)a = y(a),$$

so $\operatorname{supp}(y, B) = \operatorname{supp}(y, A)$. Thus, B witnesses that $\langle \ker P, H\rangle$ is a Plichko pair. □

Now we prove Theorem 22.9.1.

Proof We construct inductively a family of sets $\{A_\alpha\}_{\alpha<\kappa}$ such that

(i) A_α is a linearly dense subset of $P_\alpha E$;
(ii) $\alpha < \beta \Rightarrow A_\alpha \subseteq A_\beta$;
(iii) $\operatorname{supp}(y, A_\alpha)$ is countable for every $y \in D$;
(iv) $P_\alpha a = 0$ whenever $a \in A_\beta \setminus A_\alpha$.

We start with a linearly dense set $A_0 \subseteq P_0E$ witnessing that $\langle P_0E, P_0'D\rangle$ is a Plichko pair. We must check (iii). Observe that $a = P_0a$ and consequently $y(a) = (P_0'y)a$ for every $a \in A_0$ and $y \in E^*$. Hence

$$\operatorname{supp}(y, A_0) = \operatorname{supp}(P_0'y, A_0)$$

is countable and (iii) holds. Now, fix an ordinal $\delta > 0$ and assume

$$\{A_\alpha\}_{\alpha<\delta}$$

has already been defined.

Suppose first that $\delta = \beta + 1$. We use Lemma 22.9.2. Let $Y = P_\delta E$ and let

$$D' = \{y \restriction Y : y \in D\} \subseteq Y'.$$

Then $\langle Y, D'\rangle$ is a Plichko pair and $P_\beta \restriction Y$ is a projection whose dual preserves D'. By Lemma 22.9.2, there is a linearly dense subset B of

$$\ker P_\beta \cap Y$$

witnessing that

$$\langle \ker P_\beta \cap Y, E\rangle$$

is a Plichko pair, where $E = D' \cap \ker[(P_\beta \restriction Y)']$. It follows that $\operatorname{supp}(y, B)$ is countable whenever

$$y \in D \cap \ker P'_\beta.$$

Define $A_\delta = A_\beta \cup B$. Conditions (i), (ii) and (iv) are obviously satisfied. It remains to check (iii). Fix $y \in D$. By the induction hypothesis, $\operatorname{supp}(y, A_\beta)$ is countable. Let $z = y - P'_\beta y$. Then

$$z \in D \cap \ker P_\beta,$$

so $\operatorname{supp}(z, B)$ is countable. Finally, given $b \in B$ we have

$$z(b) = y(b) - (P'_\beta y)b = y(b) - y(P_\beta b) = y(b),$$

therefore $\operatorname{supp}(y, B) = \operatorname{supp}(z, B)$ is countable. This shows (iii), because

$$\operatorname{supp}(y, A_\delta) = \operatorname{supp}(y, A_\beta) \cup \operatorname{supp}(y, B).$$

Suppose now that δ is a limit ordinal. Define

$$A_\delta = \bigcup_{\xi<\delta} A_\xi.$$

Clearly, conditions (i) and (ii) are satisfied. Condition (iv) is obvious, so it remains to check (iii). Fix $y \in D$. There is nothing to prove if δ has a countable cofinality, because then

$$\operatorname{supp}(y, A_\delta) = \bigcup_{\xi<\delta} \operatorname{supp}(y, A_\xi).$$

Assume $\operatorname{cf}\delta > \aleph_0$. Since $A_\delta \subseteq P_\delta E$, we see that

$$\operatorname{supp}(y, A_\delta) = \operatorname{supp}(P'_\delta y, A_\delta).$$

Thus, we may assume that $y = P'_\delta y$. Continuity of the projectional sequence implies that

$$y = \lim_{\xi<\delta} P'_\xi y,$$

where the limit is with respect to the $weak^*$ topology. We know that P'_δ is contained in a Σ-space; therefore, it is $weak^*$ countably tight. It follows that there exists $\alpha < \delta$ such that $P'_\xi y = y$ for $\alpha \leqslant \xi < \delta$. In particular,

$$\operatorname{supp}(y, A_\delta) = \operatorname{supp}(y, A_\alpha),$$

because if $a \in A_\delta \setminus A_\alpha$ then

$$y(a) = (P'_\alpha y)a = y(P_\alpha a) = 0,$$

by (iv). By the induction hypothesis, $\operatorname{supp}(y, A_\delta)$ is countable. This shows (iii).

Finally, set

$$A = \bigcup_{\alpha<\kappa} A_\alpha.$$

By (i), A is linearly dense in E. It remains to check that $\operatorname{supp}(y, A)$ is countable for every $y \in D$. Fix $y \in D$. By the assumption, $y \in P'_\alpha D$ for some $\alpha < \kappa$. Thus, $P'_\xi y = y$ whenever

$$\alpha \leqslant \xi < \kappa,$$

and we conclude like in the limit case of the above construction. This completes the proof. □

We are now ready to characterize Plichko spaces in terms of projectional skeletons. A projectional skeleton $\{P_s\}_{s\in\Sigma}$ is *commutative* if

$$P_s \circ P_t = P_t \circ P_s$$

holds for every $s, t \in \Sigma$.

Lemma 22.9.3 *Let $\langle E, D\rangle$ be a Plichko pair and assume D is a Σ-subspace of E'. Then there exists a commutative projectional skeleton $\mathfrak{s} = \{P_s\}_{s\in\Gamma}$ on E such that $D = D(\mathfrak{s})$.*

Proof Fix a big enough regular cardinal θ and let Γ be the family of all countable elementary substructures M of

$$\langle H(\theta), \in \rangle$$

such that $\langle E, D\rangle \in M$. By Theorem 22.8.4 and the proof of Theorem 22.6.2, we know that $\mathfrak{s} = \{P_M\}_{M\in\Gamma}$ is a projectional skeleton in E such that $D \subseteq D(\mathfrak{s})$, where P_M is induced by $\langle E, D, M\rangle$, i.e., $\operatorname{im} P_M = E_M$ and $\ker P_M = {}^{\perp}(D \cap M)$. Theorem 22.6.6(b) says that $D(\mathfrak{s})$ is $weak^*$ countably tight and that it is $weak^*$ countably closed, i.e., $\overline{A} \subseteq D$ whenever $A \subseteq D$ is countable and the closure is in $\sigma(E', E)$. Thus $D = D(\mathfrak{s})$.

It remains to show that $\mathfrak{s}$ is commutative. Given $M \in \Gamma$, define $r_M = P_M \restriction G$, where $G \subseteq E$ is a linearly dense set witnessing that D is a Σ-space. We may assume that $G \in M$. We claim that $(*)$

$$r_M(x) = \begin{cases} x & x \in G \cap M, \\ 0 & x \in G \setminus M \end{cases}$$

Indeed,

$$G \cap M \subseteq \overline{(E \cap M)} = \operatorname{im} P_M.$$

If $x \in G \setminus M$ then for $y \in D \cap M$ we have that

$$x \notin \operatorname{supp}(y, G),$$

because $\operatorname{supp}(y, G) \subseteq M$. Hence $y(x) = 0$ for $y \in D \cap M$ and therefore $x \in {}^{\perp}(D \cap M) = \ker P_M$.

Using (*), we see that $r_M \circ r_N = r_{M\cap N}$ for $M, N \in \Gamma$. Since G is linearly dense in E, this shows that

$$P_M \circ P_N = P_{M\cap N} = P_N \circ P_M$$

for every $M, N \in \Gamma$. This completes the proof. □

Theorem 22.9.4 ([382]) *Let E be a Banach space and let $D \subseteq E'$ be a k-norming subspace. The following are equivalent.*

(a) $\langle E, D\rangle$ is a Plichko pair.
(b) E has a commutative projectional skeleton $\mathfrak{s}$ such that $D \subseteq D(\mathfrak{s})$.

Proof (a) ⇒ (b) is contained in Lemma 22.9.3.

(b) ⇒ (a) Suppose we have proved this implication for spaces of density $< \kappa$ and fix a Banach space E of density κ with a commutative k-projectional skeleton

$$\{P_s\}_{s\in\Gamma}.$$

By (the proof of) Theorem 22.5.5, there exists a k-projectional resolution of the identity

$$\mathfrak{p} = \{P_\alpha\}_{\alpha<\kappa}$$

on E such that for each $\alpha < \kappa$ there is a directed set $S_\alpha \subseteq \Gamma$ with $P_\alpha x = \lim_{s\in S_\alpha} P_s x$ for $x \in E$ (to be formal, we need to assume that $\Gamma \cap \kappa = \emptyset$). Observe that, by continuity,

$$P_s \circ P_\alpha = P_\alpha \circ P_s$$

holds for every $s \in \Gamma$ and $\alpha < \kappa$. Let D be the norming space induced by $\mathfrak{p}$. Fix $y \in D$ and fix $\alpha < \kappa$. Let $s \in \Gamma$ be such that $y = P'_s y$. Then

$$P'_\alpha y = P'_\alpha P'_s y = P'_s P'_\alpha y \in D.$$

Hence $P'_\alpha D \subseteq D$. Now use Theorem 22.9.1. In case where $\operatorname{cf}\kappa = \aleph_0$, it may happen that

$$D \neq \bigcup_{\alpha<\kappa} P'_\alpha D,$$

but we may replace D by

$$\bigcup_{\alpha<\kappa} P'_\alpha D,$$

still having a k-norming space. By Theorem 22.9.1, $\langle E, D\rangle$ is a Plichko pair. □

Corollary 22.9.5 *Let E be a Banach space of density $\aleph_1$. The following properties are equivalent.*

(a) E is a 1-Plichko space.
(b) E has a projectional resolution of the identity.
(c) $E = \bigcup_{\alpha<\omega_1} E_\alpha$, where $\{E_\alpha\}_{\alpha<\omega_1}$ is a continuous chain of closed separable subspaces such that E_α is 1-complemented in $E_{\alpha+1}$ for each $\alpha < \omega_1$.

Proof Implication (b) $\Longrightarrow$ (c) is trivial and (c) $\Longrightarrow$ (b) is a consequence of Theorem 22.5.4. The fact that (a) $\Longrightarrow$ (b) follows from Theorem 22.5.5 together with Theorem 22.9.4. Finally, (b) $\Longrightarrow$ (c) follows again from Theorem 22.9.4, because a PRI of length ω_1 is a commutative projectional skeleton. □

We now turn to the announced result on Markushevich bases.

Theorem 22.9.6 *Given a Banach space E and $k \geqslant 1$, the following conditions are equivalent.*

(a) E is a k-Plichko space.
(b) E has a countably k-norming Markushevich basis.

Proof (b) $\Longrightarrow$ (a) This follows directly from the definitions. Namely, let $\{\langle x_s, y_s\rangle\}_{s\in\Gamma}$ be a countably norming Markushevich basis and let D consists of all $y \in E'$ such that $\{s\colon y(x_s) \neq 0\}$ is countable. By definition, D is norming and $\{x_s : s \in \Gamma\}$ witnesses that $\langle E, D\rangle$ is a Plichko pair.

(a) $\Longrightarrow$ (b) We use induction on the density of E. If E is separable then we use Markushevich's theorem (Theorem 22.8.1 above). Fix a k-Plichko space E of density $\kappa > \aleph_0$ and assume that (a) $\Longrightarrow$ (b) holds for Banach spaces of density $< \kappa$. Actually, we need to have a stronger induction hypothesis: given a k-Plichko pair $\langle E, D\rangle$, there is a countably k-norming Markushevich basis $\{\langle x_s, y_s\rangle\}_{s\in\Gamma}$ such that $y_s \in D$ for every $s \in \Gamma$. Note that, by the proof of Markushevich's theorem, this holds for any pair $\langle E, D\rangle$ where E is separable and D is norming.

Let $D \subseteq E'$ be a Σ-subspace and let $\{P_\alpha\}_{\alpha<\kappa}$ be a PRI induced by a commutative projectional skeleton in E. Then $P'_\alpha D \subseteq D$. Indeed, $D = \bigcup_{s\in\Sigma} P'_s E'$, where $\{P_s\}_{s\in\Sigma}$ is a fixed commutative projectional skeleton. Given $y \in D$, find $t \in \Sigma$ such that $y = P'_t y$. Then $P'_\alpha y = P'_\alpha P'_t y = P'_t P'_\alpha y$ so $y \in P'_t E' \subseteq D$.

It suffices to show that a countably k-norming Markushevich basis $\mathcal{M}_\alpha = \{\langle x_s, y_s\rangle\}_{s\in\Gamma_\alpha}$ in E_α such that $y_s \in D$ for $s \in \Gamma_\alpha$ can be extended to a Markushevich basis $\mathcal{M}_{\alpha+1} = \{\langle x_s, y_s\rangle\}_{s\in\Gamma_{\alpha+1}}$ in $E_{\alpha+1}$ which is countably k-norming for $E_{\alpha+1}$. Let $F = \ker P_\alpha \cap E_{\alpha+1}$ and let $H = D\ker(P'_\alpha \restriction E_{\alpha+1})$. By Lemma 22.9.2, $\langle F, H\rangle$ is a Plichko pair. By inductive hypothesis, there is a Markushevich basis $\{\langle x_s, y_s\rangle\}_{s\in T}$ such that $y_s \in H$ for every $s \in T$. We may assume that $T \cap \Gamma_\alpha = \emptyset$. Set $\Gamma_{\alpha+1} = \Gamma_\alpha \cup T$. Since $E_{\alpha+1} = E_\alpha \oplus F$, $\{x_s\}_{s\in\Gamma_{\alpha+1}}$ is linearly dense in $E_{\alpha+1}$. Formally, by the proof of Lemma 22.9.2, H is $2k$-norming for $E_{\alpha+1}$, but on the other hand, $P'_\alpha E' + H$ is k-norming. Thus finally $\mathcal{M}_{\alpha+1}$ is countably k-norming for $E_{\alpha+1}$. □

We finish with the following result, perhaps first noticed by Plichko (see Theorem 6.3 in [360] and the comment preceding it), bringing many examples of Plichko spaces.

Theorem 22.9.7 *Every order continuous Banach lattice is a* 1*-Plichko space.*

Recall that a Banach lattice is *order continuous* if every decreasing net with infimum 0 converges to 0 in the norm topology.

Proof We only sketch the idea of the proof. More details are given in the proof of Theorem 6.3 in [360].

Let E be an order continuous Banach lattice. By [410, Proposition 1.a.9], E can be decomposed into an unconditional sum of mutually disjoint ideals, each having a weak unit. In fact, this is an ℓ_1-sum of such ideals. In order to conclude that E is 1-Plichko, it suffices to show that each ideal in the decomposition is (WCG). Theorem 22.9.1 implies that an arbitrary ℓ_1-sum of (WCG) Banach spaces is 1-Plichko.

Let X be an order continuous lattice with the weak unit e. This means that $e \wedge x = 0$ holds only if $x = 0$. The interval $[0, e]$ is weakly compact by [410, Theorem 1.b.16]. Further, given $x \geqslant 0$, let $x_n = x \wedge ne$. Then $x_n = n(\frac{x}{n} \wedge e)$ belongs to the linear span of $[0, e]$ and the sequence $\{x_n\}_{n\in\omega}$ is increasing, bounded by x. By order

continuity, it converges to some $y \in X$. Since e is a weak unit, necessarily $y = x$. Thus, the closed linear span of $[0, e]$ contains the positive cone of X which shows that X is (WCG) and completes the proof. □

Corollary 22.9.8 *Given a compact K, the space $C(K)'$ is 1-Plichko.*

Recall that $C(K)$ spaces may not be Plichko: perhaps the simplest counterexample is $C(\beta\mathbb{N}) = \ell_\infty$.

Corollary 22.9.9 *Given a semi-finite non-negative measure μ, the space $L_1(\mu)$ is 1-Plichko.*

Recall that a measure μ is *semi-finite* if for every measurable set A with $\mu(A) = +\infty$ there is a measurable subset $B \subseteq A$ such that $0 < \mu(B) < +\infty$.

Chapter 23
Spaces of Continuous Functions over Compact Lines

Abstract In this chapter, we discuss selected properties of Banach spaces of type $C(K)$, where K is a linearly ordered compact space, called a compact line for short. In particular, we present Nakhmanson's theorem stating that if K is a compact line such that $C_p(K)$ is a Lindelöf space, then K is second-countable. We also discuss the separable complementation property in the context of compact lines. Compact lines are relatively easy to investigate, yet they form a rich class of spaces and provide several interesting examples. A very special case is the smallest uncountable well-ordered space ω_1+1, which appeared several times in the previous chapters. Its space of continuous functions turns out to be a canonical example for several topological and geometric properties of Banach spaces. More complicated compact lines provide examples related to Plichko spaces.

23.1 General Facts

In this chapter, we discuss selected properties of Banach spaces of type $C(K)$, where K is a linearly ordered compact space, called shortly a *compact line*.

To be more precise, recall that a linearly ordered set X is a topological space when endowed with the *interval topology* generated by open intervals of the form

$$(\leftarrow, b) := \{x \in X : x < b\}$$

and

$$(a, \rightarrow) := \{x \in X : a < x\},$$

where $a, b \in X$. We use standard notation concerning intervals, e.g.

$$(a, b] = \{x \in X : a < x \leqslant b\}$$

and $(a, b) = (\leftarrow, b) \cap (a, \rightarrow)$. Recall that a linearly ordered space is compact if and only if all its subsets have suprema and infima (the supremum of the empty

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set is, by definition, the minimal element of the space). Given a compact line K, we shall always denote by 0_K and 1_K the minimal and the maximal element of K respectively.

A map $f : K \to L$ between linearly ordered sets is *increasing* if $f(x) \leqslant f(y)$ whenever $x \leqslant y$. If it is moreover injective, we say that f is *strictly increasing*. We shall be interested in continuous increasing maps.

Proposition 23.1.1 *Let $f\colon K \to L$ be an increasing surjection of linearly ordered spaces. If K is compact and $f^{-1}(p)$ is closed for every $p \in L$ then f is continuous. In particular, L is compact.*

Proof Fix an increasing surjection $f\colon K \to L$, where K, L are compact lines. Fix $p \in L$ and let $q = \min f^{-1}(p)$ (here we use the compactness of K). Now

$$f^{-1}[[p, \to)] = [q, \to)\,.$$

Similarly,

$$f^{-1}[(\leftarrow, p]] = (\leftarrow, q]\,.$$

Thus f is continuous, because intervals of the form $[p, \to)$ and $(\leftarrow, p]$ $(p \in L)$ form a closed subbase for the space L. □

Recall that $\mathbb{I}$ denotes the real closed unit interval, a "generic" compact line in the sense that for every compact line K continuous increasing maps from K into $\mathbb{I}$ separate points. More precisely:

Lemma 23.1.2 *Let K be a compact line, $a < b$ in K. Then there exists a continuous increasing map $f : K \to \mathbb{I}$ such that $f(a) = 0$ and $f(b) = 1$.*

Proof By Urysohn's lemma, there is a continuous map $g\colon [a, b] \to \mathbb{I}$ such that $g(a) = 0$ and $g(b) = 1$. Define f to be constant 0 on $(\leftarrow, a]$, constant 1 on $[b, \to)$ and

$$f(t) = \max_{s \in [a,t]} g(s)$$

for $t \in [a, b]$. It is clear that f is increasing and it is easy to check that it is continuous. □

Proposition 23.1.3 *For every compact line K, continuous increasing functions on K with values in $\mathbb{I}$ form a linearly dense set in $C(K)$.*

Proof Let A consist of all such functions. Note that $1 \in A$ and, by Lemma 23.1.2, A separates the points of K. Further, A is closed under multiplication, i.e., $f \cdot g \in A$ whenever $f, g \in A$. Let B be the linear span of A. Then B is a subring of $C(K)$ which separates points and contains the constant functions. By the Stone–Weierstrass theorem, B is dense in $C(K)$. □

It is easy to determine when a compact line is connected: its order must be dense in the sense that for every $x < y$, there must exist z such that $x < z < y$. It is also easy to determine when a given compact line K is zero-dimensional.

Namely, we say that an interval $[a, b]$ is a *jump* in K is $a < b$ and $(a, b) = \emptyset$. It is clear that a compact line K is zero-dimensional if and only if for every $x < y$ in K, there is a jump $[a, b]$ such that $x \leqslant a < b \leqslant y$. Indeed, if there is no jump in $[x, y]$, then this interval is connected.

Given a compact line K, we shall denote by $\mathbb{L}(K)$ the set of all $p \in K$ such that $[q, p]$ is a jump for some q. The unique point q will be denoted by p^-. In case K is zero-dimensional, it can be recovered from the set $\mathbb{L}(K)$ endowed with the induced linear order. Namely, K is just the Dedekind compactification of $\mathbb{L}(K) \times \{0, 1\}$ endowed with the lexicographic order.

We now describe a useful operation of making a compact line connected. Fix a compact line K and define the following relation:

$$x \sim y \iff [x, y] \text{ is scattered.}$$

Since the property of being scattered is stable under subsets and finite unions, $\sim$ is clearly an equivalence relation with convex equivalence classes. It is not hard to check that the closure of an increasing sequence of scattered intervals is scattered, therefore the equivalence classes of $\sim$ are closed intervals. Let $L = K/\sim$ be endowed with the natural linear order (induced from K).

By Proposition 23.1.1, the canonical increasing quotient map $q\colon K \to L$ is continuous, and L is compact. Finally, L is connected because otherwise it would contain a jump $[a, b]$ but on the other hand

$$q^{-1}(a) \cup q^{-1}(b)$$

would be a scattered interval which contradicts the fact that $x \not\sim y$ whenever $q(x) = a$ and $q(y) = b$. Summarizing we have the following:

Proposition 23.1.4 *Given a compact line K, there exist a connected compact line L and a continuous increasing quotient map $q : K \to L$ such that $q^{-1}(t)$ is scattered for every $t \in L$. Furthermore, for every connected compact line, for every continuous increasing map $f : K \to X$, there exists a unique increasing map $h : L \to X$ satisfying $h \circ q = f$.*

The map $q : K \to L$ will be called the *connectification* of K. By the second statement, it is defined uniquely up to an order isomorphism.

Proof Only the second statement requires an argument. Fix $t \in L$. Since $q^{-1}(t)$ is scattered, so is its image under f. Since X is connected, we conclude that

$$f[q^{-1}(t)]$$

is a singleton. Denote it by $h(t)$. This shows that there is a unique way to define h so that $h \circ q = f$. Clearly, h is increasing. □

A similar procedure can be applied to a first countable compact line for getting a nowhere separable one. Recall that a topological space is *nowhere separable* if all its separable subsets are nowhere dense. For a linearly ordered space, this is equivalent to saying that non-degenerate intervals are not separable.

Proposition 23.1.5 *Let K be a first countable compact line. Then there exists a continuous increasing quotient map $q\colon K \to L$ such that L is nowhere separable and $q^{-1}(t)$ is separable for every t. Furthermore, for every nowhere separable compact line, for every continuous increasing map $f : K \to X$, there exists a unique increasing map $h : L \to X$ satisfying $h \circ q = f$.*

Proof Define

$$x \sim y \iff [x, y] \text{ is separable.}$$

Then $\sim$ is an equivalence relation with convex classes. As K is first countable, the equivalence classes of $\sim$ are closed and separable. Let $L = K/\sim$. Then L is a compact line, and the natural quotient $g\colon K \to L$ is continuous and increasing.

We need to show that L is nowhere separable. For suppose $a < b$ in K is such that $I = [q(a), q(b)]$ is separable and fix a countable dense set D in I. For each $y \in D$ choose a countable set E_y which is dense in $q^{-1}(y)$. Then $\bigcup_{y\in D} E_y$ is dense in $[a, b]$ and hence, $a \sim b$, i.e. $q(a) = q(b)$.

Finally, fix a nowhere separable compact line X and fix a map $f\colon K \to X$. The only possibility for defining $h\colon L \to X$ is setting $h(q(x)) = f(x)$. It remains to check that this is correct. So, suppose $x \sim y$ and $x < y$. Then $f(x) \leqslant f(y)$ and $[f(x), f(y)] = f[[x, y]]$ is separable. Hence $f(x) = f(y)$, because X is nowhere separable. □

23.2 Nakhmanson's Theorem

The following result is due to Nakhmanson [462] (see also [32, Theorem IV.10.1]).

Theorem 23.2.1 *Let K be a compact line such that $C_p(K)$ is a Lindelöf space. Then K is second countable.*

Proof Let M consist of all continuous increasing maps of the form $f : K \to \mathbb{I}$. Clearly, M is a closed subset of $C(K)$ with the pointwise topology. Fix $f \in M$. We are going to define a neighbourhood $U(f) \in \tau_p$.

Suppose first that $\frac{1}{2} \in f[K]$ and let

$$F = f^{-1}\left(\frac{1}{2}\right).$$

Then F is a closed interval of the form $[a, b]$ (possibly $a = b$). Define

$$U(f) = \left\{ x \in M : x(a) > \frac{1}{4} \text{ and } x(b) < \frac{3}{4} \right\}.$$

Suppose now that

$$\frac{1}{2} \notin f[K].$$

Then $K = (\leftarrow, a] \cup [b, \rightarrow)$, where $a < b$ and $f(a) < \frac{1}{2}$, $f(b) > \frac{1}{2}$. Define

$$U(f) = \left\{ x \in M : x(a) < \frac{1}{2} < x(b) \right\}.$$

In both cases $U(f)$ is a τ_p-neighbourhood of f in M. Applying the Lindelöf property, there is a sequence

$$\{f_n\}_{n\in\omega} \subseteq M$$

such that $M = \bigcup_{n\in\omega} U(f_n)$. We shall check that f_n's separate the points of K, which will prove that K is second countable.

Fix $p < q$ in K. By Lemma 23.1.2, there is $g \in M$ such that $g(p) = 0$ and $g(q) = 1$. Fix k so that $g \in U(f_k)$. Let $a < b$ be the points used in the definition of $U(f_k)$.

If $\frac{1}{2} \in f_k[K]$, then $g(a) > \frac{1}{4}$ and $g(b) < \frac{3}{4}$. Since g is increasing, we conclude that $p < a \leqslant b < q$. Finally,

$$[a, b] = f^{-1}\left(\frac{1}{2}\right),$$

therefore $f_k(p) < \frac{1}{2}$ and $f_k(q) > \frac{1}{2}$.

If $\frac{1}{2} \notin f_k[K]$, then $g(a) < \frac{1}{2} < g(b)$ and hence $p \leqslant a < b \leqslant q$ because there are no points between a and b. Finally,

$$f_k(p) < \frac{1}{2} < f_k(q).$$

In both cases $f_k(p) < f_k(q)$, which completes the proof. □

The above proof trivializes when K is 0-dimensional. Namely, in that case the set of all continuous increasing maps from K into $\{0, 1\}$ is closed and discrete in $C_p(K)$. On the other hand, the cardinality of this set equals the weight of K. Thus the Lindelöf number of $C_p(K)$ equals the weight of K. The last fact holds for arbitrary compact lines, which was actually stated and proved by Nakhmanson in [462]. In

fact, our proof can be easily adapted to show the general situation. We have decided to present Nakhmanson's theorem in the special case of Lindelöf number $\aleph_0$, since we shall not use the general version.

23.3 Separable Complementation

We have already seen a negative result concerning the existence of averaging operators for increasing quotients of compact lines, see Theorem 22.4.5. It turns out that looking at the Lindelöf property of the weak topology one can find an example of a $C(K)$ space, with K a compact line, which fails the separable complementation property.

We need to recall the notion of a *double arrow line*. Fix a set $A \subseteq (0, 1)$, usually it is supposed to be dense and uncountable. Define $\mathbb{D}(A)$ to be the set

$$(\mathbb{I} \times \{0\}) \cup (A \times \{1\}),$$

endowed with the lexicographic order. The space $\mathbb{D}(A)$ is easily seen to be compact. It is zero-dimensional whenever A is dense in $(0, 1)$. It is homeomorphic (even order isomorphic) to the Cantor set exactly in case where A is countable and dense in $(0, 1)$. Otherwise it is not second countable, called a *double arrow* line/space (induced by A). Double arrow space is sometimes called the *split interval*, especially in case where $A = (0, 1)$,

Given a double arrow $K = \mathbb{D}(A)$, the space $C(K)$ has a copy of $C(\mathbb{I})$ induced by the obvious quotient map $q\colon K \to \mathbb{I}$, where $q(x, i) = x$ for $\langle x, i\rangle \in K$. We shall say that $q^*C(\mathbb{I})$ is the *natural copy* of $C(\mathbb{I})$ in $C(K)$. The following result originates in Corson [147]. We actually repeat some of the arguments from the proof of Theorem 8.1.5, using a slightly different language.

Theorem 23.3.1 *Let $A \subseteq (0, 1)$ be uncountable and let $K = \mathbb{D}(A)$ be the double arrow induced by A. The natural copy of $C(\mathbb{I})$ in the space $C(K)$ is not contained in any separable complemented space. In particular, $C(K)$ fails to have the separable complementation property.*

Note that Theorem 22.4.5 already implies that the natural copy of $C(\mathbb{I})$ is not complemented in $C(K)$, even without assuming that A be uncountable.

Proof We first need to identify the space $C(K)/C(\mathbb{I})$. Namely, we claim that it is isomorphic to $c_0(A')$, where $A' = \{0\} \cup A$. Given $a \in A$, define $a^- = \langle a, 0\rangle$ and $a^+ = \langle a, 1\rangle$. Given $f \in C(K)$, define

$$(Tf)(a) = f(a^+) - f(a^-).$$

Finally, let $(Tf)(0) = f(0)$. Observe that $Tf \in c_0(A')$. Indeed, given $\varepsilon > 0$, there is a finite cover $\mathcal{U}$ of K consisting of clopen intervals such that the oscillation of f on each member of $\mathcal{U}$ is $< \varepsilon$. Now if

$$\{a^-, a^+\} \subseteq U$$

for some $U \in \mathcal{U}$, then $|(Tf)(a)| < \varepsilon$. It follows that $T : C(K) \to c_0(A')$. It is clear that T is linear and $\|T\| \leqslant 2$. Further, $f \in \ker T$ if and only if f is constant on each set of the form $\{a^-, a^+\}$, where $a \in A$. This means that $f = f' \circ q$, where $q : K \to \mathbb{I}$ is the canonical quotient, i.e., f belongs to the natural copy of $C(\mathbb{I})$.

On the other hand, given $v \in c_0(A')$ let f be the map on K defined by

$$f(t) = \sum_{a \in A',\, a^+ \leqslant t} v(a).$$

It is easy to check that f is continuous, finite-valued, and $Tf = v$. Thus T is a surjection.

Now suppose that

$$P : C(K) \to C(K)$$

is a bounded projection such that $\operatorname{im} P$ is separable and $\ker T \subseteq \operatorname{im} P$. Then

$$C(K) = \operatorname{im} P \oplus Y,$$

where $Y = \ker P$. Then $T \restriction Y$ is an isomorphic embedding. This gives a contradiction since Y, being isomorphic to a subspace of $c_0(A')$ is weakly Lindelöf determined and consequently $C(K)$ would be weakly Lindelöf, contrary to Nakhmanson's Theorem 23.2.1. □

In the original work of Corson [147], as well as in the proof of Theorem 8.1.5, the space of all bounded right-continuous functions which have finite left limit at each point has been considered. This space can be naturally identified with the space of all continuous functions on the split interval.

We now turn to a positive result on separable complementation. Recall that a linear operator $T : C(K) \to C(K)$ is *regular* if $T1 = 1$ and $Tf \geqslant 0$ whenever $f \geqslant 0$. Notice that a regular operator has necessarily norm $\leqslant 1$.

We shall say that $C(K)$ has the *regular separable complementation property* (briefly: *regular SCP*) if for every separable subset A there exists a regular projection onto a separable subspace containing A.

Theorem 23.3.2 ([380]) *Let K be a compact line in which all separable subsets are second countable. Then $C(K)$ has the regular separable complementation property.*

Before proving this theorem, we need to develop some properties of continuous functions on compact lines. Let K be a compact line and let $f : K \to \mathbb{R}$ be a function. We say that $p \in K$ is *irrelevant* for f if one the following conditions holds.

(1) $p = 0_K$ and $f \restriction [p, b]$ is constant for some $b > p$,
(2) $p = 1_K$ and $f \restriction [a, p]$ is constant for some $a < p$,
(3) $0_K < p < 1_K$ and $f \restriction [a, b]$ is constant for some $a < p < b$.

Otherwise, p will be called *essential* for f. We shall denote by $\mathrm{ess}(f)$ the set of all essential points of f.

Lemma 23.3.3 *Let $f : K \to \mathbb{I}$ be a continuous increasing function, where K is a compact line. Then* $\mathrm{ess}(f)$ *is separable.*

Proof Let $Y = \mathrm{ess}(f)$ and let

$$Z = f[Y] = f[K].$$

Observe that $f \restriction Y$ is an increasing two-to-one map onto a second countable linearly ordered space $Z \subseteq \mathbb{R}$. Fix a countable dense set $D \subseteq f[Y]$ which contains all external points of $f[Y]$. Then

$$D' = Y \cap f^{-1}[D]$$

is countable. We claim that D' is dense in Y. For fix a non-empty open interval $U \subseteq Y$.

If $f \restriction U$ is not constant, there are $x < y$ in U with $f(x) < f(y)$. If $(f(x), f(y))$ is non-empty, it contains some $r \in D$. Thus

$$f^{-1}(r) \subseteq D' \cap U,$$

therefore $D' \cap U \neq \emptyset$. If $(f(x), f(y))$ is empty, then $f(y) \in D$; hence $y \in D' \cap U$.

If $f \restriction U$ is constant, then $U = \{x, y\}$ with $x \leqslant y$. If $y = 1_Y$, then $y \in D'$; if $x = 0_Y$ then $x \in D'$. Suppose that $0_Y < x$ and $y < 1_Y$. Then $x \in \mathbb{L}(Y)$ and $y = p^-$ for some $p \in \mathbb{L}(Y)$. If $x < y$ then $f(y) < f(p)$ and hence $y \in D'$. If $x = y$ then either $f(x^-) < f(x)$ or $f(x) = f(y) < f(p)$. In both cases $x \in D'$. □

Let X be a closed subset of a compact space K. An *extension operator* is a linear operator $T : C(X) \to C(K)$ satisfying $(Tf) \restriction X = f$ for every $f \in C(X)$.

Lemma 23.3.4 *Let X be a closed subset of a compact line K. Then there exists a regular extension operator $T : C(X) \to C(K)$ such that every $f \in C(K)$ satisfying* $\mathrm{ess}(f) \subseteq X$ *belongs to the range of T.*

Proof For each $a \in \mathbb{L}(X)$ choose a continuous increasing function

$$h_a \colon \left[a^-, a\right] \to \mathbb{I}$$

such that $h_a(a^-) = 0$ and $h_a(a) = 1$. Define

$$(Tf)(p) = \begin{cases} f(p) & p \in X, \\ f(0_X) & p < 0_X, \\ f(1_X) & p > 1_X, \\ (1 - h_a(p))\, f(a^-) + h_a(p)\, f(a) & p \in \left(a^-, a\right) \text{ for some } a \in \mathbb{L}(X). \end{cases}$$

It is straight to check that $Tf \in C(K)$ for every $f \in C(X)$. Further, $(Tf) \restriction X = f$, $T1 = 1$ and T is positive, therefore it is a regular extension operator. Note that Tf is constant both on

$$[0_K, 0_X]$$

and

$$[1_X, 1_K]\,.$$

Finally, if $\operatorname{ess}(f) \subseteq X$ then $T(f \restriction X)$ is constant on each interval of the form $\left[a^-, a\right]$ where

$$a \in \mathbb{L}(X) \setminus \mathbb{L}(K)\,,$$

therefore $f = T(f \restriction X)$. This shows that f is contained in the range of T. □

Proof of Theorem 23.3.2 Fix a separable set $A \subseteq C(K)$. By Proposition 23.1.3, there is a countabe set $F \subseteq C(K)$ consisting of increasing functions, such that A is contained in the closed linear span of F. Let X be the closure of

$$\bigcup_{f \in F} \operatorname{ess}(f).$$

Notice that

$$\{g \in C(K) : \operatorname{ess}(g) \subseteq X\}$$

is a closed linear subspace of $C(K)$. Thus, $\operatorname{ess}(g) \subseteq X$ for every $g \in A$. By Lemma 23.3.3, X is separable. By our asumption, X is second countable and hence $C(X)$ is a separabe Banach space. Let

$$T : C(X) \to C(K)$$

be a regular extension operator satisfying the condition from Lemma 23.3.4. In particular, $T(f \restriction X) = f$ for every $f \in F$. Finally, the composition of T with the restriction operator $f \mapsto f \restriction X$ is a regular projection onto the range of T, isomorphic to the separable space $C(X)$. □

One can prove more here: under the assumptions of Theorem 23.3.2, the Banach space $C(K)$ has the controlled separable projection property (CSPP). Indeed, it suffices to observe that the support of every regular Borel measure on a compact line is separable and if T is a regular extension operator for $X \subseteq K$, then T' fixes all measures whose support is contained in X.

Recalling that $\omega_1 + 1$ satisfies the assumptions of Theorem 23.3.2, we obtain another proof of Corollary 12.5.10.

An *Aronszajn continuum* is by definition, a non-metrizable first countable compact line K in which all separable subsets are second countable. Necessarily, K has weight $\aleph_1$ and consequently $C(K)$ has density $\aleph_1$. For more information concerning Aronszajn lines, we refer to [600].

By a standard argument, passing to a quotient we may assume that K is nowhere separable. More precisely, define a relation $x \sim y$ if and only if $[x, y]$ is separable (in particular second countable). Then $\sim$ is an equivalence relation with closed convex classes. The quotient $K/\sim$ is a nowhere separable Aronszajn continuum.

Theorem 23.3.5 *Let K be a nowhere separable Aronszajn continuum. Then $C(K)$ has the regular SCP, yet it is not a Plichko space.*

Proof The first part follows from Theorem 23.3.2. In order to show that $C(K)$ is not Plichko, fix a big enough regular cardinal θ and fix a countable $M \preceq H(\theta)$ such that $K \in M$. Then K/M is a second countable compact line. Let $q\colon K \to K/M$ be the canonical quotient map. Clearly, the set

$$\{p \in K/M\colon |q^{-1}(p)| > 1\}$$

is everywhere dense, because otherwise there would exist a non-trivial interval $[a, b] \subseteq K/M$ on which q^{-1} would be one-to-one; consequently

$$\left[q^{-1}(a), q^{-1}(b)\right]$$

would be a non-trivial second countable interval in K.

By Theorem 22.4.5, q does not admit any averaging operator, therefore $\overline{(C(K) \cap M)}$ is not complemented in $C(K)$. This shows that $C(K)$ does not have a projectional skeleton, therefore it is not Plichko. □

Chapter 24
Compact Spaces Generated by Retractions

Abstract This chapter presents several classes of non-metrizable compact spaces that correspond to well-known classes of Banach spaces with many projections. In particular, we discuss the class of Valdivia compact spaces and its subclasses: Corson and Eberlein compact spaces. We discuss a general class of compact spaces obtained by limits of continuous retractive sequences. We also introduce the notion of a retractional skeleton, dual to projectional skeletons in Banach spaces. The last section contains an overview of Eberlein compact spaces with some classical results and examples relevant to the subjects of previous chapters.

24.1 Retractive Inverse Systems

Recall that a *retraction* in a topological space X is a continuous map $r\colon X \to X$ such that $r \circ r = r$. Equivalently: $r \restriction r[X] = \mathrm{id}_X$. Related to a retraction is the notion of a *right-invertible* map: $f\colon X \to Y$ is right-invertible if there is a map $j\colon Y \to X$ such that $r \circ j = \mathrm{id}_Y$. In fact, if $f \circ j = \mathrm{id}_Y$ then $j \circ f$ is a retraction and if $f\colon X \to X$ is a retraction, then the same map f treated as a surjection $f\colon X \to f[X]$ is right-invertible. Abusing notation, we shall sometimes say that f is a retraction if it is right-invertible and we shall sometimes say that f is an *internal retraction* to stress out that its range is a subset of its domain and its right inverse is inclusion.

Inverse systems/sequences provide an important tool for investigating non-metrizable compact spaces. We shall be mostly interested in retractive inverse systems which are defined below.

Recall that an *inverse system* of topological spaces is a pair of the form

$$\langle \{X_s\}_{s\in\Sigma}, \{p^t_s\}_{s,t\in\Sigma,\ s\leqslant t}\rangle,$$

where Σ is an up-directed partially ordered set, each X_s is a topological space and each p^t_s for $s \leqslant t$ is a map of X_t into X_s (called a *bonding map*) and the following conditions are satisfied.

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(1) $p_s^s = \mathrm{id}_{X_s}$,
(2) $p_s^r = p_s^t \circ p_t^r$ whenever $s \leqslant t \leqslant r$.

We shall denote this inverse system shortly by $\langle X_s; p_s^t; \Sigma\rangle$.

In the language of category theory, an inverse system of topological spaces is just a contravariant functor from a directed poset category into the category of topological spaces.

Given an inverse system $\mathbb{S} = \langle X_s; p_s^t; \Sigma\rangle$, its limit will be denoted, as usual, by $\varprojlim \mathbb{S}$. Formally, $\varprojlim \mathbb{S}$ consists of a topological space X and a family of maps

$$\{p_s : X \to X_s\}_{s\in\Sigma}$$

called *projections*, satisfying the following condition:

(3) Given a space Y and a family $\{f_s : Y \to X_s\}_{s\in\Sigma}$, there exists a unique map $h : Y \to X$ such that $p_s \circ h = f_s$ for every $s \in \Sigma$.

The space X can be represented as the subspace of the product $\prod_{s\in\Sigma} X_s$ consisting of all functions x satisfying $x(s) = p_s^t(x(t))$ for every $s < t$. The projections are defined by $p_s(x) = x(s)$. We shall abuse the notation and have in mind the space X itself as the limit of $\mathbb{S}$.

An *inverse sequence* is an inverse system whose index set Σ is an ordinal. An inverse sequence $\mathbb{S} = \langle X_\alpha; p_\alpha^\beta; \lambda\rangle$ is *continuous* if for each limit ordinal $\delta < \lambda$ the space X_δ together with the collection

$$\{p_\alpha^\delta\}_{\alpha<\delta}$$

is the limit of

$$\mathbb{S} \restriction \delta := \langle X_\alpha; p_\alpha^\beta; \delta\rangle.$$

An inverse system $\mathbb{S} = \langle X_s; r_s^t; \Sigma\rangle$ is *retractive* if there is a family of maps

$$\{i_s^t\}_{s,t\in\Sigma,\ s\leqslant t}$$

such that $i_s^t : X_s \to X_t$ and the following conditions are satisfied:

(4) $r_s^t \circ i_s^t = \mathrm{id}_{X_s}$ for every $s \leqslant t$,
(5) $i_t^u \circ i_s^t = i_s^u$ whenever $s \leqslant t \leqslant u$.

Since $r_s^s = \mathrm{id}_{X_s}$, condition (3) implies that $i_s^s = \mathrm{id}_{X_s}$. This and condition (4) imply that $\langle X_t; i_t^s; \Sigma\rangle$ is an inductive system. The collection

$$\{i_s^t\}_{s,t\in\Sigma,\ s\leqslant t}$$

will be called a *right inverse* of $\mathbb{S}$.

Lemma 24.1.1 *Let $\mathbb{S} = \langle X_s; r_s^t; \Sigma\rangle$ be a retractive system of topological spaces and let $X = \varprojlim \mathbb{S}$. Then there are maps $i_s\colon X_s \to X$ $(s \in \Sigma)$ such that $r_s \circ i_s = \mathrm{id}_{X_s}$ for every $s \in \Sigma$, where $r_s\colon X \to X_s$ is the projection.*

Proof Fix $u \in \Sigma$. We shall use the fact that $X = \varprojlim(\mathbb{S} \restriction \Sigma')$, where $\Sigma' = \{t \in \Sigma\colon u \leqslant t\}$. The family $\{i_u^s\}_{s\in\Sigma'}$ satisfies

$$r_s^t \circ i_u^t = r_s^t \circ i_s^t \circ i_u^s = i_u^s$$

for every $u < s < t$. Thus, by the definition of the limit, there exists a unique map $i_u\colon X_u \to X$ satisfying $r_u \circ i_u = \mathrm{id}_{X_u}$. □

Lemma 24.1.2 *Let λ be an ordinal, and let $\mathbb{S} = \langle X_s; r_s^t; \lambda\rangle$ be a continuous inverse sequence of topological spaces such that each $r_\alpha^{\alpha+1}$ is a retraction. Then $\mathbb{S}$ is retractive.*

Proof We use induction. Assume $\beta < \lambda$, and we have constructed appropriate maps

$$i_\xi^\eta\colon X_\xi \to X_\eta$$

$(\xi < \eta < \beta)$ satisfying conditions (4), (5).

If β is a limit ordinal, use Lemma 24.1.1 and the fact that $X_\beta = \varprojlim(\mathbb{S} \restriction \beta)$.

Finally, suppose that $\beta = \alpha + 1$ and choose a right inverse k for r_α^β. Define

$$i_\xi^\beta = k \circ i_\xi^\alpha.$$

Then

$$r_\xi^\beta \circ i_\xi^\beta = r_\xi^\alpha \circ r_\alpha^\beta \circ k \circ i_\xi^\alpha = \mathrm{id}_{X_\xi}$$

and given $\xi < \eta < \beta$ we have

$$i_\eta^\beta \circ i_\xi^\eta = k \circ i_\eta^\alpha \circ i_\xi^\eta = k \circ i_\xi^\alpha = i_\xi^\beta.$$

This completes the proof. □

It turns out that every totally disconnected compact space is the limit of a retractive system. This is a consequence of the following fact.

Proposition 24.1.3 *Assume $\mathbb{S} = \langle X_s; r_s^t; \Sigma\rangle$ is an inverse system of finite metric spaces such that each bonding map r_s^t is a surjection. Then $\mathbb{S}$ has a right inverse.*

Proof Fix a well ordering $\preceq$ on $X = \varprojlim \mathbb{S}$. Each projection $r_s\colon X \to X_s$ induces a partition of X into clopen sets. Let $i_s(r_s(x))$ be defined as the $\preceq$-minimal element of $r_s^{-1}r_s(x)$. If $s \leqslant t$ then the partition induced by r_t refines the one induced by r_s. More precisely,

$$r_t^{-1}r_t(x) \subseteq r_s^{-1}r_s(x),$$

whenever $s \leqslant t$ and $x \in X$. Thus, setting $i_s^t(p) = r_t(i_s(x))$ we obtain a map

$$i_s^t \colon X_t \to X_s$$

which is a right inverse of r_s^t. Finally, if $s \leqslant t \leqslant u$, then $i_s^u = i_t^u i_s^t$. □

Recall that spaces of the form $\varprojlim \mathbb{S}$, where $\mathbb{S}$ is as above, form precisely the class of totally disconnected compact spaces.

Sometimes we shall need to represent a retractive inverse system "inside" a compact space as a certain collection of internal retractions. The way how to do it is explained in the following two lemmas [385].

Lemma 24.1.4 *Let $\mathbb{S} = \langle X_s; r_s^t; \Sigma\rangle$ be an inverse system of topological spaces with a right inverse $\{i_s^t\}_{s,t\in\Sigma\ s<t}$ and let $X = \varprojlim \mathbb{S}$, $R_s = i_s \circ r_s$. Then*

(1) $s \leqslant t \Rightarrow R_s \circ R_t = R_s = R_t \circ R_s$.
(2) $x = \lim_{s\in\Sigma} R_s(x)$ for every $x \in X$.

Proof Property (1) is clear. Fix a neighbourhood u of x. We can assume that $u = (r_s)^{-1}[v]$ for some $s \in \Sigma$ and an open set $v \subseteq X_s$. Fix $t \geqslant s$. Then

$$r_s(R_t(x)) = r_s(i_t \circ r_t(x)) = r_s^t \circ r_t \circ i_t \circ r_t(x) = r_s^t \circ r_t(x) = r_s(x) \in v.$$

Thus, $R_t(x) \in u$ for every $t \geqslant s$. This shows (2). □

Lemma 24.1.5 *Assume $\{R_s : s \in \Sigma\}$ is a family of internal retractions of a compact space K such that Σ is a directed partially ordered set and conditions (1), (2) of Lemma 24.1.4 hold. Then $K = \varprojlim\langle K_s; R_s^t; \Sigma\rangle$, where $K_s = R_s[K]$ and $R_s^t = R_s \restriction K_t$.*

Proof If $r < s < t$ then

$$R_r^s \circ R_s^t = (R_r \restriction K_s) \circ (R_s \restriction K_t) = (R_r \circ R_s) \restriction K_t = R_r \restriction K_t = R_r^t,$$

thus $\mathbb{S} = \langle K_s; R_s^t; \Sigma\rangle$ is indeed an inverse system and if $s < t$ then $R_s^t[K_t] = K_s$, because $K_s \subseteq K_t$ by condition (1). Let

$$R_s' : \varprojlim \mathbb{S} \to K_s$$

denote the projection. The collection

$$\{R_s : s \in \Sigma\}$$

is compatible with $\mathbb{S}$ which determines a unique continuous map

$$f \colon K \to \varprojlim \mathbb{S}$$

such that

$$R'_s \circ f = R_s$$

for every $s \in \Sigma$. Condition (2) implies that f is one-to-one. It remains to show that $f[K] = \varprojlim \mathbb{S}$. Fix $y \in \varprojlim \mathbb{S}$ and choose

$$x \in \bigcap_{t\in\Sigma} \overline{\{R'_s(y) : s \geqslant t\}}.$$

The above set is non-empty by the compactness of K and by the directedness of Σ. Observe that if $t < s$, then

$$R_t \circ R'_s(y) = R^s_t \circ R'_s(y) = R'_t(y),$$

whence

$$R_t(x) \in \overline{R_t[\{R'_s(y) : s \geqslant t\}]} = \{R'_t(y)\}.$$

Thus

$$R'_t(f(x)) = R_t(x) = R'_t(y),$$

which means that $f(x) = y$. □

We shall need later the following simple factorization property of σ-directed retractive inverse systems.

Lemma 24.1.6 *Let $\{r_s\}_{s\in\Gamma}$ be an inverse system of internal retractions of a compact space K (in the sense of Lemma 24.1.5) such that K is its limit and Γ is a σ-directed poset. Then for every continuous map $f : K \to L$ with L second countable, there exists $t \in \Gamma$ such that*

$$f = f \circ r_t.$$

Proof Fix a big enough cardinal θ and fix a countable $M \preceq H(\theta)$ so that $f \in M$ and $\{r_s\}_{s\in\Gamma} \in M$. By elementarity, M "knows" a fixed countable base of L. Let $t \in \Gamma$ be such that $s < t$ for every $s \in \Gamma \cap M$. We claim that $f = f \circ r_t$. For the proof, we must show that

$$r_t(x) = r_t(y) \implies f(x) = f(y)$$

for every $x, y \in K$. Fix x, y so that $f(x) \neq f(y)$. Find disjoint basic open sets $U, V \subseteq L$ which are elements of M and such that $f(x) \in U$, $f(y) \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. Observe that

$$(\exists\, s \in \Gamma)\ r_s[U] \cap r_s[V] = \emptyset. \tag{24.1}$$

Indeed, otherwise for each $s \in \Gamma$ choose $p_s \in r_s[U] \cap r_s[V]$; by compactness find an accumulation point p of the net $\{p_s\}_{s\in\Gamma}$. Knowing that $p = \lim_{s\in\Gamma} r_s(p)$, we deduce that $p \in \overline{U} \cap \overline{V}$, a contradiction.

Now observe that all parameters of the formula (24.1) are in M. Thus, by elementarity, there is $s \in \Gamma \cap M$ satisfying (24.1). In particular, $r_s(x) \neq r_s(y)$. Finally, $r_t(x) \neq r_t(y)$, because $s < t$, and consequently, $r_s = r_s \circ r_t$. □

24.2 Monolithic Sets

A subset D of a topological space X is *monolithic* in X if for every countable $A \subseteq D$ the space $\overline{A}$ (closure in X) is second countable and is contained in D. This notion was introduced by Arkhangel'skii, see [32], when studying dyadic compact spaces. In the literature "monolithic set" is usually called "ω-monolithic" or "$\aleph_0$-monolithic". It is obvious how to generalize this notion for larger cardinals in place of $\aleph_0$. Also, often in the above definition it is asserted that $\overline{A}$ must have a countable network instead of countable weight. Since we are going to use this notion for compact spaces only, there will be no confusion.

Denote by $\mathcal{M}$ the class of all compact spaces that have a dense monolithic set.

Proposition 24.2.1 *Class $\mathcal{M}$ is closed under arbitrary products, finite sums, and continuous images.*

Proof It is clear that $\mathcal{M}$ is closed under finite sums. Let

$$\{K_s : s \in S\}$$

be a collection of spaces in $\mathcal{M}$ and let $K = \prod_{s\in S} K_s$. For each $s \in S$ choose a dense monolithic set $D_s \subseteq K_s$. Also, fix $d_s \in D_s$. Define

$$D = \{x \in \prod_{s\in S} D_s : |\{s \in S : x(s) \neq d_s\}| \leqslant \aleph_0\}.$$

Then D is dense in K, and it is easy to see that it is monolithic.

Now assume $K \in \mathcal{M}$ and $f : K \to Y$ is a surjection. Then $f[D]$ is monolithic whenever $D \subseteq K$ is monolithic. Thus, $\mathcal{M}$ is closed under continuous images. □

Proposition 24.2.2 *Let $K \in \mathcal{M}$ and let $E \subseteq K$ be a countable dense set such that $t(p, K) \leqslant \aleph_0$ for every $p \in E$. Then K is second countable.*

Proof Let $D \subseteq K$ be dense and monolithic. Since each $p \in E$ is in the closure of $\leqslant \aleph_0$ many elements of D, we have $E \subseteq D$. Thus $w(K) = w(\overline{E}) \leqslant \aleph_0$. □

Corollary 24.2.3 *If $\beta X \in \mathcal{M}$ then X is pseudocompact.*

Proof It is well known that a non-psuedocompact space admits a continuous map onto a dense subset of $\mathbb{R}$, which extends to a continuous map from its Čech–Stone

compactification onto $\beta\mathbb{R}$. Thus it is enough to show that $\beta\mathbb{R} \notin \mathcal{M}$. This follows from Proposition 24.2.2, because $w(\beta\mathbb{R}) = 2^{\aleph_0}$ and $\mathbb{Q}$ is a dense countable subset of $\beta\mathbb{R}$ which consists of G_δ points. □

Finally we note the following

Corollary 24.2.4 *Let $K \in \mathcal{M}$ be a compactification of $\mathbb{N}$. Then K is metrizable.*

24.3 The Classes $\mathcal{R}$ and $\mathcal{RC}$

In this section, we discuss a general class of compact spaces obtained by limits of continuous retractive sequences. We also introduce the notion of a retractional skeleton, dual to projectional skeletons in Banach spaces.

Class $\mathcal{R}$, introduced in [108], is defined to be the smallest class of compact spaces which contains all metrizable ones and which is closed under limits of continuous retractive inverse sequences.

There is a natural hierarchy on the class $\mathcal{R}$. Namely, given a compact space K, define $\mathrm{rk}_{\mathcal{R}}(K) = 0$ if and only if K is a metric compact. Further, given a positive ordinal β, let $\mathrm{rk}_{\mathcal{R}}(K) \leqslant \beta$ if $K = \varprojlim \mathbb{S}$, where $\mathbb{S} = \langle K_i; r_i^j; \lambda\rangle$ is a continuous retractive sequence such that $\mathrm{rk}_{\mathcal{R}}(K_i) < \beta$ for every $i < \lambda$. Finally, define $\mathrm{rk}_{\mathcal{R}}(K) = \beta$ if and only if $\mathrm{rk}_{\mathcal{R}}(K) \leqslant \beta$ and $\mathrm{rk}_{\mathcal{R}}(K) \not\leqslant \alpha$ for any $\alpha < \beta$. Set $\mathrm{rk}_{\mathcal{R}}(K) = \infty$ if and only if for no ordinal β we have that $\mathrm{rk}_{\mathcal{R}}(K) \leqslant \beta$. It is clear that

$$\mathcal{R} = \{K\colon\ \mathrm{rk}_{\mathcal{R}}(K) < \infty\}.$$

Some properties of class $\mathcal{R}$. are valid for continuous images and even for a larger class defined in the similar spirit, adding continuous images. Namely, we define class $\mathcal{RC}$ as the smallest class which contains all metric compact spaces and which is closed under continuous images and inverse limits of transfinite sequences of retractions.

Class $\mathcal{RC}$ can also be defined inductively as $\mathcal{RC} = \bigcup_{\xi\in\mathfrak{Ord}} \mathcal{RC}^\xi$, where $\mathcal{RC}^0$ is the class of all metric compact spaces and $\mathcal{RC}^\beta$ consists of all compact spaces $K = f[\varprojlim \mathbb{S}]$, where

$$\mathbb{S} = \langle K_\alpha; r_\alpha^\beta; \kappa\rangle$$

is a continuous retractive sequence and each K_α belongs to

$$\bigcup_{\xi<\beta} \mathcal{RC}^\xi.$$

Given $K \in \mathcal{RC}$ denote by $\mathrm{rk}_{\mathcal{RC}}(K)$ the minimal β such that $K \in \mathcal{RC}^{\beta}$. We call $\mathrm{rk}_{\mathcal{RC}}(K)$ the *$\mathcal{RC}$-rank of K*. We shall write $\mathrm{rk}_{\mathcal{RC}}(K) = \infty$ if $K \notin \mathcal{RC}$. It is clear that

$$\mathrm{rk}_{\mathcal{RC}}(K) \leqslant \mathrm{rk}_{\mathcal{R}}(K)$$

for every compact K.

Below is a simple statement on convergent sequences. The proof shows that the above rank together with transfinite induction is very useful. Later on we shall prove a stronger statement concerning embeddability of the ordinal segment $[0, \omega_1] = \omega_1 + 1$.

Proposition 24.3.1 *Every infinite member of class $\mathcal{RC}$ contains a non-trivial convergent sequence.*

Proof The statement is true if $\mathrm{rk}_{\mathcal{RC}}(K) = 0$. Suppose $\beta > 0$ and the statement is true for spaces of $\mathcal{RC}$-rank $< \beta$ and fix $K \in \mathcal{RC}$ with $\mathrm{rk}_{\mathcal{RC}}(K) = \beta$. So $K = f[Y]$, where $Y = \varprojlim \mathbb{S}$ and

$$\mathbb{S} = \langle Y_\xi; r^\eta_\xi; \kappa \rangle$$

is a continuous inverse sequence of retractions such that $\mathrm{rk}_{\mathcal{RC}}(Y_\xi) < \beta$ for every $\xi < \kappa$. If some

$$f[Y_\xi]$$

is infinite then by induction hypothesis it contains a non-trivial convergent sequence, so assume that each $f[Y_\xi]$ is finite. Thus,

$$\{f[Y_\xi] \colon \xi < \kappa\}$$

is an increasing sequence of finite subsets of K. This sequence does not stabilize, so $\mathrm{cf}(\kappa) = \aleph_0$. We may assume that $\kappa = \aleph_0$. Now observe that

$$K \neq \bigcup_{n \in \omega} f[Y_n]$$

because otherwise K would be metrizable, and we would have $\mathrm{rk}_{\mathcal{RC}}(K) = 0 < \beta$. Fix $x = f(y) \in K$ such that $x \notin f[Y_n]$ for any $n \in \omega$. By the continuity of f and by Lemma 24.1.4(b) we have

$$x = \lim_{n \to \infty} f(r_n(y)).$$

Also $f(r_n(y)) \neq x$ for every $n \in \omega$. Thus, K contains a non-trivial convergent sequence. □

24.4 Stability

It is easy to see that the classes $\mathcal{R}$ and $\mathcal{RC}$ are stable under typical functors. Further, we prove a more precise statement involving rank.

Recall that a functor F defined on the category of compact spaces is *continuous* if it preserves limits of arbitrary inverse sequences. Typical examples are: the probability measures functor P, the superextension functor λ, the Vietoris hyperspace functor exp. Another example is the functor which assigns to K the dual unit ball $B_{(C(K))'}$ with the *weak** topology. For the next lemma see [379].

Lemma 24.4.1 (Kubiś) *Let F be a continuous covariant functor on compact spaces and assume δ is an ordinal such that $\mathrm{rk}_{\mathcal{R}}(F(K)) \leqslant \delta$ for every compact metric space K. Then*

$$\mathrm{rk}_{\mathcal{R}}(F(K)) \leqslant \delta + \mathrm{rk}_{\mathcal{R}}(K)$$

for every compact space K.

Proof We use induction on $\alpha = \mathrm{rk}_{\mathcal{R}}(K)$. The statement is true for $\alpha = 0$. Assume $K = \varprojlim \mathbb{S}$, where $\mathbb{S} = \langle K_\xi; r^\eta_\xi; \kappa\rangle$ is a continuous retractive sequence such that $\mathrm{rk}_{\mathcal{R}}(K_\xi) < \alpha$ for every $\xi < \kappa$. By induction hypothesis,

$$\mathrm{rk}_{\mathcal{R}}(F(K_\xi)) \leqslant \delta + \mathrm{rk}_{\mathcal{R}}(K_\xi) < \delta + \alpha.$$

Here we use the fact that $\delta + \xi < \delta + \alpha$ whenever $\xi < \alpha$ (it may happen that

$$\xi + \delta = \alpha + \delta$$

for $\xi < \alpha$). Notice that

$$F[\mathbb{S}] = \langle F(K_\xi), F(r^\eta_\xi), \kappa\rangle$$

is again a continuous retractive sequence with limit $F(K)$. Thus $\mathrm{rk}_{\mathcal{R}}(F(K)) \leqslant \delta + \alpha$. □

Corollary 24.4.2 *Let F be a continuous covariant functor on compact spaces such that $F(X) \in \mathcal{R}$ for every compact metric X. Then $F(K) \in \mathcal{R}$ for every $K \in \mathcal{R}$.*

Proof There are continuum many homeomorphic types of metric compact spaces, therefore there is an ordinal δ greater than all ordinals $\mathrm{rk}_{\mathcal{R}}(F(X))$ where X is compact metric. Thus, Lemma 24.4.1 applies. □

Corollary 24.4.3 *Let $K \in \mathcal{R}$. Then the spaces $P(K), \lambda(K), \exp(K), B_{C(K)'}$ belong to $\mathcal{R}$ and have the $\mathcal{R}$-rank not greater than $\mathrm{rk}_{\mathcal{R}}(K)$.*

Similar results hold for the class $\mathcal{RC}$:

Corollary 24.4.4 *Let F be a continuous covariant functor on compact spaces which preserves surjections. Assume further that $\mathrm{rk}_{\mathcal{RC}}(F(K)) \leqslant \delta$ for every compact metric K, where δ is a fixed ordinal. Then*

$$\mathrm{rk}_{\mathcal{RC}}(F(K)) \leqslant \delta + \mathrm{rk}_{\mathcal{RC}}(K)$$

for every compact space K.

Proof Repeat the arguments of the proof of Lemma 24.4.1, using additionally the fact that F preserves surjections. □

Corollary 24.4.5 *Let F be a continuous covariant functor on compact spaces which preserves surjections and such that $F(X) \in \mathcal{RC}$ for every compact metric X. Then $F(K) \in \mathcal{RC}$ for every $K \in \mathcal{RC}$.*

Corollary 24.4.6 *Let $K \in \mathcal{RC}$ and let $L \in \{P(K), \lambda(K), \exp(K), B_{C(K)'}\}$. Then $L \in \mathcal{RC}$ and moreover $\mathrm{rk}_{\mathcal{RC}}(L) \leqslant \mathrm{rk}_{\mathcal{RC}}(K)$.*

Corollary 24.4.7 *Let X be a compact metric space. Then for every compact space K it holds that*

$$\mathrm{rk}_{\mathcal{R}}(K \oplus X) \leqslant \mathrm{rk}_{\mathcal{R}}(K), \quad \mathrm{rk}_{\mathcal{R}}(K \times X) \leqslant \mathrm{rk}_{\mathcal{R}}(K).$$

The same inequalities hold for $\mathrm{rk}_{\mathcal{RC}}$ as well.

Proof Let $F(K) = K \oplus X$ and $F(f) = f \oplus \mathrm{id}_X$ (that is, $F(f)(x) = f(x)$ for $x \in K$ and $F(f)(x) = x$ for $x \in X$). Further, let

$$G(K) = K \times X, \quad G(f) = f \times \mathrm{id}_X$$

(that is, $G(f)(p, x) = \langle f(p), x\rangle$ for $\langle p, x\rangle \in K \times X$). It is clear that F, G are continuous covariant functors which preserve surjections and metric compact spaces. Thus, the above inequalities follow from Lemmas 24.4.1 and 24.4.4, respectively. □

Theorem 24.4.8 ([379]) *The classes $\mathcal{R}$ and $\mathcal{RC}$ are stable under arbitrary products and one-point compactifications of disjoint sums.*

Proof An infinite product is the limit of a continuous retractive sequence of smaller products; therefore, it suffices to show that

$$K \times L \in \mathcal{R}$$

($K \times L \in \mathcal{RC}$) whenever $K, L \in \mathcal{R}$ (respectively $K, L \in \mathcal{RC}$). Fix L and define $F_L(K) = K \times L$. Then F_L is a continuous covariant functor and by Corollary 24.4.7 we have that $F_L(K) \in \mathcal{R}$ whenever $K \in \mathcal{R}$ and $F_L(K) \in \mathcal{RC}$ whenever $K \in \mathcal{RC}$. This shows the statement concerning products.

Let K be the one-point compactification of

$$\bigoplus_{\alpha<\kappa} K_\alpha$$

and assume $K_\alpha \in \mathcal{R}$ for each α. Define $r_\alpha\colon K \to K$ by $r_\alpha(x) = x$ for

$$x \in \bigoplus_{\xi<\alpha} K_\xi$$

and $r_\alpha(x) = \infty$ otherwise, where ∞ denotes the "point at infinity" that compactifies the disjoint sum. By Lemma 24.1.5, $\{r_\alpha\}_{\alpha<\kappa}$ induces a continuous retractive sequence of one-point compactifications of disjoint sums $\bigoplus_{\xi<\alpha} K_\xi$. Thus, by induction it is enough to show that $K \oplus L \in \mathcal{R}$ whenever $K, L \in \mathcal{R}$. However, this is proved by a similar argument as for products, using the functor $G_L(K) = K \oplus L$. Almost the same arguments can be applied to class $\mathcal{RC}$. □

The class $\mathcal{R}$ is obviously not stable under subspaces, because all Tikhonov cubes are in $\mathcal{R}$, while not all compact spaces are in $\mathcal{R}$. However, we show the following.

Theorem 24.4.9 *Let $K \in \mathcal{R}$, and let U be a clopen subset of K. Then $U \in \mathcal{R}$ and* $\mathrm{rk}_{\mathcal{R}}(U) \leqslant \mathrm{rk}_{\mathcal{R}}(K)$.

Proof Fix $K \in \mathcal{R}$ of rank β and suppose the statement is true for compact spaces of $\mathcal{R}$-rank $< \beta$. Fix a clopen set $U \subseteq K$ and let $K = \varprojlim \mathbb{S}$, where

$$\mathbb{S} = \langle K_\xi; r^\eta_\xi; \kappa\rangle$$

is a continuous retractive sequence and $\mathrm{rk}_{\mathcal{R}}(K_\xi) < \beta$ for every $\xi < \kappa$.

As V is clopen, there exist $\alpha < \kappa$ and a clopen set $V \subseteq K_\alpha$ such that $U = {r_\alpha}^{-1}[V]$. Cutting-off an initial part of $\mathbb{S}$, we may assume that $\alpha = 0$. Now let $U_\xi = {(r^\xi_0)}^{-1}[V]$ and given $\xi < \eta$ let $s^\eta_\xi\colon U_\eta \to U_\xi$ be the restriction of r^η_ξ. Then $\langle U_\xi; s^\eta_\xi; \kappa\rangle$ is a continuous retractive inverse sequence with limit U. By induction hypothesis, $\mathrm{rk}_{\mathcal{R}}(U_\xi) \leqslant \mathrm{rk}_{\mathcal{R}}(K_\xi)$ for every $\xi < \kappa$. Thus $\mathrm{rk}_{\mathcal{R}}(U) \leqslant \mathrm{rk}_{\mathcal{R}}(K)$. □

24.5 Some Examples

Further, we present few simple examples related to classes $\mathcal{R}$ and $\mathcal{RC}$. We also prove a result on cutting P-points.

Lemma 24.5.1 *Assume $K \in \mathcal{R}$ and $p \in K$. Let $\{p_n\}_{n\in\omega}$ be a one-to-one sequence disjoint from K and let $L = K \cup \{p_n\}_{n\in\omega}$ be such that each p_n is isolated and $p = \lim_{n\to\infty} p_n$. Then $L \in \mathcal{R}$ and* $\mathrm{rk}_{\mathcal{R}}(L) \leqslant \mathrm{rk}_{\mathcal{R}}(K) + 1$.

Proof Let

$$K_n = K \oplus \{p_0, \ldots, p_{n-1}\}$$

and define $r_n : L \to K_n$ by

$$r_n \restriction K_n = \mathrm{id}_K$$

and $r_n(p_i) = p$ for $i \geqslant n$. By Lemma 24.4.1 we have $\mathrm{rk}_{\mathcal{R}}(K_n) = \mathrm{rk}_{\mathcal{R}}(K)$. Observe that

$$r_n \circ r_m = r_m \circ r_n = r_n$$

for $n < m$ and

$$x = \lim_{n\to\infty} r_n(x)$$

for every $x \in L$ there is n_0 such that $x = r_n(x)$ for $n \geqslant n_0$. By Lemma 24.1.5 one gets that L is the limit of a retractive sequence induced by $\{r_n\}_{n\in\omega}$. Thus

$$\mathrm{rk}_{\mathcal{R}}(L) \leqslant \mathrm{rk}_{\mathcal{R}}(K) + 1.$$

□

By the above lemma, the linearly ordered space

$$K = \omega_1 + 1 + \omega^{-1},$$

where X^{-1} denotes the line X with reversed order, belongs to class $\mathcal{R}$. It turns out that the space

$$\omega_1 + 1 + \omega_1^{-1}$$

does not belong to class $\mathcal{R}$. Further, we prove a more general statement about it, involving P-points.

Recall that an element p of a topological space X is a *P-point* if it is not isolated and

$$p \in \mathrm{int} \bigcap_{n\in\omega} u_n$$

whenever $\{u_n\}_{n\in\omega}$ is a sequence of neighbourhoods of p. An element $p \in X$ is a *cutting P-point* if $X = A \cap B$ so that $\{p\} = A \cap B$ and p is a P-point both in A and B. A typical example of a cutting P-point is

$$\omega_1 \in \omega_1 + 1 + \omega_1^{-1}.$$

For the next result we refer to [379].

Theorem 24.5.2 (Kubiś) *No space in class $\mathcal{R}$ can contain cutting P-points.*

Proof Suppose otherwise and let $K \in \mathcal{R}$ be a space of minimal $\mathcal{R}$-rank that contains a cutting P-point p and let A, B witness this property. Using certain absoluteness arguments from logic, we may assume that $w(K) = \aleph_1$, see [379, proof of Theorem 4.5] for details.

Let

$$\{u_\alpha : \alpha < \omega_1\}$$

denote a basis of p in K such that

$$\alpha < \beta \Rightarrow \overline{u_\beta} \subseteq u_\alpha.$$

Such a sequence exists, because p is a P-point in K and $\chi(p, K) = \aleph_1$. The assumptions on A, B mean in particular that both sets $u_\alpha \setminus B, u_\alpha \setminus A$ are non-empty for every $\alpha < \omega_1$.

Now suppose

$$\{r_\alpha : \alpha < \kappa\}$$

is a sequence of internal retractions in K such that $K_\alpha = r_\alpha[K]$ has $\mathcal{R}$-rank $<$ $\mathrm{rk}_{\mathcal{R}}(K)$ for every $\alpha < \kappa$ and

$$K = \overline{(\bigcup_{\alpha<\kappa} K_\alpha)}.$$

Assuming κ is regular, we deduce that

$$\kappa \in \{\aleph_0, \aleph_1\}.$$

Suppose $\kappa = \aleph_0$. For each $\alpha < \omega_1$ find $n(\alpha) < \omega$ such that

$$u_\alpha \cap A \cap K_{n(\alpha)} \neq \emptyset \neq u_\alpha \cap B \cap K_{n(\alpha)}.$$

This is possible, because p is an accumulation point of both A and B and therefore, $u_\alpha \setminus B$ and $u_\alpha \setminus A$ are non-empty open sets. Now there is $n \in \omega$ such that

$$\{\alpha < \omega_1 : n(\alpha) = n\}$$

is uncountable. Then p is an accumulation point of both $A \cap K_n$ and $B \cap K_n$ which implies that K_n is a counterexample to the theorem. This contradicts the minimality of $\mathrm{rk}_{\mathcal{R}}(K)$. It follows that $\kappa = \aleph_1$.

The same argument as above, using the minimality of $\mathrm{rk}_{\mathcal{R}}(K)$, shows that either

$$(\forall\, \alpha < \omega_1)(\exists\, \beta < \omega_1)\ K_\alpha \cap u_\beta \setminus B = \emptyset \tag{1}$$

or the same holds for A in place of B. Moreover, if

$$p \notin \bigcup_{\alpha<\omega_1} K_\alpha$$

then (1) holds both for A and B. Clearly, we may assume that either

$$(\forall\, \alpha < \omega_1)\ r_\alpha(p) \in B \tag{2}$$

or the same holds for A in place of B. Thus, without loss of generality, we may assume that both (1) and (2) hold (interchanging the roles of A and B, if necessary).

Define

$$\overline{u}_\alpha = \bigcap_{\xi<\alpha} \overline{u_{\xi+1}}.$$

Then $u_\alpha \subseteq \overline{u}_\alpha$ and $\bigcap_{\alpha<\omega_1} \overline{u}_\alpha = \{p\}$, thus every neighbourhood of p contains some $\overline{u}_\alpha$. Moreover, for a limit ordinal $\delta < \omega_1$ we have

$$\overline{u}_\delta = \bigcap_{\xi<\delta} u_\xi = \bigcap_{\xi<\delta} \overline{u_\xi}.$$

Fix $\alpha < \omega_1$. Observe that by (1) and (2) the set $B \cap K_\alpha$ is a neighbourhood of $r_\alpha(p)$ in K_α. By the continuity of r_α, there is $\xi(\alpha) < \omega_1$ such that

$$r_\alpha[\overline{u}_{\xi(\alpha)}] \subseteq B. \tag{3}$$

On the other hand, the set $u_\alpha \setminus B$ is open and non-empty, therefore there is $\eta(\alpha) > \alpha$ such that

$$K_{\eta(\alpha)} \cap u_\alpha \setminus B \neq \emptyset. \tag{4}$$

Fix a limit ordinal $\delta < \omega_1$ such that $\xi(\alpha) < \delta$ and $\eta(\alpha) < \delta$ whenever $\alpha < \delta$. Then, by (3), we have

$$r_\alpha[\overline{u}_\delta] \subseteq B$$

for every $\alpha < \delta$ and hence

$$r_\delta[\overline{u}_\delta] \subseteq B,$$

because $r_\delta(x) = \lim_{\alpha<\delta} r_\alpha(x)$ for every $x \in K$.

Now for each $\alpha < \delta$ fix $q_\alpha \in K_\delta \cap u_\alpha \setminus B$, which exists by (4). Let q be an accumulation point of the sequence

$$\{q_\alpha : \alpha < \delta\}.$$

Then $q \in K_\delta$ and

$$q \in \bigcap_{\alpha<\delta} u_\alpha = \overline{u}_\delta.$$

Furthermore, $q \neq p$, because p is a P-point in A. Hence $q \notin B$ and $r_\delta(q) = q$, which contradicts the fact that

$$r_\delta[\overline{u}_\delta] \subseteq B.$$

□

24.6 The First Cohomology Functor

It turns out that the first cohomology group of a space in $\mathcal{R}$ has a very simple structure, as we prove further. This will be used for showing that Abelian compact groups in class $\mathcal{R}$ are products of metric compact spaces. In particular, this will show that class $\mathcal{R}$ is not stable under open maps. The results presented here come from [387] and [381].

The first cohomology functor $H^1 = H^1(\,\cdot\,, \mathbb{Z})$ in the sense of the theory of sheaves [279] or the Čech theory [572, Chapter 6] has the following properties:

(1) $H^1(X)$ is countable whenever X is second countable.
(2) $H^1(f)$ is a monomorphism whenever f is a continuous surjection.
(3) If

$$X = \varprojlim \langle X_\xi;\, p_\xi^\eta;\, \kappa \rangle$$

then $H^1(X)$ is the inductive limit of the sequence of groups

$$\langle H^1(X_\xi),\, H^1(p_\xi^\eta),\, \kappa \rangle.$$

For the proofs we refer to [572] or [387]. Another important feature of the above functor is that $H^1(G)$ coincides with the Pontryagin dual of G, whenever G is a compact connected Abelian group, see [387] for more details.

Theorem 24.6.1 (Kubiś [381]) *Assume that K is a retract of a space from class $\mathcal{R}$. Then $H^1(K)$ is isomorphic to a direct sum of countable groups.*

Before proving Theorem 24.6.1, we need two statements concerning discrete Abelian groups. Let us denote by $\mathcal{H}$ the class of all Abelian groups which are isomorphic to a direct sum of countable groups. We need the following two properties of this class.

Lemma 24.6.2 *Assume that $\{G_\alpha\}_{\alpha<\kappa}$ is a continuous increasing chain of subgroups of an Abelian group G such that $G = \bigcup_{\alpha<\kappa} G_\alpha$, $G_0 = 0$ and $G_{\alpha+1} = G_\alpha \oplus H_\alpha$ for every $\alpha < \kappa$. Then*

$$G = \bigoplus_{\alpha<\kappa} H_\alpha.$$

In particular, if $\{H_\alpha : \alpha < \kappa\} \subseteq \mathcal{H}$ then also $G \in \mathcal{H}$.

Proof Denote by G' the algebraic sum

$$\sum_{\alpha<\kappa} H_\alpha \subseteq G,$$

i.e. G' consists of all elements of the form

$$x_0 + \cdots + x_{k-1},$$

where $x_i \in H_{\alpha_i}$, $i < k$. Then $G_0 \subseteq G'$ and $G_\xi \subseteq G'$ implies $G_{\xi+1} \subseteq G'$. Thus, by induction and by the continuity of the chain, we deduce that $G' = G$.

Now suppose $x_0 + \cdots + x_k = 0$, where $x_i \in H_{\alpha_i}$ for $i \leqslant k$. Assume

$$0 < \alpha_0 < \alpha_1 < \cdots < \alpha_k$$

and α_i is a minimal ordinal α such that $x_i \in H_\alpha$ $(i \leqslant k)$. By the continuity of the chain, we have $\alpha_k = \xi + 1$ and hence

$$x_0 + \cdots + x_{k-1} \in G_\xi$$

and $x_k \in H_\xi$. Thus $x_k = 0$, because $G_\xi \cap H_\xi = 0$. This shows that

$$H_\eta \cap \sum_{\alpha\neq\eta} H_\alpha = 0$$

for every $\eta < \kappa$. □

Lemma 24.6.3 *Class $\mathcal{H}$ is closed under direct summands. That is, if $G = H \oplus K \in \mathcal{H}$ then $H \in \mathcal{H}$ and $K \in \mathcal{H}$.*

Proof We use induction on the cardinality of the group. Of course, the claim is true for countable groups. Fix $\kappa > \aleph_0$ and suppose that the statement holds for all groups of cardinality $< \kappa$. Fix

$$G = \bigoplus_{\alpha<\kappa} G_\alpha,$$

where each G_α is a countable Abelian group. Let H be a direct summand of G and let $h\colon G \to H$ be a group epimorphism such that $h \restriction H = \mathrm{id}_H$. Fix a regular cardinal χ big enough so that $h \in H(\chi)$ and $G \subseteq H(\chi)$. Fix an increasing continuous chain

$$\{M_\alpha\}_{\alpha<\kappa}$$

of elementary substructures of $\langle H(\chi), \in\rangle$ such that $h \in M_0$,

$$H \subseteq \bigcup_{\alpha<\kappa} M_\alpha$$

and $|M_\alpha| < \kappa$ for every $\alpha < \kappa$. Let $H_\alpha = H \cap M_\alpha$. Then

$$\{H_\alpha\}_{\alpha<\kappa}$$

is an increasing continuous chain of subgroups of H and $H = \bigcup_{\alpha<\kappa} H_\alpha$.

Fix $\alpha < \kappa$ and let $M = M_\alpha$. Observe that

$$G \cap M = \bigoplus_{\xi\in\kappa\cap M} G_\alpha.$$

Indeed, if $\xi \in \kappa \cap M$ then $G_\xi \subseteq M$, because $|G_\xi| \leqslant \aleph_0$. Thus $\bigoplus_{\xi\in\kappa\cap M} G_\xi \subseteq M$. On the other hand, if $x \in G \cap M$ and $x = x_0 + \cdots + x_{k-1}$, where $x_i \in G_{\xi_i}$, then

$$\{\xi_i : i < k\} \subseteq M$$

and hence $x \in \bigoplus_{\xi\in\kappa\cap M} G_\alpha$.

Let $p_\alpha\colon G \to G \cap M$ denote the canonical projection. By elementarity, $g_\alpha = h \circ p_\alpha$ is a homomorphism of G onto H_α which is identity on $H_\alpha \subseteq G \cap M$. Thus

$$H_{\alpha+1} = H_\alpha \oplus K_\alpha,$$

where $K_\alpha = H_{\alpha+1} \cap \ker(g_\alpha)$. Applying Lemma 24.6.2, we get

$$H = H_0 \oplus \bigoplus_{\alpha<\kappa} K_\alpha.$$

By inductive hypothesis, $H_0, K_\alpha \in \mathcal{H}$, because both of these groups are direct summands of $G \cap M_{\alpha+1}$ and $|G \cap M_{\alpha+1}| < \kappa$. Hence $H \in \mathcal{H}$, which completes the proof. □

We are now ready to prove Theorem 24.6.1.

Proof By Lemma 24.6.3, it suffices to show that $H^1(K) \in \mathcal{H}$ whenever $K \in \mathcal{R}$. We use induction on $\mathrm{rk}_{\mathcal{R}}$. By property (1) of the functor H^1, the statement is true for spaces of $\mathcal{R}$-rank 0. Fix an ordinal $\beta > 0$ and assume $H^1(K) \in \mathcal{H}$ whenever $\mathrm{rk}_{\mathcal{R}}(K) < \beta$. Fix $K \in \mathcal{R}$ with $\mathrm{rk}_{\mathcal{R}}(K) = \beta$ and let $\mathbb{S} = \langle K_\xi; r^\eta_\xi; \kappa\rangle$ be a continuous retractive sequence with $K = \varprojlim \mathbb{S}$ and such that $\mathrm{rk}_{\mathcal{R}}(K_\xi) < \beta$ for every $\xi < \kappa$. Let $G = H^1(K)$ and let $G_\xi = H^1(K_\xi)$. By properties (2) and (3), we may identify each G_ξ with a subgroup of G so that

$$\{G_\xi\}_{\xi<\kappa}$$

becomes a continuous increasing chain with

$$G = \bigcup_{\xi<\kappa} G_\xi.$$

Further, each G_ξ is a direct summand of G. By the inductive hypothesis, $\{G_\xi : \xi < \kappa\} \subseteq \mathcal{H}$. By Lemma 24.6.2, we have that

$$G = G_0 \oplus \bigoplus_{\xi<\kappa} H_\xi,$$

where H_ξ is such that

$$G_{\xi+1} = G_\xi \oplus H_\xi, \quad \xi < \kappa.$$

Since H_ξ is a direct summand of G_ξ, we deduce, using Lemma 24.6.3, that

$$\{H_\xi : \xi < \kappa\} \subseteq \mathcal{H}.$$

Thus $G \in \mathcal{H}$. □

Theorem 24.6.4 (Kubiś [381]) *Assume that G is a compact connected Abelian group which is at the same time a topological retract of some space from class $\mathcal{R}$. Then G is isomorphic, in the category of topological groups, to a product of metrizable compact groups.*

Proof Let H denote the Pontryagin dual of G. Since G is connected, H is discrete and torsion-free, therefore isomorphic to $H^1(G)$, see [368, Proposition 2.5] for a proof of this well-known fact. By Theorem 24.6.1, H can be decomposed into a direct sum of countable groups. Thus, G is a product of metric groups, because G is the dual of H and Pontryagin duality turns direct sums into products. □

Corollary 24.6.5 (Kubiś–Uspenskij [387]) *There exists a compact connected Abelian group G of weight $\aleph_1$ which is an open image of a product of metric compact spaces, while on the other hand G is not a retract of any space from class $\mathcal{R}$.*

Proof There exists a torsion-free Abelian group A of cardinality $\aleph_1$ which is indecomposable [259, Sections 88 and 89], that is, A has no proper direct summands. Let G be the Pontryagin dual of A. We claim that the compact group G has the required properties.

The duals of torsion-free discrete groups are connected [310, Theorem 24.25], so X is connected. It is a well-known fact (for the proof see [368, Proposition 2.5]) that the cohomology group $H^1(G, \mathbb{Z})$ is isomorphic to A (in general it is the torsion-free part of A). In particular, $H^1(G, \mathbb{Z})$ is indecomposable. Theorem 24.6.1 implies that G is not a retract of any space from $\mathcal{R}$.

Finally, it is well known that every compact group is an epimorphic image of a product of metric compact group and every epimorphism between compact groups is an open map. □

One should compare the above result with Theorem 22.8.5: denoting by G the compact group from Corollary 24.6.5, the Banach space $C(G)$ is 1-Plichko and consequently the space of probability measures $P(G)$ as well as the dual unit ball of $C(G)$ with the $weak^*$ topology are Valdivia compact.

24.7 Compact Lines

We already know that the space

$$\omega_1 + 1 + \omega_1^{-1}$$

is not in class $\mathcal{R}$, while $\omega_1+1+\omega^{-1} \in \mathcal{R}$. Further, we characterize scattered linearly ordered spaces which are in $\mathcal{R}$. Let K be a linearly ordered compact space. It is not hard to check that $p \in K$ is a cutting P-point in K if and only if p is a P-point both in $(\leftarrow, p]$ and $[p, \rightarrow)$.

Theorem 24.7.1 ([379]) *Let K be a scattered linearly ordered space. Then $K \in \mathcal{R}$ if and only if K has no cutting P-points.*

Proof The necessity follows from Theorem 24.5.2. Fix a compact scattered linearly ordered space K with no cutting P-points. We use the idea from the proof of Hausdorff theorem on the structure of scattered linearly ordered sets. Define the following relation on K.

$$x \sim y \iff (\forall\, a, b \in [x, y])\ [a, b] \in \mathcal{R}.$$

We check that $\sim$ is an equivalence relation. Only transitivity requires an argument. So assume $x < y < z$ and $x \sim y$, $y \sim z$. Fix $a \in [x, y)$, $b \in (y, z]$. If y is isolated from, say, right-hand side, then $(y, b] = [y', b] \in \mathcal{R}$, $[a, y] \in \mathcal{R}$ and hence $[a, b] = [a, y] \oplus [y', b] \in \mathcal{R}$, by Theorem 24.4.8. Otherwise, reversing the order if necessary, we may assume that $y = \inf\{y_n\}_{n\in\omega}$, where $\{y_n\}_{n\in\omega}$ is strictly decreasing. Assume

$y_0 = b$. Since K is compact and 0-dimensional, we may also assume that each y_n has an immediate successor z_{n-1}, i.e. $y_n < z_{n-1}$ and $(y_n, z_{n-1}) = \emptyset$. Now

$$[y, b] = \{y\} \cup \bigcup_{n\in\omega} [z_n, y_n]$$

and the intervals $[z_n, y_n]$ are pairwise disjoint. Further, by assumption $[a, y] \in \mathcal{R}$ and hence, setting

$$K_n = [a, y] \cup \bigcup_{i<n} [z_i, y_i],$$

we have that $K_n \in \mathcal{R}$, by Theorem 24.4.8. Define $r_n : [a, b] \to K_n$ by

$$r_n \restriction K_n = \mathrm{id}_{K_n}$$

and $r_n(x) = y$ for $x \in [a, b] \setminus K_n$. Then $\{r_n : n \in \omega\}$ is an inverse sequence of internal retractions in $[a, b]$. This shows that $[a, b] \in \mathcal{R}$ and completes the proof of the transitivity of $\sim$.

By definition, the equivalence classes of $\sim$ are convex. We show that they are also closed. Fix $C \in K/\sim$ and let $b = \sup C$. Fix $a \in C$. Let

$$\{b_\alpha : \alpha < \kappa\} \subseteq [a, b)$$

be strictly increasing, continuous and such that $b = \sup_{\alpha<\kappa} b_\alpha$. Then $[a, b_\alpha] \in \mathcal{R}$, because $a, b_\alpha \in C$. Define $r_\alpha : [a, b] \to [a, b_\alpha]$ by

$$r_\alpha(x) = \min\{x, b_\alpha\}.$$

Then $\{r_\alpha : \alpha < \kappa\}$ is an inverse sequence of internal retractions in $[a, b]$. Hence $[a, b] \in \mathcal{R}$. This shows that $x \sim b$ for every $x \in C$, i.e. $b \in C$. By the same argument, $\inf C \in C$, i.e. C is a closed interval.

Finally, it remains to show that there is only one $\sim$-equivalence class. Suppose not, and fix $C, D \in K/\sim$ with $\max C = c < d = \min D$. Then $c \not\sim d$ and in particular $(c, d) \neq \emptyset$. Thus, there exists $E \in K/\sim$ such that $C < E < D$, where $A < B$ means $\sup A < \inf B$. It follows that the quotient ordering of $K/\sim$ is dense, i.e. $K/\sim$ is not scattered. This contradicts the fact that K is scattered. □

We have the following

Corollary 24.7.2 *Let K be a scattered compact linearly ordered space. Then there exists a scattered compact linearly ordered space $L \in \mathcal{R}$ which has a two-to-one continuous order preserving map onto K. In particular $K \in \mathcal{RC}$.*

Proof Let $L = K \cdot 2$, the lexicographic product of K by $\{0, 1\}$. Then L is scattered, compact and it has no cutting P-points (every point of L is isolated from at least one side). Thus $L \in \mathcal{R}$ and of course the natural projection $f : L \to K$ is two-to-one and continuous. □

It is not known how to characterize or classify compact linearly ordered spaces in classes $\mathcal{R}$ and $\mathcal{RC}$. Further, we prove two results concerning the structure of these spaces.

Theorem 24.7.3 *Assume $K \in \mathcal{R}$ is a linearly ordered space and $p \in K$ is a P-point in $(\leftarrow, p]$. Then for every $q > p$ the interval $[p, q]$ is disconnected. In particular, $\omega_1 + [0, 1] \notin \mathcal{R}$.*

Proof Suppose the statement is not true and let K and $p \in K$ form a counterexample such that $\mathrm{rk}_{\mathcal{R}}(K)$ is minimal. Let $q > p$ be such that $[p, q]$ is connected. Let λ denote the character of p in $(\leftarrow, p]$. Then λ is an uncountable regular cardinal. Let

$$\{r_\alpha : \alpha < \kappa\}$$

be an inverse sequence of internal retractions in K such that

$$\mathrm{rk}_{\mathcal{R}}(K_\alpha) < \mathrm{rk}_{\mathcal{R}}(K)$$

for every $\alpha < \kappa$, where $K_\alpha = r_\alpha[K]$. We may assume that $\kappa = \mathrm{cf}\,\kappa$ and that there exists $r \in (p, q) \cap K_0$ (recall that $\bigcup_{\alpha<\kappa} K_\alpha$ is dense in K). Let $\{p_\alpha\}_{\alpha<\lambda}$ be a strictly increasing continuous sequence with $p = \sup_{\alpha<\lambda} p_\alpha$. Striving for a contradiction, we shall consider two cases.

Case 1: $\kappa \neq \lambda$.

For each $\alpha < \lambda$ find $\delta(\alpha) < \kappa$ such that

$$K_{\delta(\alpha)} \cap (p_\alpha, p) \neq \emptyset.$$

If $\kappa < \lambda$ then, by the regularity of λ we have

$$\delta = \sup_{\alpha<\kappa} \delta(\alpha)$$

is an ordinal $< \lambda$. If $\kappa > \lambda$ then, by the regularity of κ, we can find $\delta < \lambda$ such that the set

$$\{\alpha < \kappa : \delta(\alpha) = \delta\}$$

has cardinality κ. In both cases,

$$p \in \overline{(K_\delta \cap (p_0, p))},$$

therefore p is a P-point in

$$K_\delta \cap (\leftarrow, p).$$

On the other hand, $A = r_\delta[[p, r]]$ is a closed and connected subset of K_δ which contains p, r. Thus $[p, r] \subseteq A \subseteq K_\delta$. It follows that K_δ is also a counterexample to our statement, which contradicts the minimality of $\mathrm{rk}_{\mathcal{R}}(K)$.

Case 2: $\kappa = \lambda$.

Then $\kappa > \aleph_0$. By Theorem 24.5.2, $p \in K_\alpha$ for some $\alpha < \kappa$, because p must have a countable character in $[p, q]$. We may assume that $p \in K_0$. As in Case 1, we deduce that $[p, r] \subseteq K_0$. Thus, by the minimality of $\mathrm{rk}_{\mathcal{R}}(K)$, p is not a P-point in K_α for any $\alpha < \kappa$. In other words,

$$p \notin \overline{(K_\alpha \cap (\leftarrow, p))}$$

for every $\alpha < \kappa$. Fix $\alpha < \kappa$. By the continuity of r_α at $p = r_\alpha(p)$, there exists $\eta(\alpha) < \kappa$ such that

$$r_\alpha[(p_\alpha, p]] \subseteq [p, \rightarrow).$$

Let $\xi(\alpha) > \alpha$ be such that $(p_\alpha, p_{\alpha+1}) \cap K_{\xi(\alpha)} \neq \emptyset$. Since

$$\kappa = \mathrm{cf}\,\kappa > \aleph_0,$$

there exists $\delta < \kappa$ such that

$$\eta[\delta] \subseteq \delta$$

and $\xi[\delta] \subseteq \delta$. Then $p_\delta \in K_\delta$, by the definition of the function ξ. Hence, $r_\delta(p_\delta) = p_\delta$. On the other hand, by the definition of the function η, we have $r_\alpha(p_\delta) \geqslant p$ for every $\alpha < \delta$ and hence also $r_\delta(p_\delta) \geqslant p$. This is a contradiction. □

24.8 Valdivia and Corson Compact Spaces

In this section, we present the class of Valdivia compact spaces, dual to Plichko spaces and Corson compact spaces, dual to (WLD) spaces. We discuss their properties, stability, and we give some examples.

A topological space is called *Valdivia compact* if it is homeomorphic to a subspace K of a cube $[0, 1]^\kappa$ so that $K \cap \Sigma(\kappa)$ is dense in K. A dense set $D \subseteq K$ will be called a Σ-*subset* of K if there is an embedding $j\colon K \to [0, 1]^\kappa$ such that $D = j^{-1}[\Sigma(\kappa)]$.

Proposition 24.8.1 *Let K be an infinite Valdivia compact embedded in $[0, 1]^\kappa$ so that $K \cap \Sigma(\kappa)$ is dense in K. Then there is $S \subseteq \kappa$ such that $|S| = w(K)$ and* $\mathrm{supp}(x) \subseteq S$ *for every $x \in K$.*

Proof Fix a sufficiently big cardinal χ and fix an elementary substructure M of $H(\chi)$ so that $K \in M$, $|M| = w(K)$ and M contains a fixed open basis of K. Let $S = \kappa \cap M$. Using Lemma 22.7.2 with $f = \mathrm{id}_K$, we conclude that $x \mid S \in K$ and next $x = x \mid S$ for every K. This just means that $\mathrm{supp}(x) \subseteq S$ for $x \in K$. □

The above proposition implies that a Valdivia compact of weight κ can be "properly" embedded into $[0,1]^\kappa$. We shall use this fact without any further reference.

As we shall see, the class of Valdivia compact spaces is not stable under continuous (even open) images. It is not known whether it is stable under retractions (see Corollary 24.9.4 for a positive direction).

Knowing that not all compact spaces are Valdivia, it becomes clear this class is not stable under closed subspaces, because all Tikhonov cubes are Valdivia. However, we have the following positive result.

Proposition 24.8.2 *The class of Valdivia compact spaces is stable under closed G_δ subspaces.*

Proof Let $K \subseteq [0,1]^\kappa$ be such that $\Sigma(\kappa) \cap K$ is dense in K and fix a closed G_δ subset L of K. It suffices to show that $\Sigma(\kappa) \cap L$ is dense in L.

For fix a sequence $\{U_n\}_{n\in\omega}$ of open subsets of $[0,1]^\kappa$ so that $\overline{(U_{n+1})} \subseteq U_n$ and $K \cap \bigcap_{n\in\omega} U_n = L$. Fix an open set $V \subseteq [0,1]^\kappa$ such that $L \cap V \neq \emptyset$. For each $n \in \omega$ choose $x_n \in \Sigma(\kappa) \cap K \cap U_n \cap V$. Let x be an accumulation point of the sequence $\{x_n\}_{n\in\omega}$. Then $x \in \Sigma(\kappa)$ and moreover $x \in K \cap \bigcap_{n\in\omega} U_n = L$. Finally, x is in the closure of V. By regularity, this shows that $\Sigma(\kappa) \cap L$ is dense in L. □

In order to get simple examples of compact spaces that are not Valdivia, it is sufficient to observe the following property.

Proposition 24.8.3 *Every Valdivia compact has a dense monolithic set.*

Proof Every relatively closed subset of a Σ-product is monolithic. □

By Corollary 24.2.4, non-metrizable compactifications of the integers cannot be Valdivia compact.

Recall that K is *Corson compact* if it is homeomorphic to a compact subset of a Σ-product $\Sigma(\kappa)$ for some κ.

Proposition 24.8.4 *A Valdivia compact space is Corson if and only if it has countable tightness.*

Proof Every Σ-product of real lines has countable tightness by Proposition 22.7.1, therefore Corson compact spaces are countably tight as well. Now assume K is a countably tight Corson compact and assume $K \subseteq [0,1]^\kappa$, where $D = \Sigma(\kappa) \cap K$ is dense in K. Fix $p \in K$. Then $p \in \overline{A}$ for some countable $A \subseteq D$. Let

$$S = \bigcup_{x\in A} \mathrm{supp}(x).$$

Then S is countable and $\mathrm{supp}(p) \subseteq S$. Thus $p \in D$. It follows that $K \subseteq \Sigma(\kappa)$, so it is Corson. □

The next properties are useful for showing that certain natural compact spaces are not Valdivia. Recall that a topological space X is *Fréchet–Urysohn* if for every $p \in \overline{A} \subseteq X$ there is a sequence $\{a_n\}_{n\in\omega} \subseteq A$ such that $p = \lim_{n\to\infty} a_n$.

Theorem 24.8.5 *Let K be a Valdivia compact and let D be its Σ-subset. Then D is countably tight, Fréchet–Urysohn and the closure in K of every countable subset of D is metrizable and contained in D, i.e. D is monolithic in K. Further, K is the Čech–Stone compactification of D.*

Proof The fact that D is countably tight follows from Proposition 22.7.1.

Let $K \subseteq [0,1]^\kappa$ be embedded so that $D = \Sigma(\kappa) \cap K$ and fix a countable set $A \subseteq D$. Let $S = \bigcup_{a\in A} \mathrm{supp}(a)$. Then S is countable and

$$K_0 = \{x \in K : \mathrm{supp}(x) \subseteq S\}$$

is closed in K, contained in D and metrizable (homeomorphic to a subset of the Cantor space $[0,1]^S$). Thus, the same applies to $\overline{A}$.

Now let $p \in \overline{B} \subseteq D$. Using the countable tightness, find a countable $A \subseteq B$ so that $p \in \overline{A}$. By the previous argument, $\overline{A}$ is metrizable therefore p is the limit of a sequence from A.

Finally, in order to show that $K = \beta D$ we use a well-known criterion: disjoint relatively closed subsets of D must have disjoint closures in K. However, this follows immediately from Corollary 22.7.3. □

Next result, relating Valdivia compact spaces to class $\mathcal{R}$, was already proved by Argyros, Mercourakis, and Negrepontis in [18], before the class of Valdivia was formally defined and studied.

Lemma 24.8.6 *Let K be a Valdivia compact suitably embedded in $[0,1]^\kappa$. Then for every infinite set $T \subseteq \kappa$ there exists $S \subseteq \kappa$ such that $T \subseteq S$, $|T| = |S|$ and $x \mid S \in K$ for every $x \in K$, where $x \mid S = x \cdot \chi_S$.*

Proof Fix an elementary substructure M of a suitably big $H(\chi)$ such that $K \in M$, $T \subseteq M$ and $|T| = |M|$. Let $S = M \cap \kappa$. By Lemma 22.7.2, we have that $x \mid S \in \overline{K} = K$ for every $x \in K$. □

Let K and S be as above, and let $r_S(x) = x \mid S$. Then $r_S : K \to K$ is an internal retraction and its range is

$$K \mid S = \{x \in K : \mathrm{supp}(x) \subseteq S\}.$$

Observe that the space $K \mid S$ is again Valdivia compact and

$$r_S[\Sigma(\kappa) \cap K]$$

is a Σ-subset of $K \mid S$.

A standard transfinite induction gives the following:

Theorem 24.8.7 *Let K be a Valdivia compact of weight $\kappa > \aleph_0$. Then $K = \varprojlim \mathbb{S}$, where $\mathbb{S} = \langle K_\xi; r^\eta_\xi; \kappa\rangle$ is a continuous retractive sequence of Valdivia compact spaces such that $w(K_\xi) \leqslant |\xi| + \aleph_0$ for every $\xi < \kappa$.*

Proof We may assume that $K \subseteq [0,1]^\kappa$ so that $\Sigma(\kappa) \cap K$ is dense in K (see Proposition 24.8.1). Construct inductively a chain of sets $\{S_\alpha\}_{\alpha<\kappa}$ such that

$$\alpha \subseteq S_\alpha, \quad |S_\alpha| \leqslant \alpha, \quad \bigcup_{\alpha<\kappa} S_\alpha = \kappa$$

and each S_α satisfies the assertion of Lemma 24.8.6, i.e. $x \mid S_\alpha \in K$ for every $x \in K, \alpha \in \kappa$. Given a limit ordinal δ, we have

$$x \mid \Big(\bigcup_{\xi<\delta} S_\xi\Big) = \lim_{\xi<\delta} x \mid S_\xi,$$

therefore, we may assume that

$$S_\delta = \bigcup_{\xi<\delta} S_\xi,$$

i.e. that the chain $\{S_\xi\}_{\xi<\kappa}$ is continuous.

Define $r_\alpha(x) = x \mid S_\alpha$. Then

$$x = \lim_{\xi<\kappa} r_\xi(x)$$

and for a limit ordinal δ we have

$$r_\delta(x) = \lim_{\xi<\delta} r_\xi(x).$$

By Lemma 24.1.5, the sequence $\{r_\alpha\}_{\alpha<\kappa}$ induces a continuous retractive sequence $\mathbb{S} = \langle K_\xi; r^\eta_\xi; \kappa\rangle$ with $K = \varprojlim \mathbb{S}$. Finally, the space $K_\xi = K \mid S_\xi$ is a Valdivia compact of weight is less than

$$|S_\xi| \leqslant |\xi| + \aleph_0.$$

This completes the proof. □

Corollary 24.8.8 *Valdivia compact spaces belong to class $\mathcal{R}$.*

It is not known whether the class of Valdivia compact spaces is stable under retractions. We show further that retracts of Valdivia compact spaces belong to $\mathcal{R}$.

Lemma 24.8.9 *Let $r\colon K \to L$ be a retraction, where K is a Valdivia compact. Let θ be a big enough regular cardinal, and let $M \preceq \langle H(\theta), \in\rangle$ be such that $r \in M$. Then the induced map $r^M\colon K/M \to L/M$ is a retraction.*

Proof As we have already mentioned, $f \mapsto f^M$ is a functor on the category of continuous maps in M. Now, if θ is big enough then, by elementarity, $M \models$ "r is right-invertible". This means that there is $j \in M$ such that $r \circ j = \mathrm{id}_L$. Finally, $r^M \circ j^M = (r \circ j)^M = (\mathrm{id}_L)^M = \mathrm{id}_{L/M}$. □

Corollary 24.8.10 *Retracts of Valdivia compact spaces belong to class $\mathcal{R}$.*

Proof We use induction on weight. Fix a retraction $r\colon K \to L$ and assume K is Valdivia compact, $\mathrm{w}(L) = \kappa$ and the statement is true for spaces of weight $< \kappa$. Fix a big enough regular cardinal θ so that $r \in H(\theta)$ and $C(L) \subseteq H(\theta)$. Let $\lambda = \mathrm{cf}\,\kappa$ and fix a continuous chain $\{M_\alpha\}_{\alpha<\lambda}$ of elementary substructures of $\langle H(\theta), \in\rangle$ so that $|M_\alpha| < \kappa$ for each $\alpha < \lambda$ and $N = \bigcup_{\alpha<\lambda} M_\alpha$ contains a point-separating subset of $C(L)$. In particular, $L/N = L$ and $\{M_\alpha\}_{\alpha<\lambda}$ induces a continuous inverse sequences with limit L and whose α-th projection is $q_L^{M_\alpha}\colon L \to L/M_\alpha$. By Lemma 22.7.2, $q_K^{M_\alpha}\colon K \to K/M_\alpha$ is a retraction and we have $q_L^{M_\alpha} \circ r = r^{M_\alpha} \circ q_K^{M_\alpha}$. By Lemma 24.8.9, r^M is a retraction. Thus $q_L^{M_\alpha}$ is a retraction too. This shows that the inverse sequence induced by $\{M_\alpha\}_{\alpha<\lambda}$ is retractive. Finally, L/M_α has weight $\leqslant |M_\alpha| < \kappa$, therefore $L/M_\alpha \in \mathcal{R}$ by induction hypothesis. Thus, $L \in \mathcal{R}$. □

We shall show later that retracts of Valdivia compact spaces of weight $\aleph_1$ are Valdivia compact, see Sect. 24.10.

Example 24.8.11 (a) Let

$$K_1 = \omega_1 + 1$$

considered with the linear order. The space K_1 is compact, and it can be represented as a subset of $\{0,1\}^{\omega_1}$ consisting of all non-increasing functions: an element $\alpha \in K_1$ corresponds to the characteristic function of $[0,\alpha)$. This shows that K_1 is Valdivia. On the other hand, $\omega_1 \in K_1$ witnesses that it is not countably tight, therefore not Corson.

(b) Let

$$K_2 = \omega_1 + 1 + \omega^{-1},$$

where X^{-1} denotes X with the reversed ordering. Clearly, K_2 is a two-to-one image of K_1. We claim that it is not Valdivia compact. For the contrary, suppose that D is a Σ-subset of K_2. Then D is dense, so it contains all isolated points of K_2. On the other hand, D is sequentially closed so $\omega_1 \in D$. This is a contradiction, because Σ-products are countably tight. On the other hand, notice that $K_2 \in \mathcal{R}$. More precisely, $\mathrm{rk}_{\mathcal{R}}(K_2) = 2$ (the fact that $\mathrm{rk}_{\mathcal{R}}(K_2) \leqslant 2$ follows from Lemma 24.5.1 and we shall see later that spaces of $\mathcal{R}$-rank 1 are Valdivia compact). Note that the space K_2 is a 2-to-1 image of $K_1 = \omega_1 + 1$.

This example, showing that Valdivia compacta are not stable under continuous images, was first noticed by Valdivia [621].

(c) Let

$$K_3 = \omega_1 + 1 + \omega_1^{-1}.$$

Then K_3 is not Valdivia compact by Theorem 24.5.2. A direct argument is as follows: the only possible Σ-set $D \subseteq K_3$ would have to consist of all countable ordinals and all reversed countable ordinals; on the other hand βD is homeomorphic to $\omega_1 + 2 + \omega_1^{-1}$, not to K_3.

(d) Let

$$K_4 = \omega_2 + 1,$$

again considered as a compact linearly ordered space. We claim that it is not Valdivia compact. For suppose $f\colon K_4 \to [0,1]^\kappa$ is an embedding such that $f^{-1}[\Sigma(\kappa)]$ is dense. Note that $f(\alpha) \in \Sigma(\kappa)$ whenever α has a countable cofinality. Thus,

$$f(\omega_2) \notin \Sigma(\kappa)$$

because of tightness. Since

$$f(\omega_2) = \lim_{\xi<\omega_2} f(\xi)$$

and the convergence in $[0,1]^\kappa$ is pointwise, there is $\alpha_0 < \omega_2$ such that $\operatorname{supp}(f(\xi))$ is uncountable for every $\xi > \alpha_0$. This is a contradiction because $\operatorname{supp}(f(\alpha_0 + 1))$ is supposed to be countable. Notice that $K_4 \in \mathcal{R}$, being the limit of a continuous retractive sequence of Valdivia compact spaces. In particular, $\operatorname{rk}_{\mathcal{R}}(K_4) = 2$.

Valdivia compact spaces are dual to Plichko spaces. More precisely, we have the following:

Theorem 24.8.12 (Kalenda [358]) *For every Valdivia compact K, the Banach space $C(K)$ is 1-Plichko. For every 1-Plichko space E, the dual unit ball $B_{E'}$ with the weak* topology is Valdivia.*

Proof Assume

$$K \subseteq [0,1]^\kappa$$

so that $D = K \cap \Sigma(\kappa)$ is dense in K. For each $\alpha \in \kappa$ let x_α be the restriction to K of the projection of $[0,1]^\kappa$ onto α-th coordinate. That is, $x_\alpha(p) = p(\alpha)$ for $p \in K$. Let

$$A = \{x_{\alpha_1} x_{\alpha_2} \dots x_{\alpha_n} : \alpha_1, \alpha_2, \dots, \alpha_n \in \kappa\}.$$

By the Stone–Weierstrass theorem, A is a linearly dense subset of $C(K)$. Identifying K with a suitable subset of $C(K)'$, it is clear that D is 1-norming and $\langle A, D\rangle$ is a Plichko pair.

Now let E be a 1-Plichko space, and let

$$T\colon E' \to \mathbb{R}^\kappa$$

be a *weak** continuous injective linear operator such that

$$D = T^{-1}[\Sigma(\kappa)]$$

is 1-norming for E. Let $K = B_{E'}$ with the *weak** topology. Then $T \restriction K$ is a homeomorphic embedding of K into $\mathbb{R}^\kappa$. It remains to check that $D_0 = D \cap K$ is dense. For suppose otherwise and fix $y \in K \setminus \overline{(D_0)}$. Note that D_0 is convex, because D is a linear space. Thus, by the separation theorem, there is $x \in E$ such that

$$y(x) > \sup_{z\in K} z(x).$$

But this contradicts the fact that D is 1-norming:

$$\|x\| = \sup_{d\in D\cap K} |d(x)|.$$

□

For spaces of continuous functions, we have one more result.

Corollary 24.8.13 *Let K be a compact such that $C(K)$ is a 1-Plichko space. Then the space of probability measures $P(K)$ is Valdivia compact.*

Proof By Theorem 24.8.12, $B_{C(K)'}$ is Valdivia compact. By Proposition 24.8.2, it suffices to observe that $P(K)$ is a (closed) G_δ subset of $B_{C(K)'}$. □

As we shall see later, $P(K)$ is Valdivia whenever K is so, but there are examples of non-Valdivia compact spaces for which $P(K)$ is Valdivia (see Corollary 24.6.5).

For certain applications, the following criterion for Valdivia compactness is useful. Recall that a family of sets $\mathcal{A}$ is T_0 *separating* in X if for every $x, y \in X$ there is $A \in \mathcal{A}$ satisfying $|\{x, y\} \cap A| = 1$. A family $\mathcal{A}$ is *point countable* on D if the set

$$\{A \in \mathcal{A}\colon p \in A\}$$

is countable for every $p \in D$.

Theorem 24.8.14 *Let K be a compact space. Then K is Valdivia if and only if there are a dense set $D \subseteq K$ and a family $\mathcal{U}$ of open F_σ subsets of K which is T_0 separating and point countable on D. If K is totally disconnected, then the family $\mathcal{U}$ may consist of clopen sets.*

Proof Assume $K \subseteq [0,1]^\kappa$ and $D = \Sigma(\kappa) \cap K$ is dense in K. Define

$$U^r_\alpha = \{x \in K : x(\alpha) > r\}$$

and let $\mathcal{U}$ consist of all sets of the form U^r_α where $\alpha \in \kappa$ and r is a positive rational. Clearly, $\mathcal{U}$ consists of open F_σ sets and it is T_0 separating: given $x \neq y$ and assuming $x(\alpha) > y(\alpha)$ for some $\alpha \in \kappa$, we have that $x \in U^r_\alpha$ and $y \notin U^r_\alpha$, where r is a rational such that $x(\alpha) > r > y(\alpha)$. Given $x \in D \cap U^r_\alpha$, it must be the case that $\alpha \in \mathrm{supp}(x)$, so there are only countably many possibilities for α and r. Thus $\mathcal{U}$ is point countable on D.

Assume now that $\mathcal{U}$ is a T_0 separating family of open F_σ subsets of K which is point countable on a dense set D. For each $U \in \mathcal{U}$ choose a continuous map $f_U : K \to [0,1]$ so that $U = f^{-1}[(0,1]]$. The diagonal map

$$f : K \to [0,1]^{\mathcal{U}},$$

defined by

$$f(x)(U) = f_U(x),$$

is an embedding (because $\mathcal{U}$ is T_0 separating) and $f[D] \subseteq \Sigma(\mathcal{U})$. This shows that K is Valdivia.

Finally, suppose K is a totally disconnected Valdivia compact and let $\mathcal{U}$ be a family of open F_σ sets having the above properties. Replace each $U \in \mathcal{U}$ be a countable sequence of clopen sets whose union is U. By this way we obtain a family of clopen sets which is T_0 separating and point countable on the same set as $\mathcal{U}$ was. □

A variant of the above result concerning Corson compact spaces reads as follows. The proof is exactly the same as of Theorem 24.8.14.

Theorem 24.8.15 *Let K be a compact space. Then K is Corson if and only if there is a point countable T_0 separating family of open F_σ subsets of K. In case where K is totally disconnected, the family may be assumed to consist of clopen sets.*

The following fact on 0-dimensional Valdivia compact spaces will be needed later.

Proposition 24.8.16 *Let K be a Valdivia compact with a dense Σ-subset D. Then there exist a totally disconnected Valdivia compact, a dense Σ-subset E of L and a continuous surjection $f : L \to K$ satisfying $f[E] = D$.*

Proof Let $K \subseteq \mathbb{I}^\kappa$ so that $D = \Sigma(\kappa) \cap K$. Let $\varphi : 2^\omega \to \mathbb{I}$ be a continuous surjection satisfying $\varphi^{-1}(0) = \{0\}$. This map induces a surjection $\Phi : 2^{\omega \cdot \kappa} \to \mathbb{I}^\kappa$ satisfying $\Phi^{-1}[\Sigma(\kappa)] = \Sigma(\omega \cdot \kappa)$. Let $E = \Phi^{-1}[D]$ and let L be the closure of E in $2^{\omega \cdot \kappa}$. Then L is a 0-dimensional Valdivia compact, E is its dense Σ-subset and $\Phi[E] = D$, $\Phi[L] = K$. □

24.9 A Preservation Theorem

In order to get a better understanding of the class of Valdivia compact spaces, one needs to take care of their Σ-subsets. Namely, we shall consider pairs of the form $\langle K, D\rangle$, where $D \subseteq K$ is dense and such that

$$j[D] \subseteq \Sigma(\kappa)$$

for some embedding $j\colon K \to [0,1]^\kappa$. We shall call $\langle K, D\rangle$ a *Valdivia pair*. This is of course analogous to the notion of a Plichko pair. Using the corresponding preservation result for Plichko spaces, we obtain the following.

Theorem 24.9.1 ([382]) *Let $\{r_\alpha\}_{\alpha<\kappa}$ be a continuous retractive sequence in a compact space K. Let $D \subseteq K$ be a dense set such that for each $\alpha < \kappa$, $\langle r_\alpha[K], r_\alpha[D]\rangle$ is a Valdivia pair and $D = \bigcup_{\alpha<\kappa} r_\alpha[D]$. Then $\langle K, D\rangle$ is a Valdivia pair.*

Proof The sequence $\{r_\alpha\}_{\alpha<\kappa}$ induces a projectional sequence $\{P_\alpha\}_{\alpha<\kappa}$ on the Banach space $C(K)$. In fact, identifying K with a suitable subset of the dual space $C(K)'$, r_α is just the restriction of P'_α to K. Now $\langle C(K_\alpha), r_\alpha[D]\rangle$ is a Plichko pair and the assumptions of Theorem 22.9.1 are satisfied. Using this theorem, we conclude that $\langle C(K), D\rangle$ is a Plichko pair. In particular, there is a one-to-one $weak^*$ continuous linear transformation $T\colon C(K)' \to \mathbb{R}^\Gamma$ such that $D \subseteq T^{-1}[\Sigma(\Gamma)]$. This shows that $\langle K, D\rangle$ is a Valdivia pair. □

Corollary 24.9.2 *The limit of a continuous retractive sequence of Corson compact spaces is Valdivia compact.*

Corollary 24.9.3 ([385]) *The limit of a continuous retractive sequence of metric compact spaces is Valdivia compact.*

Corollary 24.9.4 ([385]) *Let K be a retract of a Valdivia compact. If $\mathrm{w}(K) \leqslant \aleph_1$, then K is Valdivia compact.*

Proof Repeat the arguments from the proof of Corollary 24.8.10. We conclude that K is the limit of a continuous retractive sequence of metric compact spaces. Thus, K is Valdivia by Corollary 24.9.3. □

Corollary 24.9.5 ([385]) *Let K be an open image of a Valdivia compact. If K is totally disconnected and $\mathrm{w}(K) \leqslant \aleph_1$ then K is Valdivia compact.*

Proof As in the proof of Corollary 24.8.10, we obtain that K is the limit of an inverse sequence of spaces K/M_α, $\alpha < \omega_1$, where $\{M_\alpha\}_{\alpha<\omega_1}$ is a continuous chain of countable elementary substructures of a big enough $H(\chi)$. We assume that M_0 "knows" a open surjection $r\colon L \to K$ from a Valdivia compact L onto K.

Fix $\alpha < \omega_1$ and let $M = M_\alpha$. We claim that r^M is an open map.

A basic open set in L/M is of the form $q_L^M[f^{-1}[U]]$, where $f \in C(L) \cap M$ and $U = (0, \to) \subseteq \mathbb{R}$. Thus $f = f^M$ and we have that

$$q_K^M[r[f^{-1}U]] = r^M[q_L^M[f^{-1}[U]]],$$

because $f^M = f$. The set on the right-hand side is open, because q_K^M is a quotient map and $r[f[U]]$ is assumed to be open.

Thus r^M is indeed an open map. Since K/M is totally disconnected and metrizable, a version of Michael's selection theorem says that $(r^M)^{-1}$ has a continuous selection, i.e. r^M is right-invertible. It follows, by the same arguments like in the proof of Corollary 24.8.10, that q_K^M is a retraction.

Finally, K is Valdivia compact, as the limit of a continuous retractive sequence of metric compact spaces. □

It has been also proved in [385] that retracts of Tichonov or Cantor cubes are Valdivia compact. Recall again that it is not known whether the class of Valdivia compact spaces is stable under retractions. It is not stable under open continuous images, because of Corollary 24.6.5.

24.10 Retractional Skeletons

Further, we introduce the notion of a retractional skeleton, dual to projectional skeleton. We characterize Valdivia compact spaces in terms of retractional skeletons. We also present Bandlow's characterization of Corson compact spaces in terms of elementary submodels.

Fix a compact space K. A *retractional skeleton* in K is a family of retractions $\{r_s\}_{s\in\Gamma}$ of K satisfying the following conditions.

(1) Γ is a σ-directed partially ordered set.
(2) $r_s \circ r_t = r_t \circ r_s = r_s$ whenever $s \leqslant t$.
(3) Each $r_s[K]$ is metrizable.
(4) $x = \lim_{s\in\Gamma} r_s(x)$ for every $x \in K$.
(5) Given $s_0 < s_1 < \ldots$ in Γ, $t = \sup_{n\in\omega} s_n$ exists and $r_t(x) = \lim_{n\to\infty} r_{s_n}(x)$ holds for every $x \in K$.

It is an easy exercise to check that, given a retractional skeleton in K, its adjoint linear transformations form a 1-projectional skeleton in $C(K)$.

A retractional skeleton $\{r_s\}_{s\in\Gamma}$ is *commutative* if $r_s \circ r_t = r_t \circ r_s$ holds for every $s, t \in \Gamma$.

Theorem 24.10.1 ([382]) *A compact space is Valdivia if and only if it admits a commutative retractional skeleton.*

Proof Suppose first that $K \subseteq \mathbb{I}^\kappa$ is such that $\Sigma(\kappa) \cap K$ is dense in K. Call a set $T \subseteq \kappa$ *admissible* if $x \cdot \chi_T \in K$ for every $x \in K$. By Lemma 24.8.6, every countable set is contained in a countable admissible set. Denote by Γ the collection of all countable admissible sets and for $T \in \Gamma$ let $r_T(x) = x \cdot \chi_T$. It is easy to see that $\{r_T\}_{T\in\Gamma}$ is a commutative retractional skeleton in K.

Now assume that K has a commutative retractional skeleton $\{r_s\}_{s\in\Gamma}$ and let $\mathfrak{p} = \{P_s\}_{s\in\Gamma}$ be the dual projectional skeleton. By Theorem 22.9.4, $C(K)$ is 1-Plichko and (by the proof of Theorem 22.9.4), $D = \bigcup_{s\in\Gamma} \operatorname{im} P_s'$ is a Σ-subspace of the dual, that is, there is a one-to-one *weak** continuous linear transformation $T\colon C(K)' \to \mathbb{R}^\kappa$ such that $D = T^{-1}[\Sigma(\kappa)]$. In particular, $\bigcup_{s\in S} r_s[K] \subseteq D$ is a dense Σ-subspace of K and T witnesses that K is Valdivia. □

The "if" part of the above statement can also be proved directly by induction on the weight of K, using Theorem 24.9.1.

Corollary 24.10.2 *Let F be a continuous weight preserving functor on compact spaces. Then $F(K)$ is Valdivia whenever K is so.*

Proof A commutative retractional skeleton can be described as a pair $\langle \mathbb{S}, \mathbb{I}\rangle$, where $\mathbb{S} = \langle K_s; r_s^t; \Sigma\rangle$ is a continuous retractive inverse system of metric compact spaces and $\mathbb{I} = \langle i_s^t, \Sigma\rangle$ is a direct system of right inverses to $\mathbb{S}$, that is, $r_s^t \circ i_s^t = \mathrm{id}_{K_s}$ and $i_t^u \circ i_s^t = i_s^u$ holds whenever $s < t < u$. Commutativity means that

$$(i_s \circ r_s) \circ (i_t \circ r_t) = (i_t \circ r_t) \circ (i_s \circ r_s)$$

holds for every $s, t \in \Sigma$, where i_s is the uniquely defined embeddding of K_s into $K = \varprojlim \mathbb{S}$ satisfying $r_t \circ i_s = i_s^t$ for $t > s$. Now it is obvious that every continuous weight preserving functor maps such a pair $\langle \mathbb{S}, \mathbb{I}\rangle$ to a pair satisfying the same conditions. □

Example 24.10.3 Let κ be a fixed uncountable cardinal and let $K = \kappa + 1$ be considered as a compact line. Let Γ consist of all closed countable sets $A \subseteq \kappa$ such that $0 \in A$ and isolated points of A are isolated in κ. In particular, A can contain only ordinals of countable cofinality. Consider Γ as a σ-directed set endowed with inclusion. Given $A \in \Gamma$, define $r_A\colon K \to K$ by setting $r_A(x) = \max(A \cap [0, x])$. It is easy to check that r_A is a continuous retraction and moreover $\{r_A\}_{A\in\Gamma}$ is a retractional skeleton in K. If $\kappa \geqslant \aleph_2$ then this skeleton is not commutative. In fact, by Example 24.8.11(d), $\aleph_2 + 1$ is not Valdivia compact, therefore it cannot have a commutative retractional skeleton. Recall also a relevant result of Kalenda: the Banach space $C(\aleph_2 + 1)$ is not Plichko.

Given a retractional skeleton $\mathfrak{p} = \{r_s\}_{s\in\Gamma}$ in a compact K, denote

$$D(\mathfrak{p}) = \bigcup_{s\in\Gamma} r_s[K].$$

As in the case of projectional skeletons, we shall call it the *dense subspace induced* by $\mathfrak{p}$. An analogue of Theorem 22.6.6 holds, namely:

Theorem 24.10.4 ([382]) *Let $\mathfrak{p} = \{r_s\}_{s\in\Gamma}$ be a retractional skeleton in a compact space K and let $D = D(\mathfrak{p})$ be the dense subset induced by $\mathfrak{p}$. Then:*

(a) D is countably tight and $\aleph_0$-monolithic.
(b) Disjoint relatively closed subsets of D have disjoint closures in K.

In particular, D is Fréchet–Urysohn and $K = \beta D$.

Proof Part (a) is a consequences of Theorem 22.6.6, since $\mathfrak{p}$ induces a projectional skeleton in $C(K)$ which in turn induces a 1-norming subspace that contains D.

For the proof (c), fix disjoint relatively closed sets $A, B \subseteq D$ and suppose there is $p \in \overline{A} \cap \overline{B}$, where the closure is taken in K. Fix a big enough cardinal θ and fix a countable $M \prec H(\theta)$ so that $A, B, \mathfrak{p} \in M$. Let $\delta = \sup(\Gamma \cap M)$. Then $r_\delta(p) \in D$, and we may assume that $r_\delta(p) \notin A$ (interchanging the roles of A and B if necessary). Recall that $K_\delta := r_\delta[K]$ is the limit of the inverse system $\langle K_s; r_s^t; \Gamma \cap M\rangle$, where r_s^t is the restriction of r_s to $K_t = r_t[K]$. Thus, there are $t \in \Gamma \cap M$ and an open set $V \subseteq K_t$ such that $U := K_\delta \cap {r_t}^{-1}[V]$ is a neighbourhood of p in K_δ disjoint from A. On the other hand, ${r_t}^{-1}[V] \cap A \neq \emptyset$, because p is in the closure of A. Now note that $K_t \in M$ is second countable, therefore M "knows" a fixed countable open base of K_t. Thus, we may assume that $V \in M$. By elementarity, there is $a \in M \cap {r_t}^{-1}[V] \cap A$. Finally, $a \in K_\delta$ (because $D \cap M \subseteq K_\delta$), so $a \in U \cap A$, a contradiction. □

Part (b) of the above theorem has a Banach space analogue which reads as follows.

Proposition 24.10.5 *Let D be the norming space induced by a projectional skeleton $\{P_s\}_{s\in\Gamma}$ in a Banach space X. Then for every weak* continuous function $f\colon D \to \mathbb{R}$ there exists $t \in \Gamma$ such that $f = f \circ P_t^* \restriction D$.*

Proof Fix $n > 0$ and consider $K_n = n\overline{\mathrm{B}}_{X^*}$, $f_n = f \restriction (D \cap K_n)$. Then $\mathfrak{p}_n = \{P_s^* \restriction K_n\}_{s\in\Gamma}$ is a retractional skeleton in K_n and $D_n = D \cap K_n$ is the dense subspace induced by $\mathfrak{p}_n$. By Theorem 24.10.4, K_n is the Čech–Stone compactification of D_n, therefore f_n extends to a continuous map $\widetilde{f_n}\colon K_n \to \mathbb{R}$. Now, since $\mathfrak{p}_n$ determines a σ-directed inverse system of metric compact spaces, by Lemma 24.1.6, there is $s_n \in \Gamma$ such that $\widetilde{f_n} = f \circ P_{s_n}^* \restriction K_n$. In particular, $f_n = f \circ P_{s_n}^* \restriction D_n$. We may assume that $s_1 \leqslant s_2 \leqslant \ldots$. Let $t = \sup_{n\in\omega} s_n$. Then $f = f \circ P_t^* \restriction D$. □

Theorem 24.10.6 *Let K be a compact space, and let D be its dense subset. The following properties are equivalent.*

(a) There exists a retractional skeleton $\mathfrak{p}$ on K such that $D \subseteq D(\mathfrak{p})$.
(b) For every big enough cardinal θ, for every countable $M \prec \langle H(\theta), \in\rangle$, the quotient map $q^M\colon K \to K/M$ is one-to-one on $\overline{(D \cap M)}$.

Proof The proof very similar to that of Theorem 22.6.2, so we only sketch the arguments.

(a) $\Longrightarrow$ (b) Let $\mathfrak{p} = \{r_s\}_{s\in\Gamma}$ be a retractional skeleton such that $D \subseteq D(\mathfrak{p})$. Fix a countable $M \preceq H(\theta)$, where θ is big enough; assume $\mathfrak{p} \in M$. It is easy to verify that the quotient map q^M induced by M equals r_t, where $t = \sup(M \cap \Gamma)$ (this is a general property of σ-continuous inverse systems of metric compact spaces). Finally, $D \cap M$ is contained in the range of r_t.

(b) $\Longrightarrow$ (a) Fix $M \preceq H(\theta)$ so that $q^M \restriction \overline{(D \cap M)}$ is one-to-one. Note that $q^M[D \cap M]$ is dense in K/M. Indeed, a basic open set in K/M is of the form $\varphi^{-1}[U]$, where $\varphi \in C(K) \cap M$ and U is an open rational interval. The density of D ad the elementarity of M together imply that $f(d) \in \varphi^{-1}[U]$ for some $d \in D \cap M$.

It follows that q^M is a retraction, its right inverse j^M is defined as follows: $j^M(y)$ is the unique element of $x \in \overline{(D \cap M)}$ such that $q^M(x) = y$. Finally, the family of all q^M's, with suitable $M \preceq H(\theta)$, forms a retractional skeleton satisfying (a). □

Corollary 24.10.7 (Bandlow [70]) *Let K be a compact space. Then K is Corson if and only if for every large enough cardinal θ, for every countable elementary substructure M of $\langle H(\theta), \in\rangle$ the quotient map $q^M\colon K \to K/M$ is one-to-one on $\overline{(K \cap M)}$.*

A direct application gives the following stability result.

Theorem 24.10.8 *The class of Corson compact spaces is stable under continuous images.*

Proof Fix a continuous surjection $f\colon K \to L$ with K Corson. Fix a countable $M \preceq H(\theta)$, where θ is big enough and $f \in M$. Fix $p \neq q$ in $\overline{(L \cap M)}$. Let $K_M = \overline{(K \cap M)}$ and let $A = f^{-1}(p) \cap K_M$, $B = f^{-1}(q) \cap K_M$.

By the fact that K is Corson, the map $r_M\colon K \to K$ defined by the conditions $r_M(x) \in K_M$ and $r_M(x) \sim_M x$ is homeomorphic to q^M_K (it induces the equivalence relation $\sim_M$).

Note that $a \not\sim_M b$ whenever $a \in A$ and $b \in B$. Thus, by elementarity and compactness, there is $\varphi \in C(K) \cap M$ such that $\varphi(a) < 1/3$ for $a \in A$ and $\varphi(b) > 2/3$ for $b \in B$. Indeed, fixing $a \in A$, for each $b \in B$ there is a function $\varphi_b \in C(K) \cap M$ such that $\varphi_b(a) < 1/3$ and $\varphi_b(b) > 2/3$. Taking the minimum of finitely many φ_b's for which the sets $\varphi_b^{-1}(2/3, +\infty)$ cover B, we obtain a single function $\psi_a \in C(K) \cap M$ such that $\psi_a(a) < 1/3$ and $\psi_a(b) > 2/3$ for every $b \in B$. A similar argument (now with the maximum of finitely many functions) gives the desired φ.

Let $C = \varphi^{-1}(-\infty, 1/3]$, $D = \varphi^{-1}[2/3, +\infty)$. Note that we have identified K/M with K_M, so $\varphi^M = \varphi \restriction K_M$. Then $C, D \in M$ are disjoint closed sets. Consequently, $C' = {r_M}^{-1}[C] = \varphi^{-1}[C]$, $D' = {r_M}^{-1}[D] = \varphi^{-1}[D]$ are closed and disjoint. Since they belong to M, there is in M a function $\psi \in C(K)$ satisfying $\psi \restriction C' = 0$ and $\psi \restriction D' = 1$.

It now remains to observe that $f[C']$ and $f[D']$ are disjoint. Indeed, if $f(c') = f(d')$ for some $c' \in C'$, $d' \in D'$ then, setting $c = r_M(c')$, $d = r_M(d')$ we would have $f(c) = f(c') = f(d') = f(d)$, but on the other hand $c \in C$, $d \in D$, which contradicts the fact that $C \cap D = \emptyset$.

Finally, $p \in f[C']$, $q \in f[D']$, and these sets are in M therefore they are separated by a continuous function from M. □

We shall also use Bandlow's characterization for proving certain dichotomy concerning continuous images (see Theorem 24.13.1). Further, we give an application to recognizing Corson compact spaces in dual Banach spaces.

Theorem 24.10.9 *Let X be a Banach space and assume $D \subseteq X'$ generates projections in X. Then every weak* compact subset of D is Corson.*

Proof Fix a *weak** compact $K \subseteq D$. Let $M \preceq H(\theta)$ be countable, $K, D \in M$ and assume θ is big enough. Fix $f \neq g$ in the *weak** closure of $K \cap M$. There is $x \in X$ such that $f(x) \neq g(x)$. Let P_M be the projection generated by $\langle X, D, M\rangle$ and let $y = P_M x$. Notice that P'_M is a projection onto the *weak** closure of $D \cap M$, therefore

$$f(y) = f P_M x = (P'_M f)x = f(x)$$

and similarly $g(y) = g(x)$. Using the continuity of both f and g and the fact that $y \in \overline{(X \cap M)}$, find $z \in X \cap M$ such that $f(z) \neq g(z)$. Now the evaluation function $\delta_z \in C(K)$, defined by $\delta_z(v) = v(z)$, is an element of M. We have shown that δ_z separates f from g, so $f \not\sim_M g$. □

24.11 Primarily Lindelöf Spaces

In this section, we prove that, given a Valdivia compact K, the space $C(K)$ endowed with the pointwise convergence on a Σ-subset has a property that is a strengthening of being Lindelöf. For Corson compact spaces, this result is due to Alster and Pol [7], the generalization is due to Kalenda [357].

Recall that $C(K)$ endowed with the topology of pointwise convergence on a dense Σ-subset is Lindelöf by Theorem 22.3.1.

The Lindelöf property does not behave well with respect to products (see e.g. [195]). There is, however, a stronger property which is stable under countable products: being primarily Lindelöf.

A space is *primarily Lindelöf* if it is a continuous image of a closed subspace of $(\mathrm{L}_\kappa)^{\aleph_0}$ for some cardinal κ, where L_κ denotes the *one-point Lindelöfication* of the discrete space κ. More precisely, $\mathrm{L}_\kappa = \kappa \cup \{\infty\}$, where all the points of κ are isolated while a neighbourhood of ∞ is of the form $U \cup \{\infty\}$, where $\kappa \setminus U$ is countable. The following fact is obvious.

Lemma 24.11.1 *The class od primarily Lindelöf spaces is stable under closed subspaces, continuous images, countable products, and countable unions.*

Lemma 24.11.2 *Every primarily Lindelöf space is Lindelöf.*

Proof It suffices to show that for every fixed uncountable cardinal κ the space $L = (\mathrm{L}_\kappa)^{\aleph_0}$ is Lindelöf. The proof will be divided into two parts.

(1) Every open family $\mathcal{U}$ in L there is a σ-disjoint open refinement.
(2) Every set $A \subseteq L$ of cardinality $\aleph_1$ has a complete accumulation point.

Recall that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if $\bigcup \mathcal{V} = \bigcup \mathcal{U}$ and for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ with $V \subseteq U$.

We now show that (1)+(2) implies that L is Lindelöf. For fix an open cover $\mathcal{U}$ of L. By (1) we may assume that $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$, where each $\mathcal{U}_n$ is a disjoint family. Then for each $p \in L$ the family $\mathcal{U}(p) = \{U \in \mathcal{U} \colon p \in \mathcal{U}\}$ is countable. Suppose $\mathcal{U}$ has no countable subcover and choose inductively points $a_\alpha \in L$ so that $a_\alpha \notin \bigcup \mathcal{U}(a_\xi)$ for $\xi < \alpha$. Since each $\mathcal{U}(a_\xi)$ is countable, this is possible until we reach $\alpha = \omega_1$. Now $A = \{a_\alpha \colon \alpha < \omega_1\}$ is a discrete set of cardinality $\aleph_1$. Let p be its complete accumulation point. Choose $V \in \mathcal{U}$ so that $p \in \mathcal{U}$. Then $a_\alpha \in V$ for some $\alpha < \omega_1$ and therefore $a_{\alpha+1} \notin V$, a contradiction.

It remains to show (1) and (2). For the proof of (1), let $L_n = (\mathrm{L}_\kappa)^n$ and observe that

(3) Every open family of subsets of L_n has a disjoint open refinement.

This is evidently true for $n = 1$. Assuming (3) holds for L_n, note that $L_{n+1} = (L_n \times \kappa) \cup (L_n \times \{\infty\})$ and κ is discrete. Thus, given an open family $\mathcal{U}$ of subsets of L_{n+1}, first find a basic clopen neighbourhood $V = V_0 \times \ldots V_n$ of $\langle \infty, \ldots, \infty \rangle$ that refines some element of $\mathcal{U}$; next, using induction hypothesis, find a disjoint open family in $(L_n \times \kappa) \setminus V$ which is a refinement of $\mathcal{U} \restriction (L_n \times \kappa) = \{U \cap (L_n \times \kappa) \colon U \in \mathcal{U}\}$. Finally, add disjoint open sets in $(L_n \times \{\infty\})$ which form a refinement of $\mathcal{U} \restriction (L_n \times \{\infty\})$. The resulting open family is a disjoint refinement of $\mathcal{U}$.

Now fix an open family $\mathcal{U}$ of L and let $\pi_n \colon L \to L_n$ denotes the canonical projection. We may assume that $\mathcal{U}$ consists of basic open sets and therefore $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$, where each $\mathcal{U}_n$ consists of sets of the form ${\pi_n}^{-1}[U]$ with $U \subseteq L_n$ open. Thus, by (3), for each n the family $\mathcal{U}_n$ has a disjoint open refinement $\mathcal{V}_n$. Finally, $\bigcup_{n\in\omega} \mathcal{V}_n$ is a σ-disjoint open refinement of $\mathcal{U}$.

It remains to show (2). Let $p_n \colon L \to \mathrm{L}_\kappa$ be the projection onto the n-th coordinate. Fix a set $A \subseteq L$ of cardinality $\aleph_1$. Construct inductively uncountable sets $A \supseteq A_0 \supseteq A_1 \supseteq \ldots$ and elements $x_0, x_1, \ldots$ in L_κ, so that x_n is a *condensation point* of $p_n[A_n]$, i.e. $U \setminus p_n[A_n]$ is countable for every neighbourhood U of x_n. This is possible because of the structure of L_κ.

Let $x = \langle x_0, x_1, \ldots \rangle \in L$. Fix a neighbourhood U of x. We may assume that $U = U_0 \times \ldots U_{n-1} \times X \times X \times \ldots$. For each $i < n$ we know that the set $C_i = A_i \setminus {p_i}^{-1}[U_i]$ is countable. Let $C = \bigcup_{i<n} C_i$. Fix $a \in A_n \setminus C$. Then $a \in A_i \setminus C_i$ and hence $p_n(a) \in U_i$. It follows that $A_n \setminus C \subseteq U$. Thus, x is a complete accumulation point of A. □

Theorem 24.11.3 (Kalenda [357]) *Let K be a Valdivia compact with a dense Σ-subset D. Then $C(K)$, endowed with the topology of pointwise convergence on D, is primarily Lindelöf.*

Proof Let $\tau_p(D)$ denote the topology of pointwise convergence on D. By Proposition 24.8.16, $\langle C(K), \tau_p(D)\rangle$ can be regarded as a closed subspace of $\langle C(K'), \tau_p(D')\rangle$, where K' is a 0-dimensional Valdivia compact and D' is its dense Σ-subset. Thus, we may assume that $K = K'$ and $D = D'$. Since $C(K) = \bigcup_{n\in\omega} C(K, [-n, n])$ and the class of primarily Lindelöf spaces is closed under countable unions, it suffices to show that $C(K, \mathbb{I})$ is primarily Lindelöf. Further, according to Proposition 22.1.7, as K is 0-dimensional, $C(K, \mathbb{I})$ is an image of $C(K, 2^\omega)$ via a map of the form $f \mapsto f \circ \varphi$. Such a map of course preserves the topology $\tau_p(D)$. Finally, $C(K, 2^\omega)$ is naturally homeomorphic to $C(K, 2)^{\aleph_0}$ and again this homeomorphism preserves $\tau_p(D)$. We have thus reduced our proof to showing that $C(K, 2)$ is primarily Lindelöf when endowed with $\tau_p(D)$.

We now assume that $K \subseteq 2^\tau$ and $D = \Sigma(\tau) \cap K$. Define $\theta\colon \mathrm{L}_\tau \to C(K, 2)$ by $\theta(\xi) = \mathrm{pr}_\xi \restriction K$ for $\xi \in \tau$ and $\theta(\infty) = 0$, where pr_ξ denotes the projection onto ξ-th coordinate. It is standard to check that θ is continuous at ∞. The fact that elements of D have a countable support is used here.

By this way we have obtained a primarily Lindelöf subspace $L = \theta[\mathrm{L}_\tau]$ of $C(K, 2)$ that separates the points of K. Elements of L can be identified with clopen subsets of K and $C(K, 2)$ corresponds to the Boolean algebra generated by L. Thus $C(K, 2)$ is obtained by closing L under Boolean algebraic operations which in terms of functions look as follows: $m(f, g) = \min\{f, g\}$ (intersection) and $c(f) = 1 - f$ (complement). In other words, $C(K, 2) = \bigcup_{n\in\omega} L_n$, where $L_0 = L$ and $L_{n+1} = L_n \cup m[L_n \times L_n] \cup c[L_n]$. This shows that $C(K, 2)$ is primarily Lindelöf. □

Corollary 24.11.4 (Alster and Pol [7], Pol [507]) *Let K be a Corson compact. Then $C_p(K)$ is Lindelöf.*

It is worth noting that Theorem 24.11.3 can be reversed. Namely, if D is a countably closed dense subset of a compact K such that $\langle C(K), \tau_p(D)\rangle$ is primarily Lindelöf then K is Valdivia and D is a Σ-subset. This was proved by Kalenda [357] and, in case $D = K$, by Pol [507].

24.12 Corson Compact Spaces and (WLD) Spaces

We now discuss the relation between Corson compact spaces and weakly Lindelöf determined Banach spaces. Recall that, according to [17], a Banach space E is *weakly Lindelöf determined* (briefly: WLD) if its dual unit ball $\overline{\mathrm{B}}_{E'}$ endowed with the *weak** topology is Corson compact.

Theorem 24.12.1 *Given a Banach space E, the following properties are equivalent.*

(a) E is (WLD).
(b) E' generates projections in E.
(c) There exists an injective weak continuous linear operator mapping E' into a Σ-product of the real lines.*

Proof (a) $\Longrightarrow$ (b) Assume $K = \overline{\mathrm{B}}_{E'}$ is Corson and fix a countable elementary submodel M of a big enough $\langle H(\theta), \in\rangle$ so that $E \in M$. Then $K \in M$. Using (the easier part of) Bandlow's characterization we deduce that the canonical quotient map $q^M\colon K \to K/M$ is one-to-one on the $weak^*$ closure of $K \cap M$. Thus, we may assume that q^M is a retraction onto $\overline{K \cap M}^*$. By Corollary 22.1.4, q^M preserves the affine structure on K and hence it is linear. More precisely, the map $Q^M\colon E' \to E'$ defined by

$$Q^M(y) = \|y\| \cdot q^M(y/\|y\|)$$

is linear. Now Q^M is a $weak^*$ continuous projection onto the $weak^*$ closed linear span of $E' \cap M$. Hence there exists a unique projection $P^M\colon E \to E$ such that Q^M is the dual of P^M. It remains to observe that $\operatorname{im} P^M = \overline{E \cap M}$ and $\ker P^M = {}^{\perp}\big(E' \cap M\big)$.

(b) $\Longrightarrow$ (c) We use Theorem 22.9.1 and transfinite induction on the density of the space. Fix a Banach space E such that E' generates projections, $\operatorname{dens} E = \kappa > \aleph_0$ and assume the implication (b) $\Longrightarrow$ (c) holds for Banach spaces of density $< \kappa$. Recall that E has a projectional skeleton $\mathfrak{p}$ such that $E' = D(\mathfrak{p})$. By Theorem 22.5.5, E has a PRI $\{P_\alpha\}_{\alpha<\kappa}$ such that each $\operatorname{im} P_\alpha$ has the same property as E, namely $\operatorname{im} P_\alpha$ has a projectional skeleton that induces the full dual space. By Theorem 22.6.6(b), E' is $weak^*$ countably tight, therefore $E' = \bigcup_{\alpha<\kappa} P'_\alpha E'$, unless κ has countable cofinality. In the first case we conclude, using Theorem 22.9.1, that $\langle E, E'\rangle$ is a Plichko pair, and hence, (c) holds. In the latter case, also using Theorem 22.9.1, we deduce that $\langle E, D\rangle$ is a Plichko pair, where $D = \bigcup_{\alpha<\kappa} P'_\alpha E'$. In particular, there exists a one-to-one $weak^*$ continuous linear operator $T\colon E' \to \mathbb{R}^\Gamma$ such that $T[D] \subseteq \Sigma(\Gamma)$. On the other hand, since κ has countable cofinality, necessarily $TE' \subseteq \Sigma(\Gamma)$. Thus, E satisfies (c).

Implication (c) $\Longrightarrow$ (a) is trivial. □

Corollary 24.12.2 *Every (WLD) Banach space has the controlled separable projection property.*

Proof Let E be a (WLD) Banach space. By the above theorem, E' generates projections in E and hence, given countable sets $A \subseteq E$, $B \subseteq E'$, taking a countable elementary submodel M of a big enough $H(\chi)$ such that $A, B \subseteq M$, we obtain a norm one projection $P\colon E \to E$ such that $A \subseteq \operatorname{im} P$ and $B \subseteq \operatorname{im} P'$. □

Following is a well-known characterization of (WLD) spaces of continuous functions, first proved by Argyros, Mercourakis, and Negrepontis.

Theorem 24.12.3 ([18]) *Let K be a compact space. The following conditions are equivalent.*

(a) $C(K)$ is (WLD).
(b) $P(K)$ is Corson compact.
(c) K is a Corson compact such that $P(K)$ is countably tight.
(d) K is Corson compact, and every positive regular Borel measure on K has a separable support.

Proof Implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) are trivial.

(c) $\Longrightarrow$ (d) Fix a positive regular Borel measure μ on K. Normalizing μ, we may assume that it belongs to $P(K)$. Now μ belongs to the $weak^*$ closure of the convex hull of $\{\delta_x : x \in K\}$, where δ_x is the probability measure supported by $\{x\}$. Applying (c), we get a countable set $D \subseteq K$ such that

$$\mu \in \overline{\mathrm{conv}\{\delta_x : x \in D\}}^{*}.$$

Clearly, the support of μ is contained in $\overline{D}$.

(d) $\Longrightarrow$ (a) Let $B = \overline{\mathrm{B}}_{C(K)'}$ be endowed with the $weak^*$ topology. Then B is Valdivia compact by Corollary 24.10.2. As K is Corson, its separable subsets are metrizable. Consequently, if $\{r_s : K \to K_s\}_{s\in\Sigma}$ is a retractional skeleton such that $\bigcup_{s\in\Sigma} K_s = K$ then $B = \bigcup_{s\in\Sigma} \overline{\mathrm{B}}_{C(K_s)'}$ is induced by a retractional skeleton, and therefore, it is countably tight by Theorem 24.10.4(a). This shows that B is a Corson compact. □

It is interesting to note that, assuming the Continuum Hypothesis, there exist non-separable Corson compact spaces carrying a strictly positive regular probability measure, see [18]. On the other hand, Martin's axiom implies that every Corson compact satisfying the countable chain condition (i.e. every collection of open sets is countable) is separable. In particular, assuming Martin's axiom, $P(K)$ is Corson compact if and only if K is Corson if and only if $C(K)$ is (WLD).

24.13 A Dichotomy

Further, we prove a dichotomy which implies that countably tight members of class $\mathcal{RC}$ are Corson. For continuous images of Valdivia compact spaces, this is due to Kalenda [356].

Theorem 24.13.1 ([379]) *Let $K \in \mathcal{RC}$. Then either $\omega_1 + 1$ embeds into K or else K is Corson compact.*

In the proof we shall use Bandlow's characterization (Corollary 24.10.7) in terms of elementary submodels. First, we need a technical fact.

Lemma 24.13.2 *Let $\mathbb{S} = \langle K_n; r_n^m; \omega\rangle$ be an inverse sequence of internal retractions of compact spaces with $K = \varprojlim \mathbb{S}$, represented so that all projections $r_n : K \to K_n$ are internal retractions. Let $f : K \to Y$ be a surjection and let, for each $n \in \omega$, $\mathcal{B}_n$ be an open base for $f[K_n]$ which is closed under finite unions. Then the family*

$$\mathcal{C} = \{Y \setminus f[K \setminus (fr_n)^{-1}[B]] : n \in \omega,\ B \in \mathcal{B}_n\}$$

is an open base for Y. In particular,

$$w(Y) = \sup_{n\in\omega} w(f[K_n]).$$

Proof Fix $y \in Y$ and its neighbourhood U. Find $m \in \omega$ and an open set $W \subseteq K_m$ such that

$$f^{-1}(y) \subseteq {r_m}^{-1}[W] \subseteq f^{-1}[U]. \tag{1}$$

Let

$$F = f[K \setminus {r_m}^{-1}[W]].$$

Then $y \notin F$, therefore $U_1 = Y \setminus F$ is a neighbourhood of y, contained in U. Now find $n \geqslant m$ and an open set $W_1 \subseteq K_n$ such that

$$f^{-1}(y) \subseteq {r_n}^{-1}[W_1] \subseteq f^{-1}[U_1].$$

Thus

$$r_n[f^{-1}(y)] \subseteq W_1$$

and $W_1 \subseteq f^{-1}[U_1]$. The latter inclusion follows from the fact that $W_1 \subseteq {r_n}^{-1}[W_1]$. Thus $fr_n[f^{-1}(y)] \subseteq U_1$. Using compactness and the fact that $\mathcal{B}_n$ is closed under finite unions, we can find $B \in \mathcal{B}_n$ such that

$$fr_n[f^{-1}(y)] \subseteq B \subseteq U_1.$$

We have

$$r_n[f^{-1}(y)] \subseteq f^{-1}[B] \subseteq {r_m}^{-1}[W],$$

where the latter inclusion follows from the equality $U_1 = Y \setminus f[K \setminus {r_m}^{-1}[W]]$. Thus,

$$f^{-1}(y) \subseteq {r_n}^{-1}[f^{-1}[B]] \subseteq {r_n}^{-1}[{r_m}^{-1}[W]] \subseteq (r_m r_n)^{-1}[W] = {r_m}^{-1}[W] \tag{2}$$

Hence, using (1) and (2), we obtain

$$f^{-1}(y) \subseteq {r_n}^{-1}[f^{-1}[B]] \subseteq f^{-1}[U].$$

Finally, set

$$V = Y \setminus f[K \setminus (fr_n)^{-1}[B]].$$

Then $V \in \mathcal{C}$ and $y \in V \subseteq U$. □

We are ready to prove Theorem 24.13.1.

Proof Every second countable compact space is Corson; therefore, the above dichotomy holds for spaces of $\mathcal{RC}$-rank 0. Fix an ordinal $\beta > 0$ and assume that the above statement is true for spaces of $\mathcal{RC}$-rank $< \beta$. Fix $X \in \mathcal{RC}$ such that $\mathrm{rk}_{\mathcal{RC}}(X) = \beta$. Let

$$\mathbb{S} = \langle K_\xi; r^\eta_\xi; \kappa \rangle$$

be a continuous inverse system of internal retractions such that $X = f[K]$ for some map $f\colon K \to X$, where $K = \varprojlim \mathbb{S}$ and $\mathrm{rk}_{\mathcal{RC}}(K_\xi) < \beta$ for every $\xi < \kappa$. We assume that κ is a regular cardinal. Let $X_\xi = f[K_\xi]$ for $\xi < \kappa$ and let $X_\kappa = X$. If X_ξ contains a copy of $\omega_1 + 1$ for some $\xi < \kappa$, then there is nothing to prove, so assume that X_ξ is Corson compact for each $\xi < \kappa$. Suppose first that

(*) $\kappa > \aleph_0$ and $X \neq \bigcup_{\xi<\kappa} X_\xi$.

Then

$$K \neq \bigcup_{\xi<\kappa} K_\xi.$$

Fix $y \in K$ so that $f(y) \notin \bigcup_{\xi<\kappa} X_\xi$. Recall that

$$y = \lim_{\xi\to\kappa} r_\xi(y).$$

Also

$$r_\delta(y) = \lim_{\xi<\delta} r_\xi(y),$$

by the continuity of the sequence $\mathbb{S}$. Thus the map $\varphi\colon \kappa + 1 \to K$, defined by $\varphi(\xi) = fr_\xi(y)$ and $\varphi(\kappa) = f(y)$, is continuous. By assumption, $\varphi(\xi) \neq f(y)$ for every $\xi < \kappa$, therefore the sequence

$$\{\varphi(\xi)\}_{\xi<\kappa}$$

does not stabilize. Find a closed cofinal set $C \subseteq \kappa$ such that $\varphi \restriction C$ is one-to-one. Then

$$Z = \varphi[C \cup \{\kappa\}] \subseteq X$$

is homeomorphic to the linearly ordered space $\kappa + 1$. Since $\kappa > \aleph_0$, this shows that X contains $\omega_1 + 1$.

Now suppose that (*) does not hold. We claim that X is Corson. For this aim, we use Bandlow's characterization. Fix a big enough cardinal χ and a countable $M \preceq H(\chi)$ such that $f, \mathbb{S} \in M$.

Let

$$\delta = \sup(\kappa \cap M)$$

(if $\kappa = \aleph_0$ then of course $\delta = \omega \subseteq M$ and $X_\delta = X$). Define

$$X_\xi^M = \overline{(X_\xi \cap M)}, \quad K_\xi^M = \overline{K_\xi \cap M)}$$

for $\xi < \kappa$. Let $q_M : X \to X/M$ denote the quotient map induced by M. In order to show that q_M is one-to-one on $X^M := \overline{(X \cap M)}$, it suffices to find a base for X^M which is contained in M.

Notice that

$$X \cap M \subseteq X_\delta.$$

Indeed, if $\kappa = \aleph_0$ then $X_\delta = X$; otherwise $X = \bigcup_{\xi<\kappa} X_\xi$ and therefore by elementarity

$$X \cap M \subseteq \bigcup_{\xi \in \kappa \cap M} X_\xi \subseteq X_\delta.$$

Now observe that

$$f[K_\xi^M] = X_\xi^M$$

for every $\xi \in \kappa \cap M$. Indeed, if $x \in X_\xi \cap M$ and $\xi \in M$ then by elementarity $x = f(y)$ for some $y \in K_\xi \cap M$, which shows that

$$X_\xi \cap M \subseteq f[K_\xi \cap M].$$

Clearly

$$f[K_\xi \cap M] \subseteq X_\xi \cap M.$$

Note that $f[K^M] = X^M$, because $f \in M$. On the other hand,

$$K^M = K_\delta^M = \overline{(\bigcup_{\xi \in \kappa \cap M} K_\xi^M)}.$$

Let

$$\{\xi_n : n \in \omega\} \subseteq \delta \cap M$$

be an increasing sequence such that $\delta = \sup_{n\in\omega} \xi_n$. Let

$$\mathbb{S}_0 = \langle Z_n;\, p_n^m;\, \omega\rangle,$$

where $Z_n = K^M_{\xi_n}$ and $p_n = r_{\xi_n} \restriction Z_n$. We apply Lemma 24.13.2 to find a suitable base for X^M. Fix $n \in \omega$. Then $X_{\xi_n} \in M$ and by assumption it is Corson compact. Thus M induces a quotient map which is one-to-one on $X^M_{\xi_n}$. In particular, the family

$$\mathcal{B}_n = \{U \in M : U \text{ open in } X_{\xi_n}\}$$

forms a base for $X^M_{\xi_n}$ (more precisely: $\{U \cap X^M_{\xi_n} : U \in \mathcal{B}_n\}$ is an open base for $X^M_{\xi_n}$). Clearly, $\mathcal{B}_n$ is closed under finite unions. Lemma 24.13.2 says that

$$\mathcal{C} = \{X^M \setminus f[K^M \setminus (fp_n)^{-1}[B]] : n \in \omega,\ B \in \mathcal{B}_n\}$$

is an open base for $X^M = f[K^M]$. Let

$$\mathcal{C}' = \{X \setminus f[K \setminus (fp_n)^{-1}[B]] : n \in \omega,\ B \in \mathcal{B}_n\}.$$

Then $\mathcal{C}' \subseteq M$ and

$$\{C \cap X^M : C \in \mathcal{C}'\} = \mathcal{C}.$$

Thus we have shown that M contains a family which forms an open base for X^M. Since M was arbitrary, this implies that X is Corson compact. □

24.14 Alexandrov Duplications

We are going to show that, contrary to Valdivia compact spaces, class $\mathcal{R}$ contains some non-metrizable compactifications of the natural numbers. Therefore $\mathcal{R}$ contains separable spaces which are not continuous images of Valdivia compact spaces.

Fix a topological space X and its subset A. Formally, assume $X \cap (X \times 2) = \emptyset$. Define the space $\mathrm{Dbl}(X, A)$ as

$$(X \times \{0\}) \cup (A \times \{1\})$$

endowed with the following topology: each point of the form $\langle x, 1\rangle$ is isolated and a neighbourhood of $\langle x, 0\rangle$ is of the form

$$(U \times \{0, 1\}) \setminus \{\langle x, 1\rangle\},$$

where U is a neighbourhood of x in X. This construction is due to Alexandrov and it is sometimes called the *Alexandrov duplication* of A in X.

Proposition 24.14.1 $\mathrm{Dbl}(K, A) \in \mathcal{R}$ *whenever* $K \in \mathcal{R}$ *and* $A \subseteq K$. *Moreover, if* A *is countable then* $\mathrm{rk}_{\mathcal{R}}(\mathrm{Dbl}(K, A)) \leqslant \mathrm{rk}_{\mathcal{R}}(K) + 1$.

Proof Fix $K \in \mathcal{R}$ and $A \subseteq K$. We use induction on the cardinality of A. Clearly $\mathrm{Dbl}(K, A) \in \mathcal{R}$ if A is finite. So assume

$$|A| = \kappa \geqslant \aleph_0$$

and $\mathrm{Dbl}(K, B) \in \mathcal{R}$ whenever $B \in [K]^{<\kappa}$. Write

$$A = \{a_\xi : \xi < \kappa\}.$$

Define $K_\xi = \mathrm{Dbl}(K, A_\xi)$, where $A_\xi = \{a_\eta : \eta < \xi\}$ and let

$$r_\xi : \mathrm{Dbl}(K, A) \to K_\xi$$

be defined by $r_\xi(x) = x$ for

$$x \in (K \times 0) \cup (A_\xi \times 1)$$

and $r_\xi(x, 1) = (x, 0)$ for $x \in A \setminus A_\xi$. Clearly r_ξ is a retraction and by inductive hypothesis, $K_\xi \in \mathcal{R}$ for every $\xi < \kappa$. We also have

$$r_\xi r_\eta = r_\xi = r_\eta r_\xi$$

whenever $\xi < \eta$.

Finally, observe that for every $x \in \mathrm{Dbl}(K, A)$ we have $r_\xi(x) = x$ for all but boundedly many $\xi < \kappa$. Thus, by Lemma 24.1.5, we have

$$\mathrm{Dbl}(K, A) = \varprojlim \langle K_\xi ; r_\xi^\eta ; \kappa \rangle,$$

where $r_\xi^\eta = r_\xi \restriction K_\eta$.

The second statement follows from the above proof and from the fact that

$$\mathrm{rk}_{\mathcal{R}}(K \oplus S) \leqslant \mathrm{rk}_{\mathcal{R}}(K)$$

whenever $|S| < \aleph_0$, see Corollary 24.4.7. □

Note that $\mathrm{Dbl}(2^{\aleph_1}, D)$ is a non-metrizable compactification of the natural numbers, whenever $D \subseteq 2^{\aleph_1}$ is countable and dense. Thus, class $\mathcal{R}$ contains non-metrizable compactifications of $\mathbb{N}$. We finish this section by exploring more properties of the space $\mathrm{Dbl}(2^{\aleph_1}, D)$.

Proposition 24.14.2 *Assume* K *is a compact space with a countable dense set consisting of* G_δ *points. If* $w(K)$ *has uncountable cofinality, then* $C(K)$ *has no PRI.*

Proof Let $\lambda = w(K)$ and suppose

$$\{P_\alpha : \alpha < \kappa\}$$

is a PRI on $C(K)$. Taking a cofinal subsequence, we may assume that $\kappa = \mathrm{cf}(\lambda)$. Then $\kappa > \aleph_0$. Let P'_α denote the adjoint operator on $C(K)'$. We have

$$\lim_{\xi\to\kappa}(P'_\xi\mu)f = \lim_{\xi\to\kappa}\mu(P_\xi f) = \mu(f),$$

which implies that $weak^*\text{-}\lim_{\xi\to\kappa} P'_\xi\mu = \mu$ for every $\mu \in C(K)'$. Moreover, each P'_α is $(weak^*, weak^*)$-continuous.

Denote by B the closed unit ball of $C(K)'$. Then

$$\{P'_\alpha B : \alpha < \kappa\}$$

is an increasing sequence of $weak^*$-closed subsets of B (we have $P_\alpha B \subseteq B$ because $\|P_\alpha\| = 1$). Moreover,

$$\bigcup_{\alpha<\kappa} P'_\alpha B$$

is $weak^*$-dense in B.

Now assume that $x \in K$ is a G_δ point and fix functions $f_n : K \to [0, 1]$ such that

$$\bigcap_{n\in\omega} \overline{f_n{}^{-1}[(0, 1]]} = \{x\}.$$

Then

$$\{\delta_x\} = \{\mu \in B : (\forall\, n \in \omega)\ \mu(f_n) = 1\}$$

and the set on the right-hand side of the above equality is G_δ in $\langle B, weak^*\rangle$ (δ_x denotes the Dirac measure of x). It follows that

$$\delta_x \in \overline{\{P'_\alpha\delta_x : \alpha \in C_x\}}$$

(the closure in $\sigma(X'X)$) for some countable set $C_x \subseteq \kappa$. Since $\mathrm{cf}(\kappa) > \aleph_0$, it follows that

$$\delta_x \in P'_{\beta(x)}B$$

for some $\beta(x) < \kappa$.

Now let $D \subseteq K$ be a countable dense set consisting of G_δ points. Then

$$\{\delta_x : x \in D\} \subseteq P'_\beta B,$$

where

$$\beta = \sup_{x \in D} \beta(x) < \kappa.$$

It remains to observe that $P'_\beta C(K)'$ is a proper $weak^*$-closed linear subspace of $C(K)'$ while at the same time the set

$$\{\delta_x : x \in D\}$$

is linearly $weak^*$-dense in $C(K)'$. This is a contradiction. □

Theorem 24.14.3 *Let $K = \mathrm{Dbl}(2^{\aleph_1}, D)$, where $D \subseteq 2^{\aleph_1}$ is countable and dense. Then*

(i) $K \in \mathcal{R}$ *and* $\mathrm{rk}_{\mathcal{R}}(K) = 2$;
(ii) K *is not a continuous image of a Valdivia compact space;*
(iii) $C(K)$ *has no PRI;*
(iv) $C(K)$ *is isomorphic to* $C\left(2^{\aleph_1}\right)$.

Proof Part 1 follows from the second part of Proposition 24.14.1; part 2 is a consequence of the fact that continuous images of Valdivia compact spaces have dense monolithic sets. Part 3 follows from Proposition 24.14.2. Part 4 is proved by a standard decomposition method, see e.g. [79]. □

24.15 Valdivia Compact Groups

We show that every Valdivia compact group is homeomorphic to a product of metric compact spaces. The case of Abelian groups (in fact a stronger statement) has already been presented in Theorem 24.6.4 and it comes from [381]. The general case has been proved by Chigogidze [136], using different arguments.

The following lemma is crucial.

Lemma 24.15.1 *Let $f : G \to H$ be a continuous epimorphism of topological groups and assume there exists a continuous map $j : H \to G$ such that $f \circ j = \mathrm{id}_H$. Then there exists a homeomorphism $\theta : G \to H \times \ker f$ such that*

$$\begin{array}{ccc} G & \xrightarrow{\ \theta\ } & H \times \ker f \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_H} & \\ & H & \end{array}$$

i.e. $f = \mathrm{pr}_H \circ \theta$, where $\mathrm{pr}_H : H \times \ker f \to H$ is the projection.

Note that θ is not necessarily a group homomorphism.

Proof Let $K = \ker f$ and define $\theta\colon G \to H \times K$ by setting

$$\theta(x) = \langle f(x),\, j \circ f(x) \cdot x^{-1} \rangle.$$

It is clear that θ is continuous and well defined since

$$f\left(j \circ f(x) \cdot x^{-1}\right) = f \circ j \circ f(x) \cdot f(x)^{-1} = f(x) \cdot f(x)^{-1} = 1,$$

where 1 denotes the unit of G. Now define $\varrho\colon H \times K \to G$ by

$$\varrho(p, q) = q^{-1} \cdot j(p).$$

Clearly, ϱ is continuous. Given $x \in G$ and $\langle p, q\rangle \in H \times K$, we have

$$\varrho(\theta(x)) = \varrho(f(x),\, j(f(x)) \cdot x^{-1})) = [j(f(x)) \cdot x^{-1})]^{-1} \cdot j(f(x)) = x$$

and

$$\begin{aligned}\theta(\varrho(p, q)) &= \langle f(q^{-1} \cdot j(p)),\, j \circ f(q^{-1} \cdot j(p)) \cdot [q^{-1} \cdot j(p)]^{-1} \rangle \\ &= \langle f(j(p)),\, j \circ f \circ j(p) \cdot j(p)^{-1} \cdot q \rangle = \langle p, q\rangle.\end{aligned}$$

Thus $\varrho \circ \theta = \mathrm{id}_G$ and $\theta \circ \varrho = \mathrm{id}_{H\times K}$, which shows that θ is a homeomorphism. Finally, $\mathrm{pr}_H \circ \theta = f$. □

Lemma 24.15.2 *Let $\mathbb{S} = \langle X_\xi;\, p^\eta_\xi;\, \kappa\rangle$ be a continuous inverse sequence of topological spaces and assume that for each $\alpha < \kappa$ there are a space Y_α and a homeomorphism $\theta_\alpha\colon X_{\alpha+1} \to X_\alpha \times Y_\alpha$ such that $p^{\alpha+1}_\alpha = \mathrm{pr}_{X_\alpha} \circ \theta_\alpha$, where pr_{X_α} is the projection. Then $\varprojlim \mathbb{S}$ is homeomorphic to*

$$X_0 \times \prod_{\alpha<\kappa} Y_\alpha.$$

Proof Let $Z_\alpha = X_0 \times \prod_{\xi<\alpha} Y_\xi$ and let $\langle Z_\xi;\, \mathrm{pr}^\eta_\xi;\, \kappa\rangle$ be the natural continuous inverse sequence of projections. Its limit is Z_κ. Define inductively homeomorphisms $\tau_\alpha\colon X_\alpha \to Z_\alpha$ which commute with both inverse sequences. The limit step is straight because of the continuity of both sequences. Suppose τ_α has been defined and let

$$\tau_{\alpha+1} = (\tau_\alpha \times \mathrm{id}_{Y_\alpha}) \circ \theta_\alpha.$$

Then $\tau_{\alpha+1}$ is a homeomorphism for which the following diagram commutes.

$$\begin{array}{ccc} X_\alpha & \xleftarrow{p_\alpha^{\alpha+1}} & X_{\alpha+1} \\ {\scriptstyle\tau_\alpha}\downarrow & & \downarrow{\scriptstyle\tau_{\alpha+1}} \\ Z_\alpha & \xleftarrow{\mathrm{pr}_\alpha^{\alpha+1}} & Z_{\alpha+1} \end{array}$$

Finally, the limit of τ_α's is the desired homeomorphism. □

Theorem 24.15.3 (Chigogidze [136]) *Every topological group which is a topological retract of a Valdivia compact is homeomorphic to a product of metric compact groups.*

Proof We use induction on the weight of G. The above statement is obvious in case $\mathrm{w}(G) \leqslant \aleph_0$. Assume $\kappa = \mathrm{w}(G)$ and the theorem is true for groups of weight $< \kappa$ which are retracts of Valdivia compact spaces.

Fix a group G with $\mathrm{w}(G) = \kappa$ and fix a retraction $r\colon K \to G$, where K is a Valdivia compact. Let $\lambda = \mathrm{cf}\,\kappa$ and fix a continuous chain $\{M_\alpha\}_{\alpha<\lambda}$ of elementary substructures of a given big enough $H(\chi)$ such that $r \in M_0$ and $\bigcup_{\alpha<\lambda} M_\alpha$ contains a fixed open base of G. Let $G_\alpha = G/M_\alpha$ and let p_α be the canonical quotient of G onto G_α. By Corollary 22.1.4, G_α is a topological group and p_α is a group homomorphism. By this way we obtain an inverse sequence of compact groups $\mathbb{S} = \langle G_\xi; p_\xi^\eta; \kappa\rangle$ whose limit is G and each p_ξ^η is a continuous epimorphism.

We claim that each p_α has a topological right inverse. Fix α and let $q_\alpha\colon K \to K/M_\alpha$ be the canonical quotient. We have the following commutative diagram.

$$\begin{array}{ccc} K & \xrightarrow{r} & G \\ {\scriptstyle q_\alpha}\downarrow & & \downarrow{\scriptstyle p_\alpha} \\ K/M_\alpha & \xrightarrow[r^{M_\alpha}]{} & G/M_\alpha \end{array}$$

By Lemma 22.7.2, q_α is a retraction. In order to conclude that p_α is right-invertible, it suffices to show that r^{M_α} is so.

For convenience, assume that r is an internal retraction, i.e. $G \subseteq K$ and $r \restriction G = \mathrm{id}_G$. Let $Z = q_\alpha[G] \subseteq K/M_\alpha$. Given $y \in G$, we have $p_\alpha(y) = p_\alpha(r(y)) = r^{M_\alpha}(q_\alpha(y))$. Thus, $r^{M_\alpha} \restriction Z$ is one-to-one and $r^{M_\alpha}[Z] = G/M_\alpha$, which shows that r^{M_α} is a retraction.

Finally, Lemmas 24.15.1 and 24.15.2 show that G is homeomorphic to the product

$$G_0 \times \prod_{\alpha<\lambda} H_\alpha,$$

where $H_\alpha = \ker(p_\alpha^{\alpha+1})$ for $\alpha < \lambda$. In particular, $G_{\alpha+1}$ is homeomorphic to $G_\alpha \times H_\alpha$ and hence H_α is a retract of $G_{\alpha+1}$. By induction hypothesis, each H_α is homeomorphic to a product of metric compact groups. □

The following question is left open: is every compact group from class $\mathcal{R}$ homeomorphic to a product of metric compact spaces?

24.16 Compact Lines in the Class $\mathcal{R}$

In this section, we come back to the discussion of compact lines in the classes $\mathcal{R}$, $\mathcal{RC}$ and their relations to Valdivia compact spaces. Recall that, by Nakhmanson's theorem (Theorem 23.2.1) and Pol's theorem (Corollary 24.11.4), Corson compact lines are necessarily metrizable.

Theorem 24.16.1 ([379]) *Valdivia compact lines have weight* $\leqslant \aleph_1$.

Proof Let K be a Valdivia compact line. An argument similar to the one showing that $\omega_2 + 1$ is not Valdivia proves that the character of K is at most $\aleph_1$.

The rest of the proof requires an argument from logic: supposing that K has weight $> \aleph_1$, one can construct a model of set theory in which there exists a non-metrizable linearly ordered Corson compact. We refer to [379] for the details. □

Theorem 24.16.2 *Linearly ordered spaces in class $\mathcal{RC}$ are monolithic.*

Before proving the above statement, we need the following fact which is a consequence of the dichotomy from Theorem 24.13.1 and Nakhmanson's theorem.

Lemma 24.16.3 *Assume $X \in \mathcal{RC}$ is linearly ordered and $f\colon X \to [0,1]$ is an increasing surjection. Then the set $S(f) = \{t \in [0,1]\colon |f^{-1}(t)| > 1\}$ is countable.*

Proof Suppose $S(f)$ is uncountable. Define

$$Y(t) = \begin{cases} [0,1] & t \in S(f),\ \ f^{-1}(t) \text{ is connected,} \\ \{0,1\} & f^{-1}(t) \text{ is disconnected,} \\ \{0\} & \text{otherwise.} \end{cases}$$

For each $t \in [0,1]$ let

$$h_t\colon f^{-1}(t) \to Y(t)$$

be an order preserving surjection. Let Y be the lexicographic product of $\{Y(t)\}_{t\in[0,1]}$. Then the family $\{h_t\}_{t\in[0,1]}$ induces an order preserving surjection $h\colon X \to Y$. Since $S(f)$ is uncountable, $Y \in \mathcal{RC}$ is a non-metrizable first countable space. This contradicts Theorem 24.13.1, because linearly ordered Corson compact spaces are metrizable. □

We are ready to prove Theorem 24.16.2

Proof Let $X \in \mathcal{RC}$ be a linearly ordered compact space. Fix a countable set $D \subseteq X$ and let $Y = \overline{D}$. Define an equivalence relation $\sim$ on X as follows:

$$x \sim y \iff [x, y] \cap Y \text{ is scattered.}$$

Note that the closure of a scattered set in a linearly ordered compact space is scattered. Observe that the $\sim$-equivalence classes are closed and convex. Let $K = X/\sim$ and let $q\colon X \to K$ be the quotient map. Then K is a connected and separable linearly ordered space, therefore order isomorphic to $[0, 1]$. By Lemma 24.16.3, the set

$$S(q) = \{t \in K\colon |q^{-1}(t)| > 1\}$$

is countable. For each $t \in S(q)$ choose a countable family $\mathcal{B}_t$ of open intervals of X which separates the points of $q^{-1}(t) \cap Y$ (recall that $q^{-1}(t) \cap Y$ is scattered, by the definition of $\sim$). Let

$$\mathcal{B} = \{q^{-1}[(r, \to)]\colon r \in Q\} \cup \bigcup_{t \in S(q)} \mathcal{B}_t,$$

where $Q \subseteq K$ is a countable dense set. Then $\mathcal{B}$ is a countable family of open sets which separates the points of Y. Thus $w(Y) \leqslant \aleph_0$. □

We now turn to connected spaces in class $\mathcal{R}$.

Recall that the *long ray* is the linearly ordered space $R^{\to} = \omega_1 \cdot [0, 1)$ (where $P \cdot Q$ denotes the lexicographic product of P with Q). Note that $R^{\to} + 1$ is a connected linearly ordered compact space with exactly one point of uncountable character (namely, the maximal point). Moreover, each proper interval which does not contain the maximal point is separable. Denote by ${}^{\leftarrow}R^{\to}$ the unique linearly ordered space $K = [a, b]$ such that for every internal point $p \in (a, b)$ the interval $[p, b]$ is order isomorphic to $R^{\to} + 1$ and the interval $[a, p]$ is order isomorphic to the inverse of $R^{\to} + 1$

Proposition 24.16.4 *The spaces $R^{\to} + 1$ and ${}^{\leftarrow}R^{\to}$ are Valdivia compact.*

Proof Let $K_\alpha = \delta \cdot [0, 1) + 1$ and define

$$r_\alpha\colon R^{\to} + 1 \to K_\alpha$$

by setting $r_\alpha(x) = x$ if $x \in K_\alpha$ and $r_\alpha(x) = \max(K_\alpha)$ otherwise. Then

$$\{r_\alpha\colon \alpha < \omega_1\}$$

is an inverse sequence of internal retractions in $R^{\to} + 1$ and each K_α is second countable. This shows that $R^{\to}+1$ is Valdivia compact. The case of ${}^{\leftarrow}R^{\to}$ is similar. □

Theorem 24.16.5 ([379]) *Let $K \in \mathcal{R}$ be a connected linearly ordered space. Then K is order isomorphic to one of the following spaces:* 1, $[0, 1]$, $R^{\to}+1$, $(R^{\to}+1)^{-1}$, ${}^{\leftarrow}R^{\to}$. *In particular, K is Valdivia compact.*

Proof Suppose the statement is false, and let K be a counterexample of a minimal possible $\mathcal{R}$-rank. Fix an inverse sequence of internal retractions

$$\{r_\alpha : \alpha < \kappa\}$$

in K such that $\mathrm{rk}_{\mathcal{R}}(K_\alpha) < \mathrm{rk}_{\mathcal{R}}(K)$ for every $\alpha < \kappa$, where $K_\alpha = r_\alpha[K]$. We assume that κ is a regular cardinal. Since K is connected, each K_α is an interval of the form $[a_\alpha, b_\alpha]$. Further, the sequence $\{a_\alpha\}_{\alpha<\kappa}$ is non-increasing and the sequence $\{b_\alpha\}_{\alpha<\kappa}$ is non-decreasing. Taking a cofinal set of indices and reversing the order of K if necessary, we may assume that

$$\{b_\alpha\}_{\alpha<\kappa}$$

is strictly increasing. Pick $p \in (a_0, b_0)$. By the minimality of $\mathrm{rk}_{\mathcal{R}}(K)$, each of the intervals $[a_\alpha, b_\alpha]$ is order isomorphic to one of the spaces listed in the theorem. It follows that

$$[p, b_\alpha] \subseteq (a_{\alpha+1}, b_{\alpha+1})$$

is order isomorphic to $[0, 1]$. Hence, $\kappa \in \{\aleph_0, \aleph_1\}$. Let $K = [a, b]$. Then $[p, b]$ is order isomorphic either to $[0, 1]$ (if $\kappa = \aleph_0$) or to $R^{\rightarrow} + 1$ (if $\kappa = \aleph_1$). By the same argument, $[a, p]$ is order isomorphic either to $[0, 1]$ or to the reversed $R^{\rightarrow} + 1$. This shows that K is not a counterexample, a contradiction. □

We now turn to Valdivia compact lines. As we have already proved, all of them have weight $\leqslant \aleph_1$. Further, we construct a universal one, namely a Valdivia compact line which maps onto any other (non-empty!) Valdivia compact line.

Let 2^ω denote, as usual, the Cantor set. Denote by $\pi : 2^\omega \to 2^\omega$ the unique increasing map satisfying the following conditions.

(1) $\pi^{-1}(t)$ is order isomorphic to the Cantor set whenever t is external in 2^ω, i.e. t is isolated from one side (in particular when $t = 0$ or $t = 1$).
(2) $|\pi^{-1}(t)| = 1$ if t is internal in 2^ω.

It is easily seen that π is a retraction. A possible right inverse $i : 2^\omega \to 2^\omega$ to π can be defined by setting $i(p) = \min \pi^{-1}(p)$, in case where p is isolated from the right, $i(p) = \max \pi^{-1}(p)$, in case where p is isolated from the left and $i(p) = p$ otherwise.

Given two increasing maps $f, g : X \to Y$, we shall say that f is *order isomorphic* to g if there are order isomorphisms $\theta_X : X \to X$ and $\theta_Y : Y \to Y$ such that $\theta_Y \circ f = g \circ \theta_X$.

Lemma 24.16.6 *(a) $\pi \circ \pi$ is order isomorphic to π.*
(b) Let $\mathbb{S} = \langle C_m; p^n_m; \omega\rangle$, where $C_n = 2^\omega$ and $p^{n+1}_n = \pi$ for each $n \in \omega$. Then $\varprojlim \mathbb{S}$ is order isomorphic to 2^ω and every projection from $\varprojlim \mathbb{S}$ is order isomorphic to π.

Proof It is clear that $\pi \circ \pi$ satisfies the above conditions (1) and (2). For the proof of (b), notice that $C = \varprojlim \mathbb{S}$ is a compact 0-dimensional metric line without isolated points, therefore it is order isomorphic to 2^ω. Finally, every projection from the limit again satisfies (1) and (2). □

The construction is based on the following crucial fact.

Lemma 24.16.7 *Let K, L be compact metric lines, let $f\colon 2^\omega \to L$ and $r\colon K \to L$ be continuous increasing quotients. Assume further that r is a retraction. Then there exists a continuous increasing quotient $g\colon 2^\omega \to K$ for which the following diagram commutes.*

$$\begin{array}{ccc} 2^\omega & \xrightarrow{g} & K \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle r} \\ 2^\omega & \xrightarrow[f]{} & L \end{array}$$

Proof We assume that $L \subseteq K$ and $r \restriction L = \mathrm{id}_L$. Let Q denote the set of all external points of the Cantor set 2^ω.

Given $z \in L$ define $F_z = \pi^{-1}[f^{-1}(z)]$. Note that F_z is a closed interval and hence, since 2^ω is dense-in-itself, F_z is either order isomorphic to the Cantor set or else $|F_z| \leqslant 2$. Now observe that the required map g must satisfy

$$g[F_z] = r^{-1}(z) \quad \text{for every } z \in L. \tag{24.2}$$

Conversely, if $g\colon 2^\omega \to K$ satisfies (24.2) then it is an order preserving surjection such that $r \circ g = f \circ \pi$ holds. Thus, it suffices to show that F_z has an order preserving map onto $r^{-1}(z)$ for every $z \in L$.

Fix $z \in L$. Then $r^{-1}(z)$ is a closed metrizable interval in K containing z (recall that r is an internal retraction). If $r^{-1}(z) = \{z\}$ then there is nothing to prove, so assume $r^{-1}(z) = [a, b]$, where $a < b$. Note that $L \cap [a, b] = \{z\}$. Thus, if $z \neq b$ then z is isolated from the right in L and consequently $\max f^{-1}(z) \in Q$. Similarly, if $z \neq a$ then $\min f^{-1}(z) \in Q$. In both cases, $f^{-1}(z) \cap Q \neq \emptyset$ and hence $F_z = \pi^{-1}[f^{-1}(z)]$ is order isomorphic to the Cantor set, therefore it has an order preserving map onto every linearly ordered metric compact, in particular onto $r^{-1}(z)$. This completes the proof. □

Let us now define $C_{\omega_1} = \varprojlim \mathbb{S}$, where $\mathbb{S} = \langle C_\xi; \pi^\eta_\xi; \omega_1 \rangle$ is a continuous inverse sequence in which each $\pi^{\xi+1}_\xi$ is isomorphic to π. Such a sequence exists by Lemma 24.16.6 and by the same lemma (together with transfinite induction) every bonding map π^η_ξ is isomorphic to π.

Theorem 24.16.8 *The space C_{ω_1} is a Valdivia compact line such that for every non-empty Valdivia compact line K there exists a continuous increasing surjection*

$$F\colon C_{\omega_1} \to K.$$

Proof That C_{ω_1} is Valdivia compact follows from the fact that $\mathbb{S}$ is a retractive inverse sequence of metric compact spaces.

Fix a non-empty Valdivia compact line K. By Theorem 24.16.1, the weight of K does not exceed $\aleph_1$. Let $\mathbb{T} = \langle K_\xi; r^\eta_\xi; \omega_1\rangle$ be a continuous inverse sequence of continuous increasing maps onto compact metric lines. We may assume that $|K_0| = 1$. Since K is Valdivia, we may also assume that each r^η_ξ is a retraction.

We construct inductively continuous surjections $f_\alpha\colon 2^\omega \to K_\alpha$ so that $r^\beta_\alpha \circ f_\beta = f_\alpha \circ \pi^\beta_\alpha$ holds for every $\alpha < \beta < \omega_1$. The limit of $\{f_\alpha\}_{\alpha<\omega_1}$ will then be the desired increasing surjection from C_{ω_1} onto K.

Fix $\alpha < \omega_1$ and suppose f_ξ have already been defined for $\xi < \alpha$. If α is a limit ordinal then we define f_α to be the limit of $\{f_\xi\}_{\xi<\alpha}$. The continuity of both sequences guarantees that f_α is a continuous increasing surjection.

Suppose now that $\alpha = \xi + 1$. By Lemma 24.16.7, there exists a continuous increasing surjection $g\colon 2^\omega \to K_\alpha$ such that $r^{\xi+1}_\xi \circ g = f_\xi \circ \pi = f_\xi \circ \pi^{\xi+1}_\xi$. Set $f_\alpha := g$. By induction hypothesis, $r^\alpha_\xi \circ f_\alpha = f_\xi \circ \pi^\alpha_\xi$ holds for every $\xi < \alpha$. □

We finish with an example answering the problem on subspaces of Plichko spaces.

Example 24.16.9 Let K_{ω_1} be the connectification of C_{ω_1} (see Proposition 23.1.4). Then $E = C\big(K_{\omega_1}\big)$ is linearly isometric to a subspace of $C\big(C_{\omega_1}\big)$. On the other hand, E is not a Plichko space.

Proof Take a countable elementary submodel of some big enough $H(\chi)$, so that $E \in M$. Then $\overline{(E \cap M)}$ is induced by the quotient map $q_M\colon C_{\omega_1} \to C_{\omega_1}/M$. Observe that no interval of C_{ω_1} is metrizable. It follows that the set

$$\{t \in C_{\omega_1}/M\colon |q_M^{-1}(t)| > 1\}$$

is everywhere dense. By Theorem 22.4.5, q_M admits no averaging operator, that is $C(K/M)$ is not is not complemented in E. This shows that E is not Plichko. □

Notice that E has the regular separable complementation property, by Theorem 23.3.2 together with Theorem 24.16.2.

24.17 More on Eberlein Compact Spaces

Recall again that a space K is called *Eberlein compact* if it is homeomorphic to a weakly compact subset of a Banach space. We recall also the following famous result of Amir and Lindenstrauss [8].

Theorem 24.17.1 *Eberlein compact spaces are precisely those compact spaces that, for some set* Γ*, embed topologically into* $\langle c_0(\Gamma), \tau_p\rangle$.

Proof Fix a set Γ. We check that the pointwise topology coincides with the weak one on every bounded subset of $c_0(\Gamma)$. For this aim, fix a bounded net $\{x_s\}_{s\in\Sigma} \subseteq c_0(\Gamma)$ pointwise converging to zero. We shall show that it converges weakly to zero. We may assume that $\|x_s\| \leqslant 1$ for every $s \in \Sigma$. Fix $y \in \ell_1(\Gamma) = c_0(\Gamma)'$ and fix $\varepsilon > 0$. There is a finite set $A \subseteq \Gamma$ such that

$$\sum_{i\in\Gamma\setminus A} |y(i)| < \varepsilon/2.$$

Let $n = |A|$ and for each $a \in A$ find $t_a \in \Sigma$ such that

$$|x_s(a)| < \varepsilon/(2n|y(a)|)$$

whenever $s \geqslant t_a$. Let $t \in \Sigma$ be such that $t_a \leqslant t$ for $a \in A$. If $s \geqslant t$ then

$$|y(x_s)| = \left|\sum_{i\in\Gamma} x_s(i)y(i)\right| \leqslant \sum_{a\in A} |x_s(a)|\cdot|y(a)| + \sum_{i\in\Gamma\setminus A} |x_s(i)|\cdot|y(i)|$$

$$< \varepsilon/2 + \sum_{i\in\Gamma\setminus A} |y(i)| < \varepsilon.$$

Hence, $|y(x_s)| < \varepsilon$ for every $s \geqslant t$. Since y was arbitrary, this shows that x_s converges weakly to zero. We conclude that the pointwise and weak topologies coincide on compact subsets of $c_0(\Gamma)$. In particular, every compact subset of $\langle c_0(\Gamma), \tau_p\rangle$ is Eberlein compact.

Now fix an Eberlein compact K inside a Banach space E with the weak topology. Without loss of generality we may assume that K is linearly dense in E. Now E is weakly compactly generated, and in particular it has a projectional skeleton by Proposition 22.2.3 and Theorem 22.6.2. Hence, by Corollary 22.5.6, there is an injective bounded linear operator $T : E \to c_0(\Gamma)$ for some set Γ. It is clear that T is continuous with respect to the weak topologies on both spaces, therefore K embeds into $c_0(\Gamma)$ with the weak topology. By the above remarks, we can replace the weak topology by the pointwise one. □

Corollary 24.17.2 *Eberlein compact spaces are Corson.*

As another corollary, we obtain the well-known covering-type characterization of Eberlein compact spaces, originally due to Rosenthal. Recall that a family $\mathcal{A}$ is T_0 *separating* in X if for every $x, y \in X$ there is $A \in \mathcal{A}$ such that $|\{x, y\} \cap A| = 1$.

Theorem 24.17.3 *Let K be a compact space. Then K is Eberlein if and only if there exist a sequence $\{\mathcal{F}_n\}_{n\in\omega}$ of families of open F_σ subsets of K such that each $\mathcal{F}_n$ is point-finite and $\bigcup_{n\in\omega} \mathcal{F}_n$ is T_0 separating in K.*

If K is a zero-dimensional Eberlein compact, then the above sequence $\{\mathcal{F}_n\}_{n\in\omega}$ may consist of clopen sets.

Proof Assume first that K has the sequence of families $\{\mathcal{F}_n\}_{n\in\omega}$ satisfying the above conditions. For each $U \in \mathcal{F}_n$ choose a continuous map

$$f_U : K \to \left[0, \frac{1}{n}\right]$$

such that

$$U = {f_U}^{-1}\left[\left(0, \frac{1}{n}\right]\right]$$

(here we use that fact that U is open and F_σ). Let $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$ and finally, let $f : K \to [0,1]^{\mathcal{F}}$ be the diagonal of $\{f_U\}_{U\in\mathcal{F}}$, i.e. $f(x)(U) = f_U(x)$ for $x \in K$, $U \in \mathcal{F}$. Since $\mathcal{F}$ is T_0 separating, f is injective. By compactness, it is a topological embedding. It remains to check that $f[K] \subseteq c_0(\mathcal{F})$. For fix $x \in K$, fix $\varepsilon > 0$ and choose $k \in \omega$ so that $\frac{1}{k} < \varepsilon$. Then $|f_U(x)| < \varepsilon$ whenever $U \in \bigcup_{n\geqslant k} \mathcal{F}_n$. On the other hand,

$$\mathcal{F}_0 \cup \cdots \cup \mathcal{F}_{k-1}$$

is point finite, therefore $|f_U(x)| < \varepsilon$ for all but finitely many $U \in \mathcal{F}$. This shows that $f(x) \in c_0(\mathcal{F})$.

Assume now that K is an Eberlein compact embedded into $c_0(\Gamma)$ with the pointwise topology. Let $\{q_n\}_{n\in\omega}$ be a fixed enumeration of all non-zero rational numbers. Given $n \in \omega$, define

$$V_{\alpha,n} = \{x \in K : x(\alpha) > q_n\}$$

if $q_n > 0$, and

$$V_{\alpha,n} = \{x \in K : x(\alpha) < q_n\}$$

if $q_n < 0$. Note that $V_{\alpha,n}$ is an open F_σ set and for each $x \in K$, for a fixed n, there exist only finitely many $\alpha \in \Gamma$ such that $x \in V_{\alpha,n}$. It follows that

$$\mathcal{F}_n = \{V_{\alpha,n} : \alpha \in \Gamma\}$$

is point finite. It remains to check that $\bigcup_{n\in\omega} \mathcal{F}_n$ is T_0 separating. For fix $x \neq y$ in K and fix α such that $x(\alpha) \neq y(\alpha)$. Suppose $x(\alpha) < y(\alpha)$. Find n so that $x(\alpha) < q_n < y(\alpha)$. If $q_n > 0$ then $y \in V_{\alpha,n}$ and $x \notin V_{\alpha,n}$ while if $q_n < 0$ then $x \in V_{\alpha,n}$ and $y \notin V_{\alpha,n}$.

Assume now that K is a zero-dimensional Eberlein compact and let $\{\mathcal{F}_n\}_{n\in\omega}$ be as above. Using zero-dimensionality and the fact that $V_{\alpha,k}$ is open F_σ, we can find clopen sets $V^n_{\alpha,k}$ such that

$$V_{\alpha,k} = \bigcup_{n\in\omega} V^n_{\alpha,k}.$$

Now let

$$\mathcal{F}_k^n = \{V_{\alpha,k}^n : \alpha \in \Gamma\}.$$

Then $\mathcal{F}_k^n$ is point finite and

$$\bigcup_{n,k\in\omega} \mathcal{F}_k^n = \bigcup_{n\in\omega} \mathcal{F}_n$$

is T_0 separating. Arranging the families $\mathcal{F}_k^n$ into a sequence completes the proof of the second statement. □

We now turn to the properties of spaces of continuous functions over Eberlein compact spaces.

Lemma 24.17.4 *Let K be an Eberlein compact. Then $C_p(K)$ contains a set homeomorphic to the one point compactification of a discrete space that separates the points of K.*

Proof By Theorem 24.17.1 we have $K \subseteq c_0(\kappa)$ for some κ, where $c_0(\kappa)$ is considered with the pointwise topology. Let $f_\alpha : K \to \mathbb{R}$ be the restriction to K of the projection onto the α-th coordinate. That is, $f_\alpha(x) = x(\alpha)$ for $x \in K$. Let

$$L = \{f_\alpha : \alpha \in \kappa\} \cup \{0\}.$$

Then $L \subseteq C_p(K)$ and it separates the points of K. Fix a subbasic neighbourhood V of $0 \in L$. That is,

$$V = \{f : |f(x)| < \varepsilon\}$$

for some $x \in K$ and $\varepsilon > 0$. There is a finite set $S \subseteq \kappa$ such that $|x(\alpha)| < \varepsilon$ for $\alpha \in \kappa \setminus S$. Thus $f_\alpha \in V$ if and only if

$$\alpha \in \kappa \setminus S.$$

This shows that L is homeomorphic to the Alexandrov compactification of the discrete set $\{f_\alpha : \alpha \in \kappa\}$. □

Because of the above lemma, it is natural to ask which compact spaces K have a compact set $L \subseteq C_p(K)$ which separates the points of K. It turns out that these are again Eberlein compact spaces. Namely:

Theorem 24.17.5 *Let K be a compact space. Then K is Eberlein if and only if there exists a compact $L \subseteq C_p(K)$ that separates the points of K. In that case, L is also Eberlein compact.*

Before proving this result, we need two lemmas which may be of independent interest.

Lemma 24.17.6 *Let K, L be compact spaces such that $L \subseteq C_p(K)$ and L separates the points of K. Then there exists $\widetilde{K} \subseteq C_p(L)$, homeomorphic to K, which separates the points of L.*

Proof Given $x \in K$, define $\widetilde{x}(f) = f(x)$ for $f \in L$. Then $\widetilde{x}$ is a continuous function on L, i.e. $\widetilde{x} \in C_p(L)$. It is clear that the map $x \mapsto \widetilde{x}$ is continuous. It is one-to-one, because L separates the points of K. Let $\widetilde{K} = \{\widetilde{x}\colon x \in K\}$. If $f \neq g$ in L then $f(x) \neq g(x)$ for some $x \in K$ and hence $\widetilde{x}(f) \neq \widetilde{x}(g)$. This shows that $\widetilde{K}$ separates the points of L. □

Lemma 24.17.7 *Let K be a compact space and let L be a compact subset of $C_p(K)$ which separates the points of K. Furthermore, let M be an elementary submodel of some big enough $\langle H(\chi), \in\rangle$ such that $K, L \in M$. Let $K_1 = \overline{(K \cap M)}$ and $L_1 = \overline{(L \cap M)}^{\tau_p}$, where τ_p denotes the pointwise convergence topology. Then:*

(a) There exists a unique retraction $r\colon K \to K$ which satisfies $r[K] = K_1$ and $f(r(x)) = f(x)$ for every $x \in K$ and for every $f \in L_1$.
(b) The adjoint operator $P\colon C(K) \to C(K)$ (that is, the operator defined by $Pf = f \circ r$) is (τ_p, τ_p)-continuous and $P[L] = L_1$.
(c) $L_1 \restriction K_1$ is separates the points of K_1 and is compact with respect to τ_p on $C(K_1)$.

The above lemma says in particular that $\overline{(K \cap M)}$ is homeomorphic to K/M and $\overline{(L \cap M)}^{\tau_p}$ is homeomorphic to L/M.

Proof Given $S \subseteq C(K)$ define $\varphi_S\colon K \to C(S)$ by setting $\varphi_S(x)(f) = f(x)$. Then φ_S is continuous with respect to the pointwise topology on $C(S)$.

Given $T \subseteq K$ we denote by π_T the restriction operator from $C(K)$ to $C(T)$. Let $K_0 = K \cap M$, $L_0 = L \cap M$. We first note that, by elementarity, $\varphi_{L_0}[K_0]$ is dense in $\varphi_{L_0}[K]$ and $\pi_{K_0}[L_0]$ is dense in $\pi_{K_0}[L]$. Indeed, a basic neighbourhood of $\varphi_{L_0}(x)$ is of the form

$$V = \{p \in \mathbb{R}^{L_0}\colon (\forall\, i < k)\ p(f_i) \in I_i\},$$

where $f_0, \ldots, f_{k-1} \in L_0$ and each I_i is a rational interval around $\varphi_{L_0}(x)$. Thus V is defined in M and by elementarity there is $x_0 \in K \cap M$ such that $x_0 \in V$. A similar argument shows the density of $\pi_{K_0}[L_0]$ in $\pi_{K_0}[L]$.

Now observe that $\varphi_{L_0}[K_1] = \varphi_{L_0}[K]$, because K_1 is compact and hence $\varphi_{L_0}[K_1]$ is a dense closed subset of $\varphi_{L_0}[K]$. Similarly, $\pi_{K_0}[L_1] = \pi_{K_0}[L]$.

We argue that

$$\varphi_{L_1}[K_1] = \varphi_{L_1}[K] \quad \text{and} \quad \pi_{K_1}[L_1] = \pi_{K_1}[L]. \tag{24.3}$$

Indeed, if $x \in K$ then

$$\varphi_{L_1}(x) \restriction L_0 = \varphi_{L_0}(x) = \varphi_{L_0}(x_1) = \varphi_{L_1}(x_1) \restriction L_0$$

for some $x_1 \in K_1$ and since L_0 is dense in L_1, it must be the case that $\varphi_{L_1}(x) = \varphi_{L_1}(x_1)$. Similarly, if $f \in L$ then $\pi_{K_0}(f) = \pi_{K_0}(f_1)$ for some $f_1 \in L_1$ and therefore $\pi_{K_1}(f) = \pi_{K_1}(f_1)$.

Fix $x_0 \neq x_1$ in K_1. There exists $f \in L$ with $f(x_0) \neq f(x_1)$. By (24.3) there is $f_1 \in L_1$ with $f \restriction K_1 = f_1 \restriction K_1$. It follows that $L_1 \restriction K_1$ separates the points of K_1. Let $h = \varphi_{L_1} \restriction K_1$. Then $h\colon K_1 \to C_p(L_1)$ is an embedding.

Define $r = h^{-1} \circ \varphi_{L_1}$. Then $r\colon K \to K_1$ is a well defined continuous map such that $f(r(x)) = \varphi_{L_1}(x)(f) = f(x)$ holds for every $f \in L_1$ and for every $x \in K$. Moreover if $x \in K_1$ then $x = h(\varphi_{L_1}(x))$, i.e. $r(x) = x$. Thus r is the desired retraction. □

Proof of Theorem 24.17.5 By Lemma 24.17.6, the assumptions are symmetric; therefore, it suffices to show that L is Eberlein. In view of Theorem 24.17.1, it is sufficient to show that $C(K)$ has a one-to-one bounded linear operator into some $c_0(\Gamma)$.

Using Lemma 24.17.7, we construct a PRI in $C(K)$ onto subspaces of the form $C(K_\alpha)$, where K_α is a compact of smaller weight, satisfying the same assumptions as K. Specifically, letting κ be the weight of K, we fix a chain $\{M_\alpha\}_{\alpha<\kappa}$ of elementary submodels of some big enough $H(\chi)$ so that $|M_\alpha| \leqslant \alpha+\aleph_0$, $K, L \in M_0$ and $\bigcup_{\alpha<\kappa} M_\alpha$ contains a linearly dense subset of $C(K)$. Lemma 24.17.7 says that $\{M_\alpha\}_{\alpha<\kappa}$ induces a PRI on $C(K)$ onto subspaces of the form $C(K/M_\alpha)$ and moreover L/M_α separates the points of K/M_α for each $\alpha < \kappa$. An argument like in the proof of Corollary 22.5.6 shows that $C(K)$ has a one-to-one bounded τ_p-continuous linear operator into some $c_0(\Gamma)$. In particular, L embeds into $c_0(\Gamma)$. This completes the proof. □

We now discuss stability of the class of Eberlein compact spaces. Straight from the definition, closed subspace of an Eberlein compact is Eberlein. It is not hard to see that a countable product of Eberlein compact spaces is Eberlein. A non-trivial result, due to Benyamini, Rudin, and Wage [78], says that the class of Eberlein compact spaces is stable under continuous images. It is interesting that the dual class of (WCG) Banach spaces is not stable under closed subspaces.

Further, we give a simple proof of the mentioned stability result in cases of totally disconnected spaces. It is based on the following lemma, which will be used later too.

Lemma 24.17.8 *Let K be a totally disconnected compact. Then K is Eberlein if and only if $C_p(K, 2)$ is σ-compact.*

Proof Assume first that K is Eberlein and let $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ be a T_0 separating family of clopen sets such that each $\mathcal{U}_n$ is point-finite (Theorem 24.17.3). Observe that, identifying clopen sets with their characteristic functions, $\mathcal{U}_n \cup \{\emptyset\}$ is homeomorphic to the one-point compactification of a discrete space ($\emptyset$ is the only possible accumulation point). Thus, $\mathcal{U}$ is σ-compact. Finally, $C(K, 2)$ is, as a Boolean algebra, generated by (the characteristic functions of) $\mathcal{U}$; therefore, it is σ-compact too.

Now assume that $C_p(K,2) = \bigcup_{n\in\omega} L_n$ where each L_n is compact. Let

$$L = \bigcup_{n\in\omega} \frac{1}{n} L_n.$$

It is easy to check that L is compact in the pointwise topology. Finally, L clearly separates the points of K. □

Theorem 24.17.9 *Let $f\colon K \to L$ be a quotient map of totally disconnected compact spaces. If K is Eberlein, then so is L.*

Proof It is clear that $C_p(L,2)$ embeds into $C_p(K,2)$ as a closed subspace. Apply Lemma 24.17.8. □

We now discuss a very special class of compact spaces introduced by Talagrand [579], which provides interesting examples concerning Eberlein compact spaces and their relatives.

We identify a Cantor cube 2^S with the power set $\mathcal{P}(S)$. By this way, any family of sets becomes a totally disconnected Hausdorff topological space.

A family of sets K is *adequate* if it satisfies the following condition

$$x \in K \iff [x]^{<\omega} \subseteq K. \tag{AD}$$

It is clear that adequate families are compact in the Cantor cube topology: the fact that $x \notin K$ is witnessed by a finite set which provides an appropriate neighbourhood.

An *adequate compact* is simply an adequate family, considered as a (compact) topological space.

Observe that every adequate compact is Valdivia. It is rather clear that an adequate family is a Corson compact if and only if it consists of countable sets (otherwise it contains a non-metrizable Cantor cube, which is not Corson).

Fix an adequate family $K \subseteq \mathcal{P}(S)$. We define the *adequate dual* of K to be the topological space $K^\# = \langle S \cup \{\infty\}, \tau_K\rangle$, where $\infty \notin S$, all points of S are isolated in τ_K and a subbasic neighbourhood of ∞ is of the form $(S \cup \{\infty\}) \setminus a$, where $a \in K$. The next lemma shows the relation between $K^\#$ and K.

Lemma 24.17.10 ([579]) *Let K be an adequate compact. Then $K^\#$ is homeomorphic to a closed subspace of $C_p(K,2)$ separating the points of K.*

Proof Let $K \subseteq \mathcal{P}(S)$ and assume $\{\alpha\} \in K$ for every $\alpha \in S$ (otherwise replace S by $\bigcup K$).

Given $\alpha \in S$, let $\alpha^+ = \{x \in K : \alpha \in x\}$. We claim that $K^\#$ is homeomorphic to $X = \{\emptyset\} \cup \{\alpha^+ : \alpha \in S\}$, where elements of X are identified with their characteristic functions (and so $X \subseteq C_p(K,2)$). We also need to show that X is closed.

For the proof, observe that K is a compact semilattice when endowed with the binary operation $\langle x, y\rangle \mapsto x \cap y$. Let L be the set of all continuous homomorphisms of $f\colon K \to 2$ satisfying $f(\emptyset) = 0$. Here, $2 = \{0,1\}$ is endowed with the usual semilattice structure: $i \wedge j = \min\{i,j\}$.

Identify elements of X (which are clopen subsets of K) with their characteristic functions. Observe that $X \subseteq L$, where $\emptyset$ corresponds to the zero homomorphism.

It is clear that L is closed in $C_p(K, 2)$, since if $f \in C(K, 2)$ is not a semilattice homomorphism then this fact is witnessed by two elements of K that provide an appropriate neighbourhood of f disjoint from L.

Now fix $\varphi \in L \setminus 0$ and let a be the minimal element of $\varphi^{-1}(1)$. This element exists by compactness and by the fact that $\varphi^{-1}(1)$ is closed under finite meet (intersection). Notice that a must be finite, since otherwise it would be an accumulation point of its finite subsets which are in $\varphi^{-1}(0)$. Now φ is the unique $f \in L$ satisfying the equations

$$f(a) = 1 \text{ and } (\forall\, b \subsetneq a)\ f(b) = 0.$$

This shows that φ is isolated in L. Finally, observe that the characteristic function of α^+ is an element of L and $\emptyset$ corresponds to the zero homomorphism. In particular, a subbasic neighbourhood of $\emptyset$ is of the form $p^- = \{x \in L\colon p \notin x\}$, where $p \in K$.

By this way we have shown that the map $h\colon K^\# \to X$, defined by $h(\alpha) = \alpha^+$ and $h(\infty) = \emptyset$, is a homeomorphism. Finally, X is closed in $C_p(K, 2)$, because $L \setminus X$ consists of isolated points. It is obvious that X separates the points of K. □

Further, we characterize adequate Eberlein compact spaces.

Theorem 24.17.11 *Let K be an adequate family of subsets of a set κ. Then K is Eberlein compact if and only if $\kappa = \bigcup_{n\in\omega} E_n$, where for each $n \in \omega$ the family $K \cap \mathcal{P}(E_n)$ consists of finite sets only.*

Proof By Lemmas 24.17.8 and 24.17.10, K is Eberlein if and only if $K^\#$ is σ-compact.

It remains to observe that, assuming $\kappa = \bigcup K$, a compact subset of $K^\#$ is either finite or has the form $A \cup \{\infty\}$, where $A \subseteq \kappa$ and $A \cap x$ is finite for every $x \in K$. Indeed, if $x \in K$ then $A \cap x$ is a closed discrete subset of $K^\#$. □

The above result gives a tool for finding non-Eberlein Corson compact spaces. Further, we describe a specific subclass of adequate compact spaces coming from partially ordered sets.

Given a poset P let $K(P)$ denote the family of all chains (i.e. linearly ordered subsets) of P. Clearly, $K(P)$ is adequate. The space $K(P)$ is Corson if and only if all chains in P are countable. The next example was found by Alster and Pol [7]; the idea goes back to Sierpiński.

Example 24.17.12 Let $S \subseteq \mathbb{R}$ be a set of cardinality $\aleph_1$ and let $\preceq$ be a fixed well order on S of order type ω_1. Define $s \trianglelefteq t$ if and only if both $s \leqslant t$ and $s \preceq t$ hold, where $\leqslant$ denotes the usual linear order on S (taken from the reals). Then $\langle S, \trianglelefteq\rangle$ is a poset in which all chains are countable, since no uncountable subset of $\mathbb{R}$ can be well ordered. Thus, the adequate family $K(S)$ consisting of all chains in $\langle S, \trianglelefteq\rangle$ is Corson compact. We claim that it is not Eberlein compact.

Suppose otherwise. By Theorem 24.17.11, $S = \bigcup_{n\in\omega} S_n$, where for each n, all chains in S_n are finite. Let $k \in \omega$ be such that S_k is uncountable. Find $C \subseteq S_k$ such that $\{x \in C : c < x\}$ is uncountable for every $c \in C$ (recall that $<$ is the usual order on the real line, so this is possible). Note that $\langle C, \preceq\rangle$ has order type ω_1. In particular, for each $c \in C$, there exist only countably many $y \in C$ satisfying $y \preceq c$. It follows that for every $c \in C$ there exists $c^* \in C$ such that both $c < c^*$ and $c \preceq c^*$. In other words, $c \trianglelefteq c^*$ and $c \neq c^*$. This shows that C contains an infinite chain with respect to $\trianglelefteq$, a contradiction.

A poset P will be called *special* if $P = \bigcup_{n\in\omega} P_n$, where each set P_n is an *antichain*, i.e. it consists of pairwise incomparable elements. The notion of being special is usually connected with trees. Recall that a *tree* is a poset $\langle T, <\rangle$ in which initial intervals $(\leftarrow, t) = \{s : s < t\}$ are well ordered. It is usually assumed (and we shall do so) that a tree is a meet semilattice, i.e. $\inf\{s, t\}$ exists for every $s, t \in T$. This is equivalent to saying that $(\leftarrow, t) = (\leftarrow, s)$ implies $t = s$ for every $s, t \in T$. In particular, every tree has the minimal element, usually denoted by 0 or 0_T.

Theorem 24.17.13 *Let T be a tree. Then the space of chains $K(T)$ is Eberlein compact if and only if T is special.*

Proof If T is special then the condition from Theorem 24.17.11 is satisfied. Assume that $K(T)$ is Eberlein and apply Theorem 24.17.11. Thus, $T = \bigcup_{n\in\omega} T_n$, where all chains in T_n are finite. It suffices to show that each T_n is a countable union of antichains. However, T_n forms a tree in which all chains are finite. Recall that the *height* of $p \in T_n$ in T_n is the order type of the set $\{x \in T_n : x < p\}$. Denote it by $\varrho_n(p)$. Notice that $\varrho_n(p)$ is finite for every $p \in T_n$ and $T_n = \bigcup_k A_{n,k}$, where $A_{n,k} = \{p \in T_n : \varrho_n(p) = k\}$. It is clear that each $A_{n,k}$ is an antichain. □

Example 24.17.14 Denote by $w\mathbb{Q}$ the tree whose elements are well ordered subsets of the rational numbers $\mathbb{Q}$ and $s < t$ if and only if s is an initial segment of t, that is $s \subseteq t$ and $t \setminus s \subseteq [\sup s, \rightarrow)$. It is clear that $w\mathbb{Q}$ is indeed a tree and it has no uncountable chains. Thus $K(w\mathbb{Q})$ is a Corson compact. That $K(w\mathbb{Q})$ is not Eberlein is a consequence of Theorem 24.17.13 together with the following result of Kurepa. The proof we present is due to Todorčević.

Theorem 24.17.15 *The tree $w\mathbb{Q}$ is not special.*

Proof We show that the subtree $\sigma\mathbb{Q}$ consisting of all bounded well ordered subsets of $\mathbb{Q}$ is not special. For suppose $\sigma\mathbb{Q} = \bigcup_{n\in\omega} A_n$, where each A_n is an antichain.

We construct inductively a sequence $t_0 \leqslant t_1 \leqslant \dots$ in $\sigma\mathbb{Q}$ and a sequence $r_0 \geqslant r_1 \geqslant \dots$ of rational numbers so that $r_n > \sup t_n$ and the following rule holds.

(*) If there exists $s \in A_n$ with $t_{n-1} \leqslant s$ and $\sup s < r_{n-1}$ then t_n has this property, $t_n \in A_n$ and $r_n < r_{n-1}$; otherwise $t_n = t_{n-1}$ and $r_n = r_{n-1}$.

Notice that the above rule says how to construct both sequences. Notice also that the sequences $\{t_n\}_{n\in\omega}$ and $\{r_n\}_{n\in\omega}$ cannot be eventually constant.

Let $t_\infty = \bigcup_{n\in\omega} t_n$. Then $t_\infty \in \sigma\mathbb{Q}$, since it is well ordered and bounded from above by any r_n. Find k such that $t_\infty \in A_k$. Then $t_{k-1} < t_\infty$ and $\sup t_\infty < r_{k-1}$, therefore by (*) we conclude that $t_k \in A_k$ has the same property. Thus $t_k < t_\infty$ are two comparable elements of A_k, a contradiction. □

Let us note that a result similar to Theorem 24.17.13 was obtained by Gruenhage [290]: the space of all initial paths (i.e. downward closed chains) of a tree T is Eberlein if and only if the tree is special. As one can guess, non-trivial here is the "only if" part. Gruenhage's result can be proved by similar arguments as for adequate compact spaces, analysing a slightly different "dual" space (analogous to the adequate dual).

In this direction, perhaps the most "extreme" Corson compact built on a certain tree was found by Todorčević. We refer to [384, Example 5.5] for its short description and further references. It is also proved in [384] that this Corson compact cannot be embedded into a cube $\mathbb{R}^\kappa$ so that its range is dense in $c_0(\kappa)$.

We have not discussed Talagrand and Gul'ko compact spaces here. We refer the readers to Fabian's book [196] for a nice presentation of these classes and corresponding (WCD) Banach spaces. One can prove that the examples described earlier are not Gul'ko compact. Examples of Gul'ko or Talagrand compact spaces that are not Eberlein can be found e.g. in [579] and [196, Chapter 8].

Chapter 25
Complementably Universal Banach Spaces

Abstract We deal with complementably universal Banach spaces. Assuming the Continuum Hypothesis, there exists a complementably universal Banach space of density $\aleph_1$ for the class of Banach spaces with a projectional resolution of the identity. Similar methods produce a universal preimage for the class of Valdivia compacta of weight $\aleph_1$.

A classical result of Pełczyński [490] says that there exists a separable Banach space E with a Schauder basis, such that every other separable space with a basis is isomorphic to a complemented subspace of E (the space E is called *complementably universal* for the class of separable spaces with a basis). When looking for non-separable versions of this result, one needs first to ask a question what kind of a basis to consider.

It turns out that, assuming the Continuum Hypothesis, the class of spaces with a countably norming Markushevich basis has a complementably universal object. We shall present its construction below.

One has to mention a negative result, due to Todorčević [602]: for every Banach space E of size continuum and with Corson's property (C), there exists a (WLD) Banach space F of size continuum which is not isomorphic to a subspace of any quotient of E. A topological counterpart, also proved in [602], reads as follows: for every countably tight compact K of weight continuum, there exists a Corson compact L of weight continuum such that no closed subspace of K maps continuously onto L. A similar result on the non-existence of universal Eberlein compact spaces was proved earlier by Argyros and Benyamini [16].

25.1 An Amalgamation Lemma

It is known that given two Banach spaces X, Y whose intersection $Z = X \cap Y$ is closed in both spaces and both norms coincide on Z, there exists a Banach space W containing an isometric copy of $X \cup Y$. To be more precise, there exist isometric

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embeddings $i\colon X \to W$, $j\colon Y \to W$ such that $i \restriction Z = j \restriction Z$. Such a space W is called an *amalgamation* of X and Y. We need a special version of this fact, where Z is 1-complemented both in X and in Y. This is proved below.

We shall use the following convention in the diagrams: we draw arrows $\rightarrowtail$ to indicate linear isometric embeddings and $\twoheadrightarrow$ for norm 1 projections.

Lemma 25.1.1 *Let X, Y be two Banach spaces with $Z = X \cap Y$ a closed subspace both in X and in Y. Assume further that $P\colon X \to X$ and $Q\colon Y \to Y$ are norm 1 projections such that* $\operatorname{im} P = Z = \operatorname{im} Q$. *Then there exist a Banach space W, isometric embeddings $i\colon X \to W$, $j\colon Y \to W$ and norm one linear operators $P'\colon W \to X$, $Q'\colon W \to Y$ such that*

$$P' \circ i = \mathrm{id}_X, \qquad Q' \circ j = \mathrm{id}_Y$$

and the following diagrams commute.

$$\begin{array}{ccc} Y & \overset{j}{\rightarrowtail} & W \\ {\scriptstyle\subseteq}\uparrow & & \uparrow{\scriptstyle i} \\ Z & \underset{\subseteq}{\rightarrowtail} & X \end{array} \qquad \begin{array}{ccc} Y & \overset{j}{\rightarrowtail} & W \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle P'} \\ Z & \underset{\subseteq}{\rightarrowtail} & X \end{array} \qquad \begin{array}{ccc} Y & \overset{Q'}{\twoheadleftarrow} & W \\ {\scriptstyle\subseteq}\uparrow & & \uparrow{\scriptstyle i} \\ Z & \underset{P}{\twoheadleftarrow} & X \end{array} \qquad \begin{array}{ccc} Y & \overset{Q'}{\twoheadleftarrow} & W \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle P'} \\ Z & \underset{P}{\twoheadleftarrow} & X \end{array}$$

Proof Let $W = (X \oplus_1 Y)/\Delta$, where $\oplus_1$ denotes the direct sum with the ℓ_1-norm (i.e. $\|\,\|\langle x, y\rangle = \|\,\|x + \|\,\|y$ for $x \in X$, $y \in Y$) and $\Delta = \{\langle z, -z\rangle\colon z \in Z\}$. Recall that the quotient norm in W is given by the formula

$$\|\,\|\langle x, y\rangle + \Delta = \inf_{z\in Z}(\|\,\|x - z + \|\,\|x + z).$$

Define $i(x) = \langle x, 0\rangle + \Delta$ and $j(y) = \langle 0, y\rangle + \Delta$. Observe that $\|\,\|i(x) \leqslant \|\,\|x$. On the other hand, $\|\,\|x - z + \|\,\|z \geqslant \|\,\|x$, therefore $\|\,\|i(x) = \|\,\|x$. Thus, i is an isometric embedding. By the same argument, j is an isometric embedding.

Observe that if $\langle x, y\rangle - \langle x', y'\rangle \in \Delta$ then $x' = x - z$, $y' = y + z$ for some $z \in Z$ and hence

$$x' + Qy' = x - z + Q(y + z) = x - z + Qy + z = x + Qy.$$

Similarly, $y' + Px' = y + Px$. Thus, the following definitions are correct.

$$P'(\langle x, y\rangle + \Delta) = x + Qy \qquad \text{and} \qquad Q'(\langle x, y\rangle + \Delta) = y + Px.$$

Clearly, P' and Q' are linear and $\operatorname{im} P' \subseteq X + Z = X$, $\operatorname{im} Q' \subseteq Y + Z = Y$. Further, the inequality

$$\|\,\|P'(\langle x, y\rangle + \Delta) \leqslant \|\,\|x' + \|\,\|y'$$

holds whenever $\langle x, y\rangle + \Delta = \langle x', y'\rangle + \Delta$. It follows that $\| \|P'(\langle x, y\rangle + \Delta) \leqslant \| \|\langle x, y\rangle + \Delta$. Hence $\| \|P' \leqslant 1$. Similarly, $\| \|Q' \leqslant 1$.

It is clear that $P' \circ i = \mathrm{id}_X$ and $Q' \circ j = \mathrm{id}_Y$. It is straight to check that the first three diagrams from the statement of our Lemma are commutative. It remains to check that $P \circ P' = Q \circ Q'$. For this aim, notice that $\mathrm{im}\, P = Z = \mathrm{im}\, Q$, therefore $QPx = Px$ and $PQy = Qy$ for every $x \in X$, $y \in Y$. Thus,

$$PP'(\langle x, y\rangle + \Delta) = P(x + Qy) = Px + Qy = Q(Px + y) = QQ'(\langle x, y\rangle + \Delta).$$

This completes the proof. □

Let us note that the first part of the proof shows that W is an amalgamation of X and Y, without assuming that Z is 1-complemented in these spaces.

25.2 Embedding-Projection Pairs

In order to shorten some statements, we shall denote by $\ddagger\mathfrak{B}$ the category of pairs of the form $f = \langle i, P\rangle$, where $i\colon X \to Y$, $P\colon Y \to X$ are linear operators of norm $\leqslant 1$ such that $P \circ i = \mathrm{id}_X$ and X, Y are separable Banach spaces. Given two compatible pairs $f = \langle i, P\rangle$ and $g = \langle j, Q\rangle$ in $\ddagger\mathfrak{B}$, their composition is defined in the obvious way:

$$f \circ g = \langle j \circ i, P \circ Q\rangle.$$

Observe that for every $\langle i, P\rangle \in \ddagger\mathfrak{B}$ we have that i is a linear isometric embedding and $i \circ P\colon Y \to Y$ is a projection of norm 1 (unless $X = \{0\}$). Conversely, given a norm 1 projection $Q\colon Y \to Y$ with $X = \mathrm{im}\, Q$, we get a $\ddagger\mathfrak{B}$-arrow $\langle i, P\rangle$, where i is the inclusion $X \subseteq Y$ and P is the corestriction of Q to its range (i.e. $P\colon Y \to X$ is defined by $Pv = Qv$ for $v \in Y$). Given any category $\mathfrak{K}$, one can consider the category $\ddagger\mathfrak{K}$, defined in the same way as above. Our particular case is based on the category $\mathfrak{B}$ of separable Banach spaces with linear operators of norm $\leqslant 1$. In general, $\ddagger\mathfrak{K}$ is called *the category of embedding-projection pairs* (see [187]).

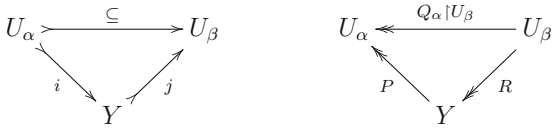
In the language of embedding-projection pairs, Lemma 25.1.1 says in particular that every pair of arrows $f, g \in \ddagger\mathfrak{B}$ with the same domain, can be amalgamated in $\ddagger\mathfrak{B}$. That is, there are $f', g' \in \ddagger\mathfrak{B}$ such that $f' \circ f = g' \circ g$. Indeed, if $f = \langle i, P\rangle$, $g = \langle j, Q\rangle$ are such that $\mathrm{dom}(i) = Z = \mathrm{dom}(j)$ and $X = \mathrm{dom}(P)$, $Y = \mathrm{dom}(Q)$, we may identify replace X and Y by isometric copies, so that $X \cap Y = Z$ and i, j become inclusions. It is important that Lemma 25.1.1 gives a stronger version of amalgamation, which will be needed for universality.

Following is the main technical lemma.

Lemma 25.2.1 *Assume* $2^{\aleph_0} = \aleph_1$. *There exists a Banach space* $U = \bigcup_{\alpha<\omega_1} U_\alpha$ *such that* $\{U_\alpha\}_{\alpha<\omega_1}$ *is a continuous chain of closed separable subspaces,* $\{Q_\alpha\}_{\alpha<\omega_1}$

is a projectional resolution of the identity on U such that $\operatorname{im} Q_\alpha = U_\alpha$ *for* $\alpha < \omega_1$. *Further, the following conditions are satisified.*

(1) $U_0 = \{0\}$ *(the trivial Banach space).*
(2) *Given* $\alpha < \omega_1$, *given a pair of norm one linear operators* $i : U_\alpha \to Y$, $P : Y \to U_\alpha$ *such that* Y *is a separable Banach space and* $P \circ i = \mathrm{id}_{U_\alpha}$, *there exist* $\beta > \alpha$ *and norm one linear operators* $j : Y \to U_\beta$, $R : U_\beta \to Y$ *such that* $R \circ j = \mathrm{id}_Y$ *and the following diagrams commute.*



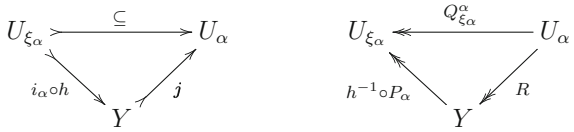
Proof Let $\mathcal{B}$ be a fixed family of separable Banach spaces such that every separable Banach space is linearly isometric to exactly one element of $\mathcal{B}$. It is well known and easy to compute that $|\mathcal{B}| = 2^{\aleph_0}$. For each separable Banach space X we fix a linear isometry $h_X : X \to Y$ such that $Y \in \mathcal{B}$. Now, let $\mathcal{F}$ denote the family of all pairs $f = \langle i, P\rangle$ such that $X, Y \in \mathcal{B}, i : X \to Y, P : Y \to X$ are linear operators of norm $\leqslant 1$ and $P \circ i = \mathrm{id}_X$. In other words, $\mathcal{F}$ consists of all arrows of $\ddagger\mathfrak{B}$ whose domain and codomain are in $\mathcal{B}$. Again, $|\mathcal{F}| = 2^{\aleph_0}$.

Using the assumption $2^{\aleph_0} = \aleph_1$, enumerate the set $\omega_1 \times \mathcal{F}$ as $\{\langle \xi_\alpha, f_\alpha\rangle\}_{\alpha<\omega_1}$ so that for each $\langle \xi, f\rangle \in \omega_1 \times \mathcal{F}$ there exist uncountably many $\alpha < \omega_1$ satisfying $\langle \xi, f\rangle = \langle \xi_\alpha, f_\alpha\rangle$.

We shall construct inductively a continuous chain $\{U_\alpha\}_{\alpha<\omega_1}$ of separable Banach spaces and a sequence $\{Q^\beta_\alpha\}_{\alpha<\beta<\omega_1}$ of norm one projections, in such a way that the following conditions are satisfied.

(3) $U_0 = \{0\}$ and $Q^\beta_\alpha : U_\beta \to U_\alpha$.
(4) $\xi < \eta < \varrho \implies Q^\varrho_\xi = Q^\eta_\xi \circ Q^\varrho_\eta$.
(5) If $\xi_\alpha < \alpha$ and $f_\alpha = \langle i_\alpha, P_\alpha\rangle$ is such that $\operatorname{dom}(i_\alpha)$ is linearly isometric to U_{ξ_α} then, setting $h = h_{U_{\xi_\alpha}}$, there exists $\langle j, R\rangle \in \ddagger\mathfrak{B}$ such that $\operatorname{dom}(R) = U_\alpha$, $h^{-1} \circ P_\alpha \circ R = Q^\alpha_{\xi_\alpha}$ and $j \circ i_\alpha \circ h$ is the inclusion $U_{\xi_\alpha} \subseteq U_\alpha$.

Condition (5) is visualized in the next two diagrams:



Condition (3) tells us how to start the construction. Fix $\alpha < \omega_1$ and suppose U_ξ and Q^η_ξ have been defined for $\xi < \eta < \alpha$. If α is a limit ordinal, let U_α be the completion of the union $\bigcup_{\xi<\alpha} U_\xi$ and let Q^α_ξ be the unique projection satisfying $Q^\alpha_\xi \restriction U_\eta = Q^\eta_\xi$ for every $\xi < \zeta < \alpha$ (see the proof of Theorem 22.5.4).

Suppose $\alpha = \zeta + 1$. If $\xi_\alpha \not< \alpha$ or $\mathrm{dom}(i_\alpha)$ is not linearly isometric to U_{ξ_α}, we set $U_\alpha = U_\zeta$ and $Q^\alpha_\xi = Q^\zeta_\alpha$ for $\xi < \alpha$. So assume that $\xi_\alpha < \alpha$ and $h\colon U_{\xi_\alpha} \to \mathrm{dom}(i_\alpha)$ is the chosen linear isometry, i.e. $h = h_{U_{\xi_\alpha}}$. Let $Y = \mathrm{dom}(P_\alpha)$. Note that $\langle i_\alpha \circ h, h^{-1} \circ P_\alpha\rangle \in \ddagger\mathfrak{B}$. Using Lemma 25.1.1, we can find W and $\langle k, T\rangle$, $\langle j, R\rangle \in \ddagger\mathfrak{B}$ such that

$$j \circ i_\alpha \circ h = k \restriction U_\alpha \qquad \text{and} \qquad h^{-1} \circ P_\alpha \circ R = Q^\zeta_{\xi_\alpha} \circ T.$$

Actually, we are using a weaker form of Lemma 25.1.1, which gives the following two commutative diagrams.

$$\begin{array}{ccc} U_{\xi_\alpha} & \xrightarrow{\subseteq} & U_\zeta \\ {\scriptstyle i_\alpha \circ h}\downarrow & & \downarrow{\scriptstyle k} \\ Y & \xrightarrow[j]{} & W \end{array} \qquad\qquad \begin{array}{ccc} U_{\xi_\alpha} & \xleftarrow{Q^\zeta_{\xi_\alpha}} & U_\zeta \\ {\scriptstyle h^{-1}\circ P_\alpha}\uparrow & & \uparrow{\scriptstyle T} \\ Y & \xleftarrow[R]{} & W \end{array}$$

We may further assume that $U_\zeta \subseteq W$ and k is the inclusion. We set $U_\alpha = W$ and $Q^\alpha_\xi = Q^\zeta_\xi \circ T$ for $\xi < \alpha$. It is clear that conditions (4) and (5) are satisfied.

The above construction gives a Banach space $U = \bigcup_{\alpha<\omega_1} U_\alpha$ (note that this space is already complete). It follows from Theorem 22.5.4 (or rather from its proof) that $\{Q^\beta_\alpha\}_{\alpha<\beta<\omega_1}$ induces a projectional resolution of the identity $\{Q_\alpha\}_{\alpha<\omega_1}$ on U such that $Q_\alpha \restriction U_\beta = Q^\beta_\alpha$ for every $\alpha < \beta < \omega_1$. It remains to check that condition (2) is satisfied.

Fix $\langle i, P\rangle \in \ddagger\mathfrak{B}$ such that $i\colon U_\alpha \to Y$ for some $\alpha < \omega_1$. Recall that there is a unique $X \in \mathcal{B}$ linearly isometric to U_α. Let $h\colon U_\alpha \to X$ be the chosen isometry, i.e. $h = h_{U_\alpha}$. Without loss of generality, we may assume that $Y \in \mathcal{B}$. Notice that $f = \langle i \circ h^{-1}, h \circ P\rangle$ is an element of $\mathcal{F}$. Recall that we have used a "rich" enumeration of $\omega_1 \times \mathcal{F}$, there exists $\beta > \alpha$ such that $\xi_\beta = \alpha$ and $f_\beta = f$. Now condition (5) tells us that that for some $\langle j, R\rangle \in \ddagger\mathfrak{B}$ we have that $j \circ (i \circ h^{-1}) \circ h = j \circ i$ is the inclusion $U_\alpha \subseteq U_\beta$ and $h^{-1} \circ (h \circ P) \circ R = P \circ R$ equals Q^β_α. This shows (3) and completes the proof. □

25.3 A Complementably Universal Banach Space

We are now ready to prove the main result of this chapter.

Theorem 25.3.1 (Kubiś [383]) *Assume the Continuum Hypothesis* $2^{\aleph_0} = \aleph_1$. *There exists a Banach space* U *with the following properties.*

(a) $\mathrm{dens}\, U = \aleph_1$ *and* U *has a projectional resolution of the identity.*
(b) Every Banach space with a projectional resolution of the identity and of density $\leqslant \aleph_1$ *is linearly isometric to a 1-complemented subspace of* U.

(c) Given two separable 1-complemented subspaces $X, Y \subseteq U$, for every bijective linear isometry $h\colon X \to Y$ there exists a bijective linear isometry $\overline{h}\colon U \to U$ such that $\overline{h} \restriction X = h$.

Proof We shall show that the space U from Lemma 25.2.1 has the desired properties. Property (a) is stated and proved in Lemma 25.2.1 already.

Fix a Banach space X of density $\leqslant \aleph_1$ and with a PRI $\{P_\alpha\}_{\alpha<\omega_1}$. Write $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a continuous chain of separable subspaces such that $X_0 = \{0\}$. If X is separable, we may assume that $X_\alpha = X$ for $\alpha > 0$. Let $Q_\alpha^\beta = Q_\alpha \restriction U_\beta$ and $P_\alpha^\beta = P_\alpha \restriction X_\beta$. We shall construct inductively a sequence of pairs $\langle j_\alpha, R_\alpha\rangle \in \ddagger\mathfrak{B}$ so that $\mathrm{dom}(j_\alpha) = X_\alpha$, $\mathrm{dom}(R_\alpha) = X_{\delta(\alpha)}$, and for each $\xi < \eta$, the following diagrams commute.

$$\begin{array}{ccc} U_{\delta(\xi)} & \xrightarrow{\subseteq} & U_{\delta(\eta)} \\ {\scriptstyle j_\xi}\uparrow & & \uparrow{\scriptstyle j_\eta} \\ X_\xi & \xrightarrow[\subseteq]{} & X_\eta \end{array} \qquad \begin{array}{ccc} U_{\delta(\xi)} & \xleftarrow{Q^{\delta(\eta)}_{\delta(\xi)}} & U_{\delta(\eta)} \\ {\scriptstyle R_\xi}\downarrow & & \downarrow{\scriptstyle R_\eta} \\ X_\xi & \xleftarrow[P^\eta_\xi]{} & X_\eta \end{array} \qquad \begin{array}{ccc} U_{\delta(\xi)} & \xrightarrow{\subseteq} & U_{\delta(\eta)} \\ {\scriptstyle R_\xi}\downarrow & & \downarrow{\scriptstyle R_\eta} \\ X_\xi & \xrightarrow[\subseteq]{} & X_\eta \end{array}$$

Since $X_0 = U_0 = \{0\}$, we put $\delta(0) = 0$ and $j_0 = \mathrm{id}_{X_0} = R_0$. Fix $\beta > 0$ and assume that $\langle j_\xi, R_\xi\rangle$ have been defined for every $\xi < \beta$. If β is a limit ordinal, we set $\delta(\beta) = \sup_{\xi<\beta}\delta(\xi)$. There is a uniquely determined pair $\langle j_\beta, R_\beta\rangle \in \ddagger\mathfrak{B}$ so that the above diagrams commute for every $\xi < \eta \leqslant \beta$.

Suppose now that $\beta = \alpha+1$. Using Lemma 25.1.1, find $\langle k, S\rangle \in \ddagger\mathfrak{B}$ and $\langle \ell, T\rangle \in \ddagger\mathfrak{B}$ with $W = \mathrm{dom}(S) = \mathrm{dom}(T)$, so that the following diagrams commute.

$$\begin{array}{ccc} U_{\delta(\alpha)} & \xrightarrow{k} & W \\ {\scriptstyle j_\alpha}\uparrow & & \uparrow{\scriptstyle \ell} \\ X_\alpha & \xrightarrow[\subseteq]{} & X_{\alpha+1} \end{array} \qquad \begin{array}{ccc} U_{\delta(\alpha)} & \xleftarrow{S} & W \\ {\scriptstyle R_\alpha}\downarrow & & \downarrow{\scriptstyle T} \\ X_\alpha & \xleftarrow[P^{\alpha+1}_\alpha]{} & X_{\alpha+1} \end{array} \qquad \begin{array}{ccc} U_{\delta(\alpha)} & \xrightarrow{k} & W \\ {\scriptstyle R_\alpha}\downarrow & & \downarrow{\scriptstyle T} \\ X_\alpha & \xrightarrow[\subseteq]{} & X_{\alpha+1} \end{array}$$

We now use condition (2) of Lemma 25.2.1 to obtain $\delta > \delta(\alpha)$ and $\langle m, A\rangle \in \ddagger\mathfrak{B}$, so that the following diagrams commute.

$$\begin{array}{ccc} U_{\delta(\alpha)} & \xrightarrow{\subseteq} & U_\delta \\ & {\scriptstyle k}\searrow \quad \nearrow{\scriptstyle m} & \\ & W & \end{array} \qquad \begin{array}{ccc} U_{\delta(\alpha)} & \xleftarrow{Q^\delta_{\delta(\alpha)}} & U_\delta \\ & {\scriptstyle S}\nwarrow \quad \swarrow{\scriptstyle A} & \\ & W & \end{array}$$

Finally, we set $\delta(\alpha+1) = \delta$ and $\langle j_{\alpha+1}, R_{\alpha+1}\rangle = \langle m \circ \ell, T \circ A\rangle$. It is straight to check that the above condition is satisfied.

The above construction leads to an embedding $j\colon X \to U$ together with a norm one operator $R\colon U \to X$ such that $R \circ j = \mathrm{id}_X$. Thus, X is isometric to a 1-complemented subspace of U. This shows (b).

Part (c) is proved by a standard back-and-forth argument. We sketch it below.

Fix a bijective linear isometry $h\colon X \to Y$ between separable 1-complemented subspaces of U. First, let us denote by u_α^β the $\ddagger\mathfrak{B}$-pair $\langle \iota_\alpha^\beta, Q_\alpha^\beta\rangle$, where $\iota_\alpha^\beta\colon U_\alpha \to U_\beta$ is inclusion and $Q_\alpha^\beta = Q_\alpha \restriction U_\beta$. We shall construct inductively $\ddagger\mathfrak{B}$-arrows $f_\xi\colon U_{\alpha(\xi)} \to U_{\beta(\xi)}$ and $g_\xi\colon U_{\beta(\xi)} \to U_{\alpha(\xi+1)}$ for $\xi < \omega_1$, so that

(i) $f_\xi = \langle i_\xi, P_\xi\rangle$, $g_\xi = \langle j_\xi, R_\xi\rangle$ and i_0 extends h,
(ii) $g_\xi \circ f_\xi = u_{\alpha(\xi)}^{\beta(\xi)}$ and $f_{\xi+1} \circ g_\xi = u_{\beta(\xi)}^{\beta(\xi+1)}$,
(iii) $\xi < \eta \implies u_{\beta(\xi)}^{\beta(\eta)} \circ f_\xi = f_\eta \circ u_{\alpha(\xi)}^{\alpha(\eta)}$.

Find $\alpha(0) < \theta < \omega_1$ so that $X \subseteq U_{\alpha_0}$ and $Y \subseteq U_\theta$. Let $\langle a, B\rangle \in \ddagger\mathfrak{B}$ be such that $a\colon X \to U_\theta$ is the composition of h with the inclusion $Y \subseteq U_\theta$ and $B\colon U_\theta \to X$ is a fixed norm 1 projection onto Y followed by h^{-1}. Using Lemma 25.1.1, find $\langle c, D\rangle \in \ddagger\mathfrak{B}$ and $\langle e, F\rangle \in \ddagger\mathfrak{B}$ so that the following diagrams commute.

$$\begin{array}{ccc} X & \xrightarrow{\subseteq} & U_{\alpha(0)} \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e} \\ U_\theta & \xrightarrow[c]{} & W \end{array} \qquad \begin{array}{ccc} X & \xleftarrow{S} & U_{\alpha(0)} \\ {\scriptstyle B}\uparrow & & \uparrow{\scriptstyle F} \\ U_\theta & \xleftarrow[D]{} & W \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\subseteq} & U_{\alpha(0)} \\ {\scriptstyle B}\uparrow & & \uparrow{\scriptstyle B} \\ U_\theta & \xrightarrow[c]{} & W \end{array} \qquad \begin{array}{ccc} X & \xleftarrow{S} & U_{\alpha(0)} \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e} \\ U_\theta & \xleftarrow[D]{} & W \end{array}$$

where $S\colon U_{\alpha(0)} \to X$ is a fixed norm 1 projection. Using property (2) of Lemma 25.2.1, we find $\beta(0) > \theta$ and $\langle c', D'\rangle \in \ddagger\mathfrak{B}$ so that $c' \circ c = \iota_\theta^{\beta(0)}$ and $D \circ D' = Q_\theta^{\beta(0)}$. We set $f_0 = \langle c' \circ e, F \circ D'\rangle$. Using condition (2) of Lemma 25.2.1, find $g_0 \in \ddagger\mathfrak{B}$ so that

$$g_0 \circ f_0 = \langle \iota_{\alpha(0)}^{\alpha(1)}, \iota_{\alpha(0)}^{\alpha(1)}\rangle.$$

Now fix $0 < \delta < \omega_1$ and suppose $\{f_\eta\}_{\eta<\delta}$ and $\{g_\eta\}_{\eta<\delta}$ have been defined. If δ is a limit ordinal, we define i_δ and P_δ to be the unique map extending $\bigcup_{\eta<\delta} i_\eta$ and $\bigcup_{\eta<\delta} P_\eta$ respectively. This defines f_δ. We define g_δ in the same way, and we easily check that conditions (ii), (iii) are satisfied.

Suppose $\delta = \xi + 1$. Using condition (2) of Lemma 25.2.1 twice, find $f_{\xi+1}$ and next $g_{\xi+1}$ so that

$$f_{\xi+1} \circ g_\xi = \langle \iota_{\beta(\xi)}^{\beta(\xi+1)}, Q_{\beta(\xi)}^{\beta(\xi+1)}\rangle \qquad \text{and} \qquad g_{\xi+1} \circ f_{\xi+1} = \langle \iota_{\alpha(\xi+1)}^{\alpha(\xi+2)}, \iota_{\alpha(\xi+1)}^{\alpha(\xi+2)}\rangle.$$

This finishes the construction.

Finally, let $\overline{h}\colon U \to U$ be such that $\overline{h} \restriction U_{\alpha(\xi)} = \iota_{\beta(\xi)} \circ i_\alpha$, where ι_α denotes the inclusion $U_\alpha \subseteq U$. The existence of $\overline{h}$ is guaranteed by condition (ii). Further, (i) says that $\overline{h}$ extends h. Finally, (iii) ensures that h is a bijective linear isometry. □

One can notice that the above construction can be generalized to other categories. In particular, assuming the Continuum Hypothesis, there exists a Valdivia compact K of density $\aleph_1$ such that every other Valdivia compact of the same density is

homeomorphic to a retract of K. For a general, category-theoretic, approach to this subject we refer to [383]. In fact, the Banach space constructed in Lemma 25.2.1 is the so-called *Fraïssé limit* of the category $\ddagger\mathfrak{B}$. Originally, the notion of a Fraïssé limit belongs to model theory; it was developed by Fraïssé in [242] and continued by Jónsson in [323]. Category-theoretic approach is due to Droste and Göbel [187], extended in Kubiś [383].

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